The Azimi Attenuation Model  
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Amplitude attenuation model: The amplitude $A(\omega, x)$ declines exponentially with distance $x$, according to an attenuation function $\alpha(\omega)$, or equivalently, the quality factor $Q(\omega)$ or tee-star $t^*(\omega)$:

$$A(\omega, x) = A_0(\omega) \exp\{-\alpha(\omega) x\} = A_0(\omega) \exp\left\{-\frac{\omega x}{2c(\omega)Q(\omega)}\right\} = A_0(\omega) \exp\left\{-\frac{\omega t^*(\omega)}{2}\right\}$$

Here $\omega$ is angular frequency, $A_0(\omega)$ is the initial amplitude at $x = 0$, and $c(\omega)$ is phase velocity. In some seismological settings, the quality factor $Q(\omega)$ is only weakly frequency-dependent, and one can speak, approximately, of a constant-$Q$ attenuation model (with quality factor $Q_0$). Similarly, while the phase velocity is dispersive, in some settings it is only weakly so, and one can speak of a non-dispersive model (with velocity $c_0$). When dispersion is negligible, the attenuation function and quality factor are related by:

$$\alpha(\omega) \approx \frac{\omega}{2c_0Q(\omega)} \quad \text{and} \quad Q(\omega) \approx \frac{\omega}{2c_0\alpha(\omega)}$$

Propagation Model: A harmonic wave with angular frequency $\omega$, initial amplitude $A_0(\omega)$ and time dependence $\exp\{-i\omega t\}$ propagates to a position $x$ via:

$$A_0(\omega) \exp\{ikx - i\omega t\} = A_0(\omega) \exp\{i\omega[s(\omega)x - t]\}$$

Here $s(\omega) = k/\omega = 1/c(\omega)$ is the phase slowness.

Causality requires that the attenuation function $\alpha(\omega)$ and phase slowness $s(\omega)$ be related through an integral equation called the Kramer-Kronig Relationship. It can be shown that no constant-$Q$ model can satisfy this relationship. Azimi found an $(\alpha, s)$ pair that satisfies the relationship and is approximately constant-$Q$, at least for frequencies much less than some corner frequency $\omega_0$:

$$a(\omega) = \frac{a_2\omega}{1 + a_3\omega} \quad \text{and} \quad \Delta s(\omega) = s(\omega) - s_0 = -\frac{2a_2 \ln a_3\omega}{\pi(1 - a_3^2\omega^2)} \quad \text{with} \quad \omega \geq 0$$

Here $a_2$ and $a_3$ are constants. Note that when we set $a_2 = 1/(2c_0Q_0)$ and $a_3 = 1/\omega_0$, the attenuation function obeys:

$$a(\omega) \approx a_2\omega \quad \text{and} \quad Q(\omega) \approx \frac{\omega}{2c_0\alpha(\omega)} = \frac{1}{2c_0a_2} = Q_0 \quad \text{for} \quad \omega \ll \omega_0$$

That is, it is constant-$Q$ for frequencies much less than the corner frequency.
A real displacement pulse \( u_0(t) = u(x = 0, t) \) can be attenuated and propagated to the position \( x \) in the following steps:

**Step 1:** Fourier transform \( u_0(t) \) to \( \tilde{u}_0(\omega) \) and focus on the non-negative frequency values of \( \tilde{u}_0(\omega) \) only.

**Step 2:** Multiply \( \tilde{u}_0(\omega) \) by \( \exp\{-\alpha(\omega)x\}\exp\{i\omega s(\omega)x\} \) to obtain \( \tilde{u}(\omega) \).

**Step 3:** Set \( \tilde{u}(\omega = 0) \) to unity.

**Step 4:** Form the negative frequency values of \( \tilde{u}(\omega) \) by taking the complex conjugate of the positive frequency values.

**Step 5:** Inverse Fourier transform \( \tilde{u}(\omega) \) back to \( u(t) \).

Sometimes, it may be convenient to replace \( s(\omega) \) with \( \Delta s(\omega) \) in Step 2, so that the pulse is only delayed by the deviation in phase velocity. In this way, several pulses can be aligned on the same plot.

