Application of Adjoint Methods to Travel Time Tomography

By Bill Menke, April 19, 2016 (with addendum March 25, 2017)

We study an acoustic problem in which the predicted travel time T depends upon slowness s. Here we discuss the use of adjoint methods to compute the partial derivative of the total error $E=(T^{obs}-T)^2$ with respect to a model parameter m that controls the slowness.

Derivation of $\partial E/\partial m$.

The Eikonal equation for traveltime T in an acoustic medium with slowness s is:

$$\nabla T\cdot\nabla T=s^2 \quad \text{or} \quad |s^{-1}\nabla T|=1$$

This equation indicates that travel time advances according to the local slowness; that is $s^{-1}\nabla T=\hat{n}$, where $\hat{n}$ is a unit vector parallel to the "ray" (direction of propagation). In this problem, all function vary with spatial coordinates (x,y,z) but have no time dependence.

Let the slowness equal a background slowness s_0 plus a small perturbation εs_1, where ε is a small parameter, and the corresponding travel time equal a background travel time T_0 plus a small perturbation εT_1. Then:

$$\nabla T\cdot\nabla T=\nabla(T_0+\varepsilon T_1)\cdot\nabla(T_0+\varepsilon T_1)=(s_0+\varepsilon s_1)^2\approx s_0^2+2\varepsilon s_0 s_1$$

Equating terms of equal order in ε yields equations for the background travel time and the perturbation in travel time:

$$\nabla T_0\cdot\nabla T_0=s_0^2 \quad \text{and} \quad s_0^{-1}\nabla T_0\cdot\nabla T_1=s_1 \quad \text{or} \quad \hat{n}_0\cdot\nabla T_1=s_1$$

The third equation indicates that the component of the ∇T_1 in the direction $\hat{n}_0$ direction is s_1. If we define $d\ell$ to be an increment of arc length along the unperturbed ray (the curve R with tangent $\hat{n}_0$), then this is just an equation involving the directional derivative:

$$\frac{dT_1}{d\ell}=s_1 \quad \text{which has solution} \quad T_1=\int_R s_1\,d\ell$$

This is exactly the result that we expect from Fermat's principle. The perturbation in travel time is the integral of the perturbation in slowness along the unperturbed ray. We abbreviate the equation for the perturbed time T_1 as:

$$\mathcal{L}T_1=s_1 \quad \text{where} \quad \mathcal{L}\equiv s_0^{-1}\nabla T_0\cdot\nabla=\hat{n}_0\cdot\nabla$$

Now suppose that we observe travel time T^{obs} everywhere in space and define the error $e=T^{obs}-T$. The total error is:

$$E=\langle e,e\rangle$$

where the inner product $\langle .,. \rangle$ is over three dimensional space. Now suppose that the slowness perturbation is a function of a model parameter m; that is $s_1 = s_1(m)$. The corresponding travel time perturbation and total error are also functions of m; that is $T = T(m)$and $E = E(m)$. . The partial derivative of the total error E is:

$$\frac{\partial E}{\partial m} = 2\,\langle e, \frac{\partial e}{\partial m}\rangle = -2\,\langle e, \frac{\partial T}{\partial m}\rangle = -2\varepsilon\,\langle e, \frac{\partial T_1}{\partial m}\rangle$$

Since $\mathcal{L}$ is not a function of m, differentiating $\mathcal{L}T_1 = s_1$ yields:

$$\mathcal{L}\frac{\partial T_1}{\partial m} = \frac{\partial s_1}{\partial m} \quad \text{or} \quad \frac{\partial T_1}{\partial m} = \mathcal{L}^{-1}\,\frac{\partial s_1}{\partial m}$$

The partial derivative of the total error is therefore:

$$\frac{\partial E}{\partial m} = -2\varepsilon\,\langle e_0, \frac{\partial T_1}{\partial m}\rangle = -2\varepsilon\,\langle e_0, \mathcal{L}^{-1}\,\frac{\partial s_1}{\partial m}\rangle$$

Here $e_0 \equiv T^{obs} - T_0$. Manipulating this equation using adjoints yields:

$$\frac{\partial E}{\partial m} = -2\varepsilon\,\langle \mathcal{L}^{-1\dagger} e_0, \frac{\partial s_1}{\partial m}\rangle = -2\varepsilon\,\langle \lambda, \frac{\partial s_1}{\partial m}\rangle$$

with an adjoint field λ that satisfies:

$$\lambda \equiv \mathcal{L}^{-1\dagger} e_0 \quad \text{or} \quad \mathcal{L}^{\dagger}\lambda = e_0$$

