1. Mathematical Preliminaries.

Consider the 2\textsuperscript{nd} order linear differential equation:

\[ \mathcal{L}y = \left\{ \frac{d}{dt} p(t) \frac{d}{dt} + q(t) \right\} y(t) = f(t) \]

with the Green function \( G(t, \tau) \) satisfying \( \mathcal{L}G(t, \tau) = \delta(t - \tau) \) so that:

\[ y(t) = \int G(t, \tau) f(\tau) \, d\tau \]

Note that \( \mathcal{L} \) is self-adjoint:

\[ \left[ \frac{d}{dt} p(t) \frac{d}{dt} + q(t) \right]^\dagger = \left[ \frac{d}{dt} p(t) \frac{d}{dt} \right]^\dagger + [q(t)]^\dagger = \frac{d}{dt} p(t) \frac{d}{dt} + q(t) \]

It follows that:

\[ \langle v, \mathcal{L}u \rangle_t = \langle u, \mathcal{L}v \rangle_t \]

for any two solutions \( u(t) \) and \( v(t) \) where the inner product is taken over all time:

\[ \langle a, b \rangle_t = \int_{-\infty}^{+\infty} a(t) \, b(t) \, dt \]

We can use this result to show that the Green function satisfied a reciprocity relationship. Set \( u = G(t, \tau_1) \) and \( v = G(t, \tau_2) \). Then:

\[ \langle G(t, \tau_2), \delta(t - \tau_1) \rangle_t = \langle G(t, \tau_1), \delta(t - \tau_2) \rangle_t \]

\[ G(\tau_1, \tau_2) = G(\tau_2, \tau_1) \]

It also follows that:

\[ \frac{\partial}{\partial t} G(t, \tau) = \frac{\partial}{\partial t} G(\tau, t) \]
(but I can’t work out a proof).

Green’s relationship:

\[
\int_{T_1}^{T_2} (u \, Lv - \nu \, Lu) \, dt = (pu v_t - pu_t v) \big|_{T_1}^{T_2}
\]

(with \( u_t \equiv du/dt \) and \( v_t \equiv dv/dt \)) follows from integration by parts:

\[
\int_{T_1}^{T_2} \left( u \, \frac{d}{dt} pv_t + qv - v \, \frac{d}{dt} pu_t + qu \right) \, dt =
\]

\[
\int_{T_1}^{T_2} \left( u \, \frac{d}{dt} pv_t + quv - v \, \frac{d}{dt} pu_t - quv \right) \, dt =
\]

\[
\int_{T_1}^{T_2} \left( u \, \frac{d}{dt} pv_t - v \, \frac{d}{dt} pu_t \right) \, dt =
\]

\[
\int_{T_1}^{T_2} p(u_t v_t - u_t v_t) \, dt - pu v_t \big|_{T_1}^{T_2} + pu_t v_t \big|_{T_1}^{T_2} =
\]

\[
(pu v_t - pu_t v) \big|_{T_1}^{T_2}
\]

2. Formula for solution involving initial conditions:

\[
y(0) = c_0 \quad \text{and} \quad y_t(0) = c_1
\]

Insert \( u = y, v = G(t, \tau), T_1 = 0 \) and \( T_2 \to \infty \) into Green’s relationship, and assume that \( y \to 0 \) and \( y_t \to 0 \) as \( t \to \infty \):

\[
\int_{T_1}^{T_2} (u \, Lv - \nu \, Lu) \, dt = (pu v_t - pu_t v) \big|_{T_1}^{T_2}
\]

\[
\int_0^\infty (y \, \delta(t-\tau) - G(t, \tau) \, f(t)) \, dt = (py G_e(t, \tau) - py_t G(t, \tau)) \big|_0^\infty
\]

\[
y(\tau) - \int_0^\infty G(t, \tau) f(t) \, dt = -(p(0) \, y(0) \, G_e(0, \tau) - p(0) \, y_t(0) \, G(0, \tau))
\]

\[
y(\tau) = \int_0^\infty G(t, \tau) f(t) \, dt - p(0) \, c_0 \, G_e(0, \tau) + p(0) \, c_1 \, G(0, \tau)
\]
Reverse $t$ and $\tau$:

$$y(t) = \int_0^\infty G(\tau, t) f(\tau) \, d\tau - p(0) \, c_0 \, G_t(0, t) + p(0) \, c_1 \, G(0, t)$$

Apply reciprocity:

$$y(t) = \int_0^\infty G(t, \tau) f(\tau) \, d\tau - p_0 c_0 \, G_t(t, 0) + p_0 c_1 \, G(t, 0)$$

with $p_0 \equiv p(0)$.

3. Force equivalent to an initial condition.

Consider

$$f(\tau) = p_0 \, c_0 \, \frac{d}{d\tau} \delta(\tau) + p_0 \, c_1 \, \delta(\tau)$$

Which yields the solution:

$$y(t) = \int_0^\infty G(t, \tau) f(\tau) \, d\tau$$

$$= p_0 c_1 \int_0^\infty G(t, \tau) \frac{d}{d\tau} \delta(\tau) \, d\tau + p_0 c_1 \int_0^\infty G(t, \tau) \delta(\tau) \, d\tau$$

$$= p_0 \, c_1 \, G_t(t, 0) + p_0 \, c_1 \, G(t, 0)$$

which matches the initial condition part of the solution in 2. Hence $f(\tau)$ above is a virtual force equivalent to an initial condition.