Let $x_i$ be a sequence of uncorrelated random numbers with zero mean and variance $\sigma^2$. The autoregressive sequence AR1 is:

$$y_i = \varphi y_{i-1} + x_i \quad i = 2, 3, 4, \ldots$$

Rearranging, we obtain the IIR filter:

$$y_i - \varphi y_{i-1} = x_i \quad i = 2, 3, 4, \ldots$$

which has z-transform

$$(1 - \varphi z) y(z) = x(z)$$

Evaluating $z$ on the unit circle $z = \exp \left( -\pi if / f_{ny} \right)$ (where $f$ is frequency and $f_{ny}$ is the Nyquist frequency), we obtain the Fourier transform:

$$
\left(1 - \varphi \exp \left( -\frac{\pi if}{f_{ny}} \right)\right) y(f) = x(f)
$$

or

$$u(f) \ y(f) = x(f) \quad \text{with} \quad u(f) \equiv 1 - \varphi \exp \left( -\frac{\pi if}{f_{ny}} \right)$$

So the spectrum of $y(f)$ is:

$$|y(f)|^2 = \frac{|x(f)|^2}{|u(f)|^2} \quad \text{with} \quad |u(f)|^2 = 1 + \varphi^2 - 2\varphi \cos \left( \frac{\pi f}{f_{ny}} \right)$$

From the point of view of the statistics of spectra, $u(f)$ is just a multiplicative constant. Thus, the

$$\mean(|y(f)|^2) = \frac{\mean(|x(f)|^2)}{|u(f)|^2} \quad \text{and} \quad \var(|y(f)|^2) = \frac{\var(|x(f)|^2)}{|u(f)|^4}$$

The power spectral density of uncorrelated white noise is:

$$\mean(|x(f)|^2) = pc \quad \text{and} \quad \var(|x(f)|^2) = 2pc^2 \quad \text{with} \quad c = \frac{f_f \sigma^2}{2 \, N_f \, \Delta f}$$
with degrees of freedom \( p = 2 \), number of frequencies \( N_f = N/2 + 1 \), number of data \( N \) and \( f_r = N^{-1} \sum w_i^2 \) a correction for the power reduction that arises from multiplication of \( x_i \) by the window function \( w_i \).

Example for \( \sigma = 100, \varphi = 0.6, N = 1024 \):

Black: Power spectral density of one realization of the AR1 process.
Green: Median of 1000 realizations of the AR1 process.
Yellow: Mean of 1000 realizations of the AR1 process.
Red: Mean predicted from formula. Note agreement with yellow curve.
Blue: 95% confidence interval of 1000 realizations of the AR1 process.
Magenta: Mean plus 2\( \sigma \) (proxy for 95% confidence) predicted from formula. Note agreement with blue curve.

Key part of MATLAB code:

\[
\begin{align*}
p &= 2; \\
f_f &= 1; \\
c &= \frac{f_f \cdot (\sigma^2)}{2 \cdot N_f \cdot D_f}; \\
xs2meantrue &= p \cdot c; \\
p &= 2; \\
xs2vartrue &= 2 \cdot p \cdot c^2; \\
denom &= (1 + (\varphi^2) - 2 \cdot \varphi \cdot \cos(2\pi \cdot 0.5 \cdot f / f_{\max})); \\
ys2mean &= \frac{xs2meantrue}{denom}; \\
y295 &= ys2mean + 2 \cdot \sqrt{xs2vartrue} / denom;
\end{align*}
\]