The operator $\mathcal{L}$, written out in terms of matrix operations, is:

$$\mathcal{L} = s_0^{-1}\nabla T_0 \cdot \nabla = s_0^{-1}\begin{bmatrix}\frac{\partial T_0}{\partial \mathrm{x}} & \frac{\partial T_0}{\partial \mathrm{y}} & \frac{\partial T_0}{\partial \mathrm{z}}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial \mathrm{x}} \\ \frac{\partial}{\partial \mathrm{y}} \\ \frac{\partial}{\partial \mathrm{z}}\end{bmatrix}$$

Using the rules $(\boldsymbol{\mathcal{L}}_1\boldsymbol{\mathcal{L}}_2)^{\dagger} = \boldsymbol{\mathcal{L}}_2^{T\dagger}\boldsymbol{\mathcal{L}}_1^{T\dagger}$ and $(\partial/\partial\mathrm{x})^{\dagger} = -\partial/\partial\mathrm{x}$, we obtain

$$so \quad \mathcal{L}^{\dagger}\lambda = -\begin{bmatrix}\frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}}\end{bmatrix}\begin{bmatrix}\frac{\partial T_0}{\partial \mathrm{x}} \\ \frac{\partial T_0}{\partial \mathrm{y}} \\ \frac{\partial T_0}{\partial \mathrm{z}}\end{bmatrix}(s_0^{-1}\lambda) = -\nabla\cdot[\nabla T_0(s_0^{-1}\lambda)] \quad \text{or } \nabla\cdot(\lambda\, s_0^{-1}\nabla T_0) = -e$$

Equivalent forms are:

$$\nabla\cdot(\lambda\,\hat{n}_0) = -e_0 \quad \text{and} \quad \hat{n}_0\cdot\nabla\lambda + \lambda\,\nabla\cdot\hat{n}_0 = -e_0$$

When expressed in terms of the arc-length ℓ along the unperturbed ray, this equation has the form of an inhomogeneous first order ordinary differential equation:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ell} + P(\ell)\,\lambda = Q(\ell) \quad \text{with} \quad P(\ell) = \nabla \cdot \hat{n}_0 \quad \text{and} \quad Q(\ell) = -e_0$$

A closed form for the solution of this equation is known and is reasonably well-behaved.

When the error is defined only at a single point, say $E_C = e^2(\mathbf{x}_R)$ (where the subscript C designates an error associated with a point), then $\partial E_C/\partial m$ is promoted to an inner product by introducing a 3D Dirac impulse function:

$$e(\mathbf{x}) = e_0\,\delta(\mathbf{x} - \mathbf{x}_R) \quad \text{with} \quad e_0 \equiv e(\mathbf{x}_R)$$

so that

$$\frac{\partial E_C}{\partial m} = 2\,\langle e, \frac{\partial e}{\partial m}\rangle = 2\,\langle e_0\delta(\mathbf{x} - \mathbf{x}_R), \frac{\partial e}{\partial m}\rangle = 2e_0\,\frac{\partial e_0}{\partial m}$$

and the adjoint field becomes:

$$\frac{\partial E_C}{\partial m} = -2\varepsilon\,\langle \lambda_C, \frac{\partial s_1}{\partial m}\rangle \quad \text{and} \quad \nabla \cdot (\lambda_C\,\hat{n}_0) = -e_C \quad \text{with} \quad e_C = e_0\,\delta(\mathbf{x} - \mathbf{x}_R)$$

When the error is defined only on a surface (say $z = z_0$), so that:

$$E_S = \iint e^2(x, y, z_0)\,\mathrm{d}x\,\mathrm{d}y$$

(where the subscript S designates an error associated with a surface), then $\partial E_S/\partial m$ is promoted to an inner product by introducing a 1D Dirac impulse function:

$$e(\mathbf{x}) = e_0(x, y)\delta(\mathrm{z} - \mathrm{z}_0) \quad \text{with} \quad e_0(x, y) \equiv e(x, y, z_0)$$

so that

$$\frac{\partial E}{\partial m} = 2\,\langle e, \frac{\partial e}{\partial m}\rangle = 2\,\langle e_0(x, y)\delta(\mathrm{z} - \mathrm{z}_0), \frac{\partial e}{\partial m}\rangle = \iint 2e(x, y, z_0)\frac{\partial e}{\partial m}\,\mathrm{d}x\,\mathrm{d}y$$

