**Sensitivity Kernel for the Squared Envelope**

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This derivation is motivated by Yuan, YO, FJ Simons, E Bozdağ, Geophysics 80, R281-R302, 2015, but is derived independently. (