Note that \( c_0 \) and \( Q_0 \) appear only in the constant \( a_2 \propto 1/(c_0Q_0) \), and not in \( a_3 \) and that \( a_2 \) appears in \( a(\omega) \) and \( \Delta s(\omega) \) only as a leading multiplicative factor. Thus, both decay rate \( a(\omega)x \) and phase delay \( \Delta s(\omega)x \) are proportional to \( x/(c_0Q_0) = t_0^* \). Therefore, the pulse shape contains only enough information to determine \( t_0^* \) and not enough to determine \( x \) and \( Q_0 \) individually.

Sample \( Q(f) \)'s for \( f_0 = 2\pi\omega_0 = 50 \text{ Hz} \), \( Q_0 = 10 \) (red) and 20 (green) and \( c_0 = 4.5 \text{ km/s} \).
Sample $c(f)$’s for $f_0 = 2\pi \omega_0 = 50$ Hz, $Q_0 = 10$ (red) and 20 (green) and $c_0 = 4.5$ km/s.
Sample $u(x,t)$ for and $x = 100$ km and $u_0(t)$ a length $N = 1024$ time-series with a sampling interval of 0.1 s and a unit spike at position $N/2$: 
The differential attenuation between the two Azimi pulses (black) and the best-fitting log-linear model (red).

The true differential $t^*$ and the one estimated via the linear fit:

$Dt^{true} = 1.111111$  $Dt^{arest} = 1.100142$
MATLAB CODE

clear all;

% spectral ratio of two azimi pulses

% azimi attenuation model has 2 parameters
Q1 = 20;  % quality factor at low frequencies
Q2 = 10;  % quality factor at low frequencies
f0 = 50;  % frequency below which Q(f) is approximately constant

N=1024;  % number of samples
Dt=0.1;  % sampling interval
c0=4.5;  % low frequency velocity
x=100;  % propagation distance

[ t, pulse0, pulse1, f, Qf1, cw1 ] = azimi( N, Dt, x, c0, Q1, f0 );
[ t, pulse0, pulse2, f, Qf2, cw2 ] = azimi( N, Dt, x, c0, Q2, f0 );

figure(1);
clf;
hold on;
axis( [40, 70, 0, 0.1] );
plot( t, pulse0, 'k-', 'LineWidth', 2 );
plot( t, pulse1, 'g-', 'LineWidth', 2 );
plot( t, pulse2, 'r-', 'LineWidth', 2);
title('azimi pulses for two different amounts of attenuation');
xlabel('t');
ylabel('u');

figure(2);
clf;
hold on;
axis( [f(1), f(end), 0.5*Q2, 2*Q1] );
plot( f, Qf1, 'g-', 'LineWidth', 2 );
plot( f, Qf2, 'r-', 'LineWidth', 2);
plot( [f(1), f(end)], [Q1, Q1], 'k:', 'LineWidth', 2 );
plot( [f(1), f(end)], [Q2, Q2], 'k:', 'LineWidth', 2 );
title('Q(f) associated with the two azimi pulses');
xlabel('f');
ylabel('Q');
% plot frequency-dependent quality factors
figure(4);
clf;
hold on;
axis( [f(1), f(end), 0, 2*c0] );
plot( f, cw1, 'g-', 'LineWidth', 2 );
plot( f, cw2, 'r-', 'LineWidth', 2 );
plot( [f(1), f(end)], [c0, c0], 'k:', 'LineWidth', 2 );
title('c(f) associated with the two azimi pulses');
xlabel('f');
ylabel('c');

% standard fft setup
fny = f(end);
Df = f(2)-f(1);
N2 = N/2+1;

% compute spectral ratio
pulse1t = fft( pulse1 );
pulse1t = pulse1t(1:N2);
A1 = abs( pulse1t );
pulse2t = fft( pulse2 );
pulse2t = pulse2t(1:N2);
A2 = abs( pulse2t );
r = A2 ./ A1;
r(1)=1; % reset zero-frequency value