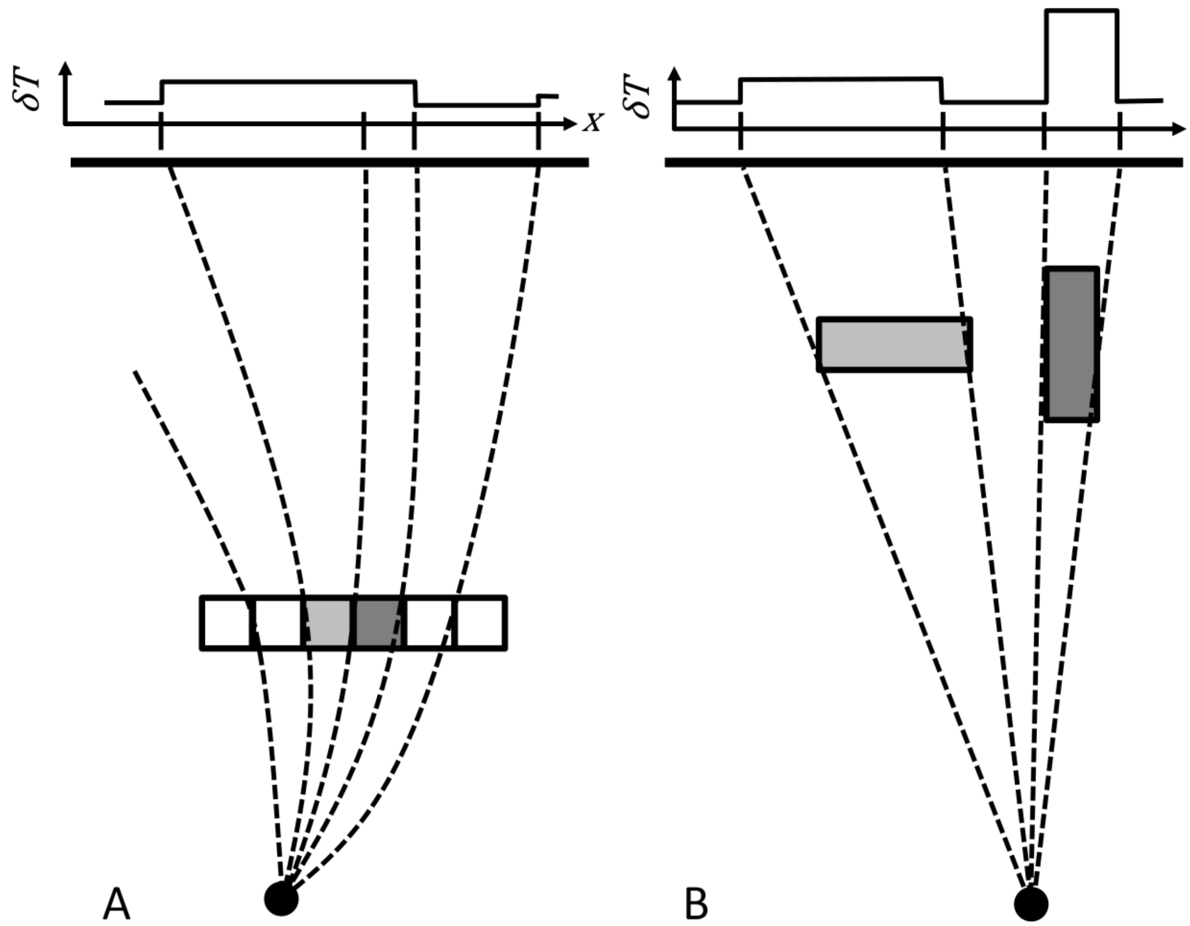
and the adjoint field becomes:

$$\frac{\partial E_S}{\partial m} = -2\varepsilon\,\langle \lambda_S, \frac{\partial s_1}{\partial m}\rangle \quad \text{and} \quad \nabla \cdot (\lambda_S\,\hat{n}_0) = -e_S \quad \text{with} \quad e_S = e_0(x, y, z_0)\delta(\mathrm{z} - \mathrm{z}_0)$$

