Nonuniqueness of Anisotropic Teleseismic Tomography

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Aki (1976) pointed out that in isotropic teleseismic P wave tomography, the slowness fields $s(x, y, z)$ and $s(x, y, z) + s^{(0)}(z)$ fit the data equally well, which implies that the earth structure can be determined only up to an additive vertically-stratified function. This nonuniqeness arises because near-source structure is not accounted for and only the differential travel times between stations recording a common event are meaningful. The vertically-stratified function $s^{(0)}$causes a constant offset T_0 in the travel time T of each event, but this offset drops out when differential travel times are computed. (Note that the offset varies with the angle of incidence of the event, but presuming parallel straight-line rays, is constant for any given event).

We now consider a simplified version of anisotropic teleseismic S wave tomography, where the earth model is presumed to have azimuthal anisotropy with a constant and known symmetry axes but varying strength of anisotropy. In this model, the travel times of two orthogonal polarizations of S waves, say T_1 and T_2 each depend on different slowness fields, say $s_1(x, y, z)$ and $s_2(x, y, z)$. The mean slowness $[s_1(x, y, z) + s_2(x, y, z)]/2$ is a measure of the isotropic structure and their difference $[s_2(x, y, z) - s_1(x, y, z)]/2$ is a measure of the strength of the anisotropy. As in the P wave case, only differential travel time between stations recording a common event are meaningful; however the splitting time T_2-$T_1$is also measurable.

While the addition of the vertically-stratified structures:

$$s_1(x, y, z) + s_1^{(0)}(z) \text{ and } s_2(x, y, z) + s_2^{(0)}(z)$$

causes no change in the differential delays of each polarization, it does change the splitting time. However, if we require that the vertical averages of $s_1^{(0)}(z)$ and $s_2^{(0)}(z)$ are equal:

$$\langle s_1^{(0)}(z)\rangle \equiv \frac{1}{z_{max}}\int_0^{z_{max}} s_1^{(0)}(z)\,dz \;=\; \frac{1}{z_{max}}\int_0^{z_{max}} s_2^{(0)}(z)\,dz \;\equiv \langle s_2^{(0)}(z)\rangle$$

then each polarization experiences the same delay and the splitting time is unchanged. The two offsets vary with the angle of incidence of the event, but presuming parallel straight-line rays, they vary proportionately, so the difference in offset between polarizations is always zero. Such structures lead to no change in splitting times.

In conclusion, in this simple form of anisotropic S wave tomography, the model is determined up to two arbitrary vertically-stratified slownesses with equal vertical averages.