% confine analysis to f<fc band
fc = 0.5;
Nc = floor(fc/Df)+1;
f = f(1:Nc);
r = r(1:Nc);
logr = log(r);

% fit straight line to log spectral ratio
G = [ones(Nc,1), f];
mest = (G'*G)
    \( G'*logr )
b = mest(2);
logrpre = G*mest;
% A = A0 exp( -w x/2Qc ) = A0 exp( -f pi tstar )
% b = -pi tstar so tstar = -b/pi

% compare true and predicted tstar
Dtstarest = -b/pi;
Dtstartrue = x/(Q2*c0) - x/(Q1*c0);
fprintf('Dtstartrue %f Dtstarest %f
', Dtstartrue, Dtstarest);

% plot spectral ratio and straight line fit
figure(3);
clf;
hold on;
axis([0, fc, -2, 1]);
plot( f, logr, 'k-', 'LineWidth', 2);
plot( f, logrpre, 'ro', 'LineWidth', 2);
title('log spectral ratio (solid) of the two pulses with linear fit (circles)');
xlabel('f');
ylabel('pulse2(f) / pulse1(f)');

function [ t, pulse0, pulse, f, Qw, cw ] = azimi( N, Dt, x, c0, Q, f0 )

% input parameters:
% f0 corner frequency of Azimi Q model, in hz (e.g. 50)
% c0 base velocity in km/s (e.g. 4.5);
% x propagation distance in km (e.g. 100)
% Q low frequency quality factor (e.g. 10)
% N number of samples in pulse (e.g. 1024);
% Dt sampling interval (e.g. 0.1)

% returned values
% t time array
% pulse0 input pulse, a unit spike at time N/2
% pulse attenuated pulse
% f frequencies in Hz
% Qw frequency dependent quality factors
% cw frequency dependent phase velocities

% time series
 t = Dt*[0:N-1]';
pulse0 = zeros(N,1);
pulse0(N/2)=1;

% standard fft setup
 fny = 1/(2*Dt);
N2 = N/2+1;
df = fny / (N/2);
f = df*[0:N2-1]';
w = 2*pi*f;
w0 = 2*pi*f0;

% attenuation factor
% \exp( -a(w) x ) = \exp( -\frac{wx}{2Qc} )
%
% propagation law with velocity c=w/k and slowness s=1/c=k/w
% \exp\{ i(kx - wt) \} = \exp\{ iw(sx - t) \}
% propagation law
% \exp\{ iwsx \}

% Azimi's second law en.wikipedia.org/wiki/Azimi_Q_models
% 
% a(w) = a2 \frac{|w|}{[ 1 + a3 |w| ]}
% note that for w<<w0 a(w) = 
%
% s(w) = s0 + 2 a2 \ln( a3 w ) / [ \pi (1 - a3^2 w^2 ) ]

% now set a3 = 1/w0 where w0 is a reference frequency
% and set a2 = 1 / (2Qc0) where c0 is a reference velocity
% so that
% a(w) = (1/2Qc0) \frac{|w|}{[ 1 + |w|/w0 ]}
% so for w/w0 << 1
% a(w) = w/(2Qc0) and Q(w) = w/(2 a c0)

a2 = 1 / (2*Q*c0);
a3 = 1 / w0;
a = a2*w ./ ( 1 + a3.*w );
Qw = w ./ (2.*a.*c0);
Qw(1) = Q;
ds = -2*a2*log(a3*w) ./ (pi*(1-(a3^2).*(w.^2 )));
ds(1)=0;
cw = 1./( (1/c0) + ds );

dt = fft(pulse0);
dp = dt(1:N2);
dp = dp .* exp(-a*x) .* exp(-complex(0,1)*w.*ds.*x);
dtnew = [dp(1:N2); conj(dp(N2-1:-1:2))]; % fold out negative frequencies
pulse = ifft(dtnew);

end