In both of these case, the error is zero except at the observation point or surface. In the zero-error region, the adjoint equation in homogeneous:

$$\hat{n}_0 \cdot \frac{\nabla \lambda}{\lambda} = -\nabla \cdot \hat{n}_0$$

This equation indicates that the fractional change in λ in the ray direction $\hat{n}_0$ is equal to the negative of the divergence of neighboring rays. This is exactly the form of the *transport equation* for the ray theoretical amplitude (see Menke and Abbott, 1989, their equation 8.7.13); the adjoint field behaves exactly like geometrical spreading. The reason for this behavior is illustrated in Part A of the figure below, where the error is defined on a surface:



Suppose that the model is parameterized in terms of voxels, so that $\partial s_1 / \partial m = 1$ inside a voxel and zero outside of it. Then $\partial E / \partial m = -2\varepsilon \langle \lambda, \partial s_1 / \partial m \rangle = -2\varepsilon V \bar{\lambda}$, where V is the volume of the voxel and $\bar{\lambda}$ is the average value of the adjoint field inside the voxel. The light grey voxel in the figure subtends a large patch of surface, whereas the dark grey voxel subtends a small patch. Although a perturbation δm in slowness causes the same perturbation δT in travel time, the one that subtends the most area results in the largest perturbation δE in the error. The adjoint field backtracks the geometrical spreading from the surface of observation to the voxel and gives the voxel that subtends the most area of the surface the most weight.

The formula $\partial E/\partial m = -2\varepsilon V\bar{\lambda}$ is independent of the shape of the voxel and its orientation relative to the unperturbed ray paths, even though, intuitively, we might expect some dependence. The perturbation δT in travel time, as depicted in part B of the figure, is indeed dependent on shape. However, an elongated voxel aligned parallel to the unperturbed ray has a large δT but subtends a small patch of the surface, while the same voxel aligned perpendicular to the unperturbed ray has a small δT but subtends a large patch of the surface. The two effects exactly cancel.

Plane Wave Example

Consider an unperturbed plane wave with travel time $T_0(x, y) = s_0 x$. The direction of propagation is along the x-axis; that is $\hat{n}_0 = s_0^{-1}\nabla T_0 = \hat{x}$. This travel time function satisfies the Eikonal equation, because $\left|s_0^{-1}\nabla T_0\right| = |\hat{x}| = 1$. Consider a perturbation in slowness:

$$s_1(x, y) = m\,\delta(x - x_H)b(y)$$

Here $b(y)$ is the unit boxcar function, which is unity inside the $(0,1)$ interval and zero outside of it. This slowness distribution is an idealization of a rectangular heterogeneity. The perturbed travel time satisfies:

$$\hat{n}_0 \cdot \nabla T_1 = s_1 \quad \text{or} \quad \frac{\partial T_1}{\partial x} = m\,\delta(x - x_H)b(y)$$

Integration parallel to the direction of propagation, together with the boundary condition that $T_1 = 0$ to the left, yields:

$$T_1(x, y) = m\,H(x - x_H)b(y)$$

where H is the Heaviside step function. This function defines a rectangular region of constant travel time perturbation to the right of the heterogeneity.

By direct differentiation, the partial derivative of travel time with respect to model parameter is:

$$\frac{\partial T_1}{\partial m} = H(x - x_H)b(y)$$

We first derive the partial derivative of error using a direct method. Suppose that the travel time is observed at one station located at (x_R, y_R), yielding error $e_C = e^{obs}(x_R, y_R)\delta(x - x_R)\delta(y - yR$. The partial derivative of error is:

$$\frac{\partial E_C}{\partial m} = -2\varepsilon\left\langle e_C, \frac{\partial T_1}{\partial m}\right\rangle = -2\varepsilon\langle e_C, H(x - x_H)b(y)\rangle$$

$$= -2\varepsilon e^{obs}(x_R, y_R)\langle \delta(x - x_R)\delta(y - y_R), H(x - x_H)b(y)\rangle$$

$$= -2\varepsilon e^{obs}(x_R, y_R)H(x_R - x_H)b(y_R)$$

The partial derivative of travel time with respect to the model parameter is zero everywhere, except in a rectangular region to the right of the heterogeneity, where it is the constant $-2\varepsilon e^{obs}(x_R, y_R)$. The error is perturbed by an amount that is independent of the position of the receiver, as long as the receiver is in the “shadow” of the heterogeneity.