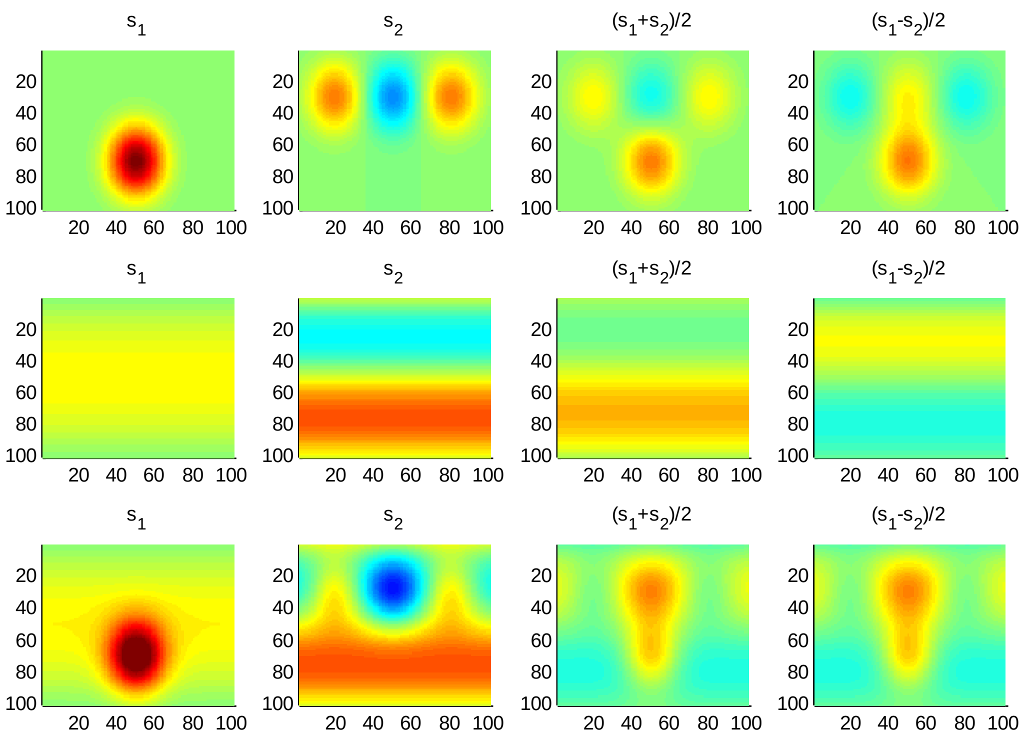

Figure 1. (Top row) Vertical slice through a hypothetical 3D earth structure (s_1, s_2) (Middle row) Vertical slice through a hypothetical vertically-stratified earth structure $(s_1^{(0)}, s_2^{(0)})$, which obeys the constraint $\langle s_1^{(0)}\rangle = \langle s_2^{(0)}\rangle$. (Bottom row) Vertical slice through the 3D earth structure $(s_1 + s_1^{(0)}, s_2 + s_2^{(0)})$.

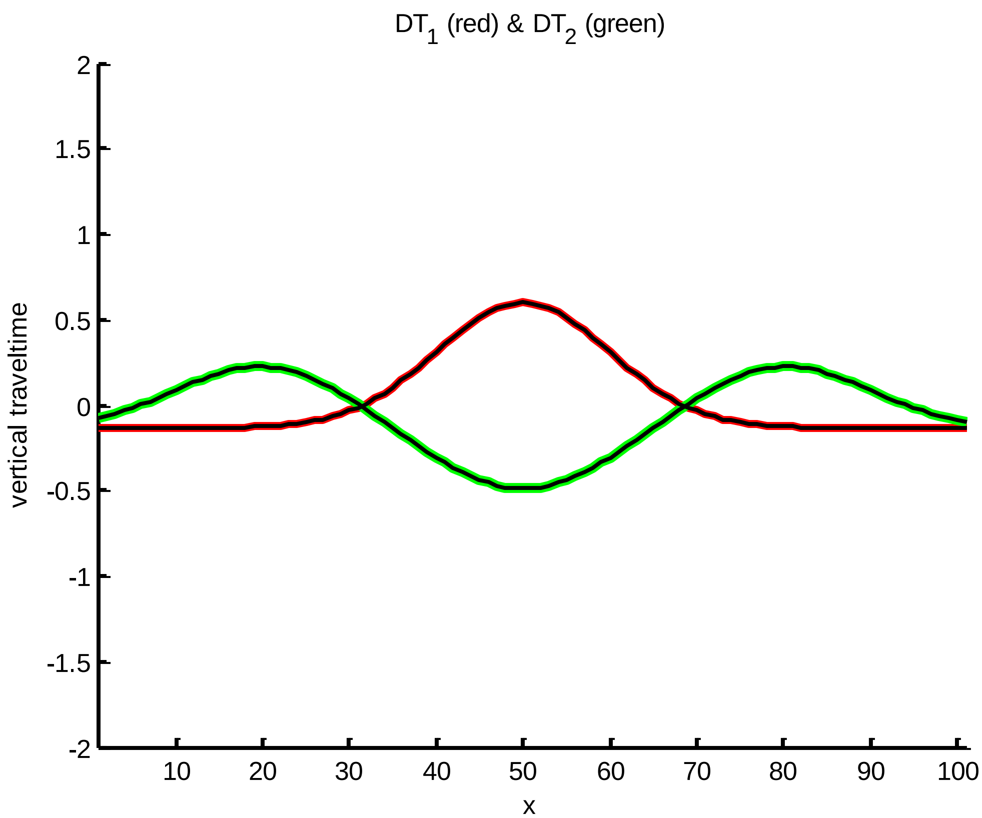

Figure 2. Differential travel times $T_1(x)$ and $T_2(x)$ for vertical rays traversing the (s_1, s_2) model (red and green, respectively) and the $(s_1 + s_1^{(0)}, s_2 + s_2^{(0)})$ model (black). An overall mean travel time of $\sum_i^N \big(T_1(x_i)+T_2(x_i)\big)/(2N)$ has been removed in both cases.

Aki, K., Y. Christoffersson and E. Husebye, Three-dimensional seismic structure under the Montana LASA, Bull. Seism. Soc. Am. 66, 501-524, 1976.

