1. We consider the 2nd order linear differential equation:

\[ \mathcal{L} y = \left( \frac{d}{dt} p(t) \frac{d}{dt} + p(t) \right) y p(t) = f(t) \]

defined on the interval \((T_1, T_2)\). The Green function \(G(t, \tau)\) satisfies \(\mathcal{L} G(t, \tau) = \delta(t - \tau)\) and homogenous boundary conditions.

2. \(\mathcal{L}\) is self-adjoint, as

\[
\left( \frac{d}{dt} p(t) \frac{d}{dt} + p(t) \right)^\dagger = \left( \frac{d}{dt} \right)^\dagger \left( p(t) \right)^\dagger + \left( p(t) \right)^\dagger = \frac{d}{dt} p(t) \frac{d}{dt} + p(t)
\]
as \(d/dt\) is anti-self-adjoint and \(p(t)\) is self-adjoint. It follows that

\[
(v, \mathcal{L} u)_t = (u, \mathcal{L} v)_t \quad \text{with} \quad (a, b)_t = \int_{T_1}^{T_2} a b \, dt
\]

for any two solutions, \(u\) and \(v\).

3. The Green function satisfies the reciprocity relationship \(G(t, \tau) = G(\tau, t)\). Inserting the Green function for source at \(\tau_1\) and \(\tau_2\):

\[
\left( G(t, \tau_2), \delta(t - \tau_1) \right)_t = \left( G(t, \tau_1), \delta(t - \tau_2) \right)_t
\]

\[
G(\tau_1, \tau_2) = G(\tau_2, \tau_1)
\]

4. Green's identity for two solutions, \(u\) and \(v\) is

\[
\int_{T_1}^{T_2} \left( u \mathcal{L} v - v \mathcal{L} u \right) dt = \left( p u v_t - p u_t v \right)_{T_2}^{T_1}
\]

with \(u_t \equiv du/dt\) and \(v_t \equiv dv/dt\). It is derived by applying integration by parts

\[
\int_{T_1}^{T_2} \left( u \mathcal{L} v - v \mathcal{L} u \right) dt = \\
\int_{T_1}^{T_2} \left( u \frac{d}{dt} p v_t + u p v - v \frac{d}{dt} p u_t - v p u \right) dt \\
= \int_{T_1}^{T_2} \left( u \frac{d}{dt} (p v_t) - v \frac{d}{dt} (p u_t) \right) dt
\]
\[
= (pu v_t - pu_t v)|_{T_1}^{T_2} - \int_{T_1}^{T_2} (pu_t v_t - pu_t v_t) \, dt
\]

5. A formula for the solution involving initial conditions:

\[
y(t = T_1) \quad \text{and} \quad y_t(t = T_1)
\]

\[
y(t = T_2) = 0 \quad \text{and} \quad y_t(t = T_2) = 0
\]

is obtained by equating \( u = y \) and \( v = G(t, \tau) \)

\[
\int_{t=T_1}^{t=T_2} (u \mathcal{L} v - v \mathcal{L} u) \, dt = p(t)(u(t)v_t(t) - u_t(t)v(t))|_{t=T_1}^{t=T_2}
\]

\[
\int_{t=T_1}^{t=T_2} (y \delta(t - \tau) - G(t, \tau) f(t)) \, dt = p(t)[y(t)G_t(t, \tau) - y_t G(t, \tau)]|_{t=T_1}^{t=T_2}
\]

Now swap the names of \( t \) and \( \tau \)

\[
y(t) = \int_{t=T_1}^{t=T_2} \frac{\partial}{\partial \tau} G(t, \tau) \, d\tau - p(\tau)[y_t G(t, \tau) - y(\tau) G_t(t, \tau)]|_{\tau=T_1}^{\tau=T_2}
\]

Apply reciprocity, noting \( G_{\tau}(\tau, t) = \frac{\partial}{\partial \tau} G(t, \tau) = \frac{\partial}{\partial \tau} G(t, \tau) \)

\[
y(t) = \int_{t=T_1}^{t=T_2} G(t, \tau) f(\tau) \, d\tau - p(\tau)[y_t G(t, \tau) - y(\tau) G_t(t, \tau)]|_{\tau=T_1}^{\tau=T_2}
\]

This results agrees with (12.1.5) of Copley (2015). By assumption, only the lower boundary condition contributes

\[
y(t) = \int_{t=T_1}^{t=T_2} G(t, \tau) f(\tau) \, d\tau + p(\tau = T_1) \left[ y_t(\tau = T_1) G(t, \tau = T_1) - y(\tau = T_1) \frac{\partial G(t, \tau = T_1)}{\partial \tau} \right]
\]
6. Inserting the force

\[ f(\tau) = A\delta(\tau - (T_1 + \varepsilon)) + B \frac{\partial}{\partial \tau} \delta(\tau - (T_1 + \varepsilon)) \]

with \( \varepsilon \ll (T_2 - T_1) \) into the Green function integral, and integrating

\[ \int_{\tau = T_1}^{\tau = T_2} G(t, \tau) f(\tau) \, d\tau = AG(t, T_1) - B \frac{\partial G(t, \tau = T_1)}{\partial \tau} \]

7. With the choices

\[ A = p(\tau = T_1) y_\tau(\tau = T_1) \quad \text{and} \quad B = p(\tau = T_1) y(\tau = T_1) \]

the force is equivalent to the boundary condition. Thus the boundary conditions at \( T_1 \) can be replaced by a virtual force acting at time \( T_1 + \varepsilon \)