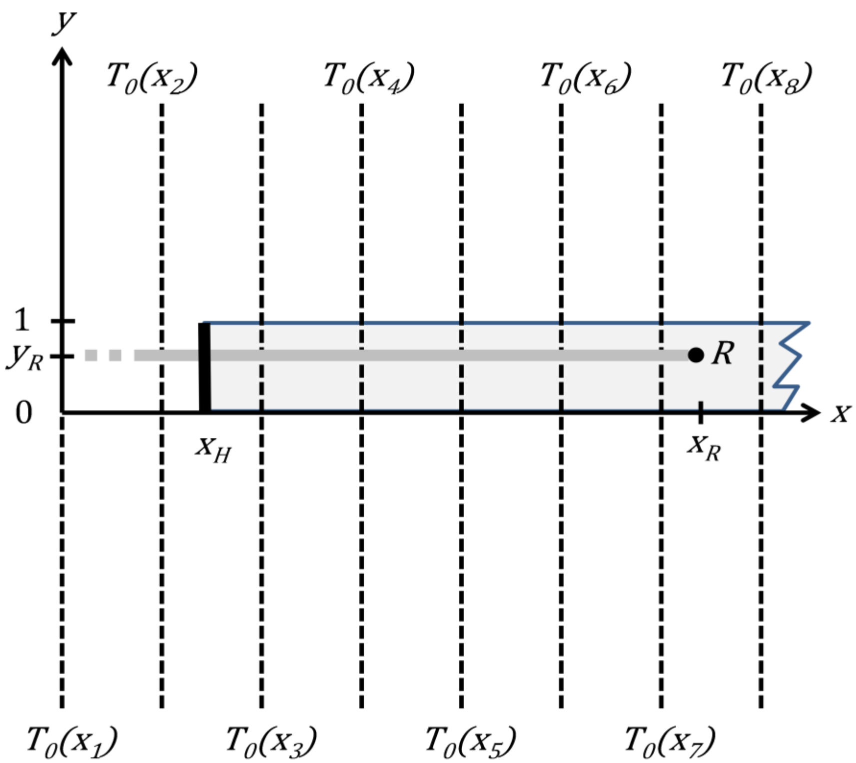


Fig.1. Surfaces of equal travel time (dotted vertical lines); heterogeneity (bold vertical bar at x_H). Region of constant travel time perturbation (grey rectangle); receiver (circle R at x_R); line segment along which the adjoint field λ is non-zero

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We now derive the same result using the adjoint equation:

$$\nabla \cdot (\lambda_C\, \hat{n}_0) = -e_C$$

$$\frac{\partial \lambda_C}{\partial x} = -e^{obs}(x_R, y_R)\delta(x - x_R)\delta(y - y_R)$$

Integration parallel to the direction of propagation, together with the boundary condition that $\lambda = 0$ to the right, yields:

$$\lambda_C = e^{obs}(x_R, y_R) H(x_R - x)\delta(y - y_R)$$

The adjoint field is zero, except on a line segment to the left of the receiver, where it is singular. The partial derivative of error is then:

$$\frac{\partial E_C}{\partial m} = -2\varepsilon \langle \lambda_C, \frac{\partial s_1}{\partial m} \rangle$$

$$= -2e^{obs}(x_R, y_R)\langle H(x_R - x)\delta(y - y_R), \delta(x - x_H)b(y)\rangle$$

$$= -2\varepsilon e^{obs}(x_R, y_R)\, H(x_R - x_H)\, b(y_R)$$

This is the same result as was achieved with the direct method. The error is perturbed by an amount that is independent of the position of the receiver, as long as the adjoint field, when "back-tracked" from the receiver along a ray, intersects the heterogeneity.

Spherical Wave Example

Consider an unperturbed plane wave with travel time $T(r, \theta, \varphi) = s_0 r$. The direction of propagation is along radially outward; that is $\hat{n}_0 = s_0^{-1}\nabla T_0 = \hat{r}$, since $\nabla f(r) = (\partial f/\partial r)\hat{r}$. This travel time function satisfies the Eikonal equation, because $|s_0^{-1}\nabla T_0| = |\hat{r}| = 1$. Consider a perturbation in slowness:

$$s_1(r, \theta, \varphi) = m\, \delta(r - r_H) b(\theta, \varphi)$$

Here $b(\theta, \varphi)$ is a two dimensional boxcar function that is unity in the range $0 < \theta < \theta_0$ and $0 < \varphi < \varphi_0$ and zero outside it. This slowness distribution is an idealization of a "rectangular" spherical patch at a radius r_H from the origin. The perturbed travel time satisfies:

$$\hat{n}_0 \cdot \nabla T_1 = s_1 \quad \text{or} \quad \frac{\partial T_1}{\partial r} = m\, \delta(r - r_H) b(\theta, \varphi)$$

Integration parallel to the direction of propagation (that is, $\int dr$), together with the boundary condition that $T_1 = 0$ at $r < r_H$, yields:

$$T_1(x, y) = m\, H(r - r_H) b(\theta, \varphi)$$

where H is the Heaviside step function. This function defines a rectangular region of non-zero travel time perturbation to the $r > r_H$ side of the heterogeneity.

