Confidence limits for the power spectral density of an AR1 process

Bill Menke, September 30, 2015

Let x_i be a sequence of uncorrelated random numbers with zero mean and variance σ^2. The autoregressive sequence AR1 is:

$$y_i = \varphi y_{i-1} + x_i \quad i = 2, 3, 4, \dots$$

Rearranging, we obtain the IIR filter:

$$y_i - \varphi y_{i-1} = x_i \quad i = 2, 3, 4, \dots$$

which has z-trasnform

$$(1 - \varphi z)\, y(z) = x(z)$$

Evaluating z on the unit circle $z = \exp\left(-\pi i f / f_{ny}\right)$ (where f is frequency and f_{ny} is the Nyquist frequency), we obtain the Fourier transform:

$$\left(1 - \varphi \exp\left(-\frac{\pi i f}{f_{ny}}\right)\right) y(f) = x(f)$$

or

$$u(f)\, y(f) = x(f) \quad \text{with} \quad u(f) \equiv 1 - \varphi \exp\left(-\frac{\pi i f}{f_{ny}}\right)$$

So the spectrum of $y(f)$ is:

$$|y(f)|^2 = \frac{|x(f)|^2}{|u(f)|^2} \quad \text{with} \quad |u(f)|^2 = 1 + \varphi^2 - 2\varphi \cos\left(\frac{\pi i f}{f_{ny}}\right)$$

From the point of view of the statistics of spectra, $u(f)$ is just a multiplicative constant. Thus, the

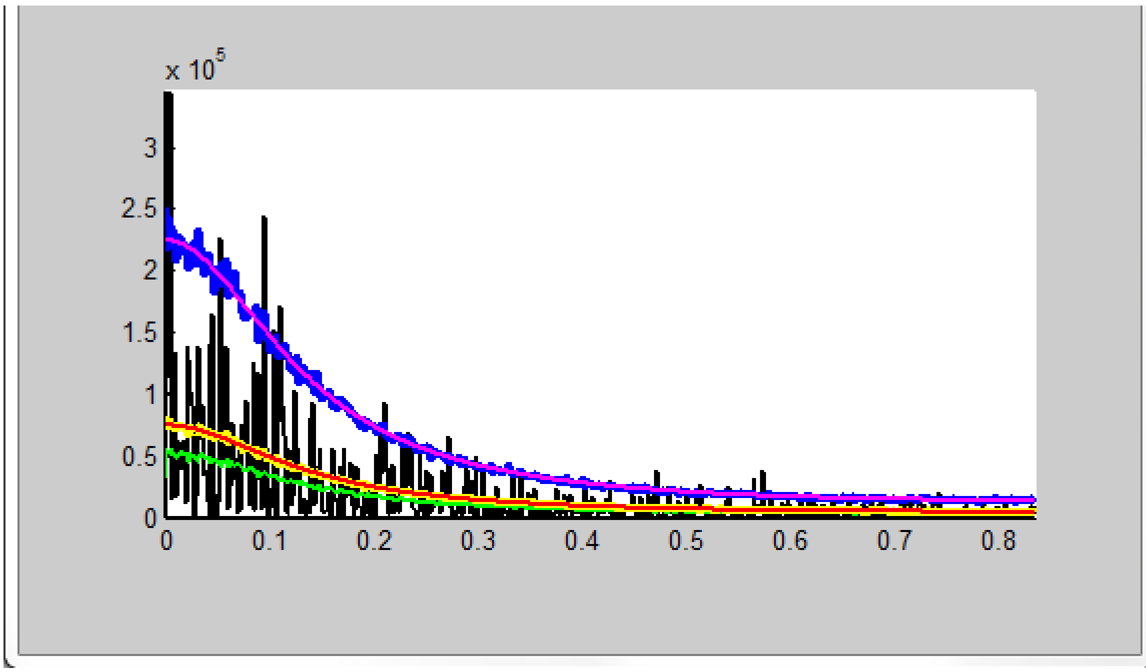
$$\text{mean}(|y(f)|^2) = \frac{\text{mean}(|x(f)|^2)}{|u(f)|^2} \quad \text{and} \quad \text{var}(|y(f)|^2) = \frac{\text{var}(|x(f)|^2)}{|u(f)|^4}$$

The power spectral density of uncorrelated white noise is:

$$\text{mean}(|x(f)|^2) = pc \quad \text{and} \quad \text{var}(|x(f)|^2) = 2pc^2 \quad \text{with} \quad c = \frac{f_f \sigma^2}{2\, N_f\, \Delta f}$$

with degrees of freedom $p = 2$, number of frequencies$N_f = N/2 + 1$, number of data N and $f_f = N^{-1} \sum w_i^2$ a correction for the power reduction that arises from multiplication of x_i by the window function w_i.

Example for$\sigma = 100$, $\varphi = 0.6$, $N = 1024$:



Black: Power spectral density of one realization of the AR1 process.
Green: Median of 1000 realizations of the AR1 process.
Yellow: Mean of 1000 realizations of the AR1 process.
Red: Mean predicted from formula. Note agreement with yellow curve.
Blue: 95% confidence interval of 1000 realizations of the AR1 process.
Magenta: Mean plus 2σ (proxy for 95% confidence) predicted from formula. Note agreement with blue curve.

Key part of MATLAB code:

```
p=2;
ff = 1;
c = ( ff*(sigma^2) ) / (2*Nf*Df);
xs2meantrue = p*c;
p=2;
xs2vartrue = 2*p*c^2;
denom = (1 + (phi^2) - 2*phi*cos(2*pi*0.5*f/fmax) );
ys2mean =  xs2meantrue ./ denom;
ys295 = ys2mean + 2*sqrt(xs2vartrue) ./ denom;
```