MatLab script

```
clear all;

Nx=101;
Nz=101;
Nt=Nx*Nz;

N = zeros(Nx,Nz);
Nn = 1;
an = [2]';
xn = [50]';
zn = [70]';
sxn = [10]';
szn = [15]';
for ix=[1:Nx]
for iz=[1:Nz]
    for j=[1:Nn]
        N(iz,ix) = N(iz,ix) + ...
            an(j)*exp(-0.5*(((ix-
xn(j))/sxn(j))^2)) ...
                 *exp(-0.5*(((iz-
zn(j))/szn(j))^2));
    end
end
end

E = zeros(Nx,Nz);
Ne = 3;
ae = [1, -1, 1]';
xe = [20, 50, 80]';
```

```
ze = [30, 30, 30]';
sxe = [10, 10, 10]';
sze = [15, 15, 15]';
for ix=[1:Nx]
for iz=[1:Nz]
    for j=[1:Ne]
        E(iz,ix) = E(iz,ix) + ...
            ae(j)*exp(-0.5*(((ix-
xe(j))/sxe(j))^2)) ...
                 *exp(-0.5*(((iz-
ze(j))/sze(j))^2));
    end
end
end

figure(1);
clf;
subplot(3,4,1);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc(N);
title('s_1');
subplot(3,4,2);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
```

```
colormap('jet');
imagesc(E);
title('s_2');
subplot(3,4,3);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc((N+E)/2);
title('(s_1+s_2)/2');
subplot(3,4,4);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc((N-E)/2);
title('(s_1-s_2)/2');

N2 = zeros(Nx,Nz);
an2 = 0.5;
kn2 = pi/Nz;
zn2 = 0;
cn2 = 0.0;
for ix=[1:Nx]
for iz=[1:Nz]
    N2(iz,ix) = N2(iz,ix) + cn2 + ...
            an2*sin( kn2*(iz-zn2) );
end
```

```
end

E2 = zeros(Nx,Nz);
ae2 = 0.85;
ke2 = 2*pi/Nz;
ze2 = 50;
ce2 = 0.0;
for ix=[1:Nx]
for iz=[1:Nz]
    E2(iz,ix) = E2(iz,ix) + ce2 + ...
            ae2*sin( ke2*(iz-ze2) );
end
end

AN2 = sum(sum(N2))/Nt;
AE2 = sum(sum(E2))/Nt;
E2 = E2 - AE2 + AN2;

subplot(3,4,5);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc(N2);
title('s_1');
subplot(3,4,6);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
```

```
caxis( [-2, 2] );
colormap('jet');
imagesc(E2);
title('s_2');
subplot(3,4,7);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc((N2+E2)/2);
title('(s_1+s_2)/2');
subplot(3,4,8);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc((N2-E2)/2);
title('(s_1-s_2)/2');

E3 = E + E2;
N3 = N + N2;

subplot(3,4,9);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
```

```
colormap('jet');
imagesc(N3);
title('s_1');
subplot(3,4,10);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc(E3);
title('s_2');
subplot(3,4,11);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc((N3-E3)/2);
title('(s_1+s_2)/2');
subplot(3,4,12);
axis ij;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, 1, Nz] );
caxis( [-2, 2] );
colormap('jet');
imagesc((N3-E3)/2);
title('(s_1-s_2)/2');

figure(2);
```

```
clf;
set(gca,'LineWidth',2);
hold on;
axis( [1, Nx, -2, 2] );
tN = sum(N,1)/Nz;
tE = sum(E,1)/Nz;
tA = (sum(tN)/Nx+sum(tE)/Nx)/2;
tN = tN - tA;
tE = tE - tA;
plot( [1:Nx]', tN, 'r-', 'LineWidth', 3 );
plot( [1:Nx]', tE, 'g-', 'LineWidth', 4 );

tN3 = sum(N3,1)/Nz;
tE3 = sum(E3,1)/Nz;
tA3 = (sum(tN3)/Nx+sum(tE3)/Nx)/2;
tN3 = tN3 - tA3;
tE3 = tE3 - tA3;
plot( [1:Nx]', tN3, 'k-', 'LineWidth', 2 );
plot( [1:Nx]', tE3, 'k-', 'LineWidth', 2 );
xlabel('x');
ylabel('vertical traveltime');
title('DT_1 (red) & DT_2 (green)');
```