By direct differentiation, the partial derivative of travel time with respect to model parameter is:

$$\frac{\partial T_1}{\partial m} = H(r - r_H) b(\theta, \varphi)$$

Now suppose that the travel time is observed at one station located at (x_R, y_R), yielding error:

$$e_C = e^{obs}(r_R, \theta_R, \varphi_R)\delta(\mathbf{r} - \mathbf{r}_R) = e^{obs}(r_R, \theta_R, \varphi_R)\frac{\delta(r - r_R)\,\delta(\theta - \theta_R)\,\delta(\varphi - \varphi_R)}{r^2 \sin\theta}$$

The partial derivative of total error is:

$$\frac{\partial E_C}{\partial m} = -2\varepsilon\,\langle e_C, \frac{\partial T_1}{\partial m}\rangle = -2\varepsilon\langle e_C, H(r - r_H)b(\theta, \varphi)\rangle$$

$$= -2\varepsilon e^{obs}(r_R, \theta_R, \varphi_R)\,\langle\frac{\delta(r - r_R)\,\delta(\theta - \theta_R)\,\delta(\varphi - \varphi_R)}{r^2 \sin\theta}, H(r - r_H)\,b(\theta, \varphi)\rangle$$

$$= -2\varepsilon\, e^{obs}(r_R, \theta_R, \varphi_R)\, H(r_R - r_H\,)b(\theta_R, \varphi_R)$$

As in the pane wave case, the derivative is a constant $-2\varepsilon\, e^{obs}$ is the receiver is in the "shadow" of the patch, and zero otherwise.

We now derive the same result using the adjoint equation:

$$\nabla \cdot (\lambda_C\,\hat{n}_0) = -e_C$$

The divergence obeys the rule $\nabla \cdot (f\,\hat{r}) = r^{-2}\,\partial(r^2 f)/\partial r$, so

$$\frac{\partial(r^2 \lambda_C)}{\partial \mathrm{r}} = -e^{obs}(r_R, \theta_R, \varphi_R)\frac{\delta(r - r_R)\,\delta(\theta - \theta_R)\,\delta(\varphi - \varphi_R)}{\sin\theta}$$

Integration parallel to the direction of propagation, together with the boundary condition that $\lambda = 0$ for $r > r_R$, yields:

$$r^2 \lambda_C = e^{obs}(r_R, \theta_R, \varphi_R)\,H(r_R - r)\frac{\delta(\theta - \theta_R)\,\delta(\varphi - \varphi_R)}{\sin\theta}$$

The adjoint field is zero, except on a radially-oriented line segment at smaller radius than the receiver, where it is singular. The partial derivative of error is then:

$$\frac{\partial E_C}{\partial m} = -2\varepsilon\,\langle\lambda_C, \frac{\partial s_1}{\partial m}\rangle$$

$$= \ -2\varepsilon\, e^{obs}(r_R, \theta_R, \varphi_R)\ \langle H(r_R - r)\frac{\delta(\theta - \theta_R)\,\delta(\varphi - \varphi_R)}{r^2 \sin\theta}, \delta(r - r_H)b(\theta, \varphi)\rangle$$

$$= \ -2\varepsilon\, e^{obs}(r_R, \theta_R, \varphi_R)\, H(r_R - r_H)\, b(\theta_R, \varphi_R)$$

As in the plane wave case, this is the same result as was achieved with the direct method. The error is perturbed by an amount that is independent of the position of the receiver, as long as the adjoint field, when "back-tracked" from the receiver along a ray, intersects the heterogeneity.

Conclusions

We have successfully used the adjoint method to construct a formula for the partial derivative of travel time error with respect to a slowness model parameter. An analysis of the formula using plane wave and spherical wave examples brings out its ray-like character. However, whether this formula has a greater utility than traditional ray-based methods is unclear.

References:

Menke, W. and D. Abbott, Geophysical Theory (Textbook), Columbia University Press, 458p, 1989.