The Wave Field Error is Invariant Under Downward Continuation

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Consider a scalar wave field $u(x, y, z, t)$ consisting of up-going traveling waves observed on the surface $z = z_0$. This wave field can be downward continued to a depth $z_1 < z_0$ through a homogenous medium with velocity c using the following process: 1) Fourier transform (x, y, t) to the frequency-wavenumber domain (k_x, k_y, ω):

$$\hat{u}(k_x, k_y, \omega, z_0) = \mathcal{F}\, u(x, y, t, z_0)$$

where $\mathcal{F}$ is the Fourier transform operator. The function $\hat{u}(k_x, k_y, \omega, z_0)$ represents the amplitudes of plane waves $\exp(ik_x x + ik_y y - i\omega t)$. 2) Advance each of these plane waves by a phase factor $\Phi \equiv \exp(i\varphi)$. For a travelling wave, $\varphi = k_z \Delta z$ where $k_z = +(\omega^2 c^{-2} - k_x^2 - k_z^2)^{1/2}$ and $\Delta z = z_0 - z_1$. By assumption, the wave field contains no evanescent waves, so that $\hat{u}(k_x, k_y, \omega) = 0$ for frequency-wavenumber combinations for which k_z is imaginary. However, in practice, observational noise may lead to spurious non-zero amplitudes. Rather than to continue exponentially growing waves, we define $\varphi_z = \mathrm{real}(k_z)\, z_0$, which has the effect of leaving unchanged the amplitudes of these spurious waves during the downward continuation process. 3) The downward continued wave field is inverse Fourier transformed back into the space/time domain.

$$u(x, y, z_1, t) \; = \mathcal{F}^{-1}\, \Phi\, \hat{u}(k_x, k_y, \omega, z_0)$$

Here $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Note that $\mathcal{F}$ and Φ are unary transformations; that is $\mathcal{F}^{-1} = \mathcal{F}^{\dagger}$ and $\Phi^{-1} = \Phi^{\dagger}$. Here † represents the Hermetian adjoint, e.g. $\langle \mathcal{L}v, w \rangle = \langle v, \mathcal{L}^{\dagger} w \rangle$ where the inner product $\langle ., . \rangle$ is defined as:

$$\langle v, w \rangle = \int \iiint v^* \, u \; dx \; dy \; dt$$

where v^* is the complex conjugate of v.

The wave field error is defines as:

$$E = \langle u^{obs} - u, u^{obs} - u \rangle$$

It is invariant under downward continuation of the wave field:

$$E(z_1) = \langle [u^{obs}(z_1) - u(z_1)], [u^{obs}(z_1) - u(z_1)] \rangle$$

$$= \langle \mathcal{F}^{-1}\, \Phi \mathcal{F} [u^{obs}(z_0) - u(z_0)], \mathcal{F}^{-1}\, \Phi \mathcal{F} [u^{obs}(z_0) - u(z_0)] \rangle$$

$$= \langle\, [u^{obs}(z_0) - u(z_0)], \mathcal{F}^{\dagger}\Phi^{\dagger}\mathcal{F}^{-1\dagger}\mathcal{F}^{-1}\;\Phi\mathcal{F}[u^{obs}(z_0) - u(z_0)] \rangle$$

$$= \langle\, [u^{obs}(z_0) - u(z_0)], \mathcal{F}^{-1}\Phi^{-1}\mathcal{F}\mathcal{F}^{-1}\;\Phi\mathcal{F}[u^{obs}(z_0) - u(z_0)] \rangle$$

$$= \langle\, [u^{obs}(z_0) - u(z_0)], [u^{obs}(z_0) - u(z_0)] \rangle = E(z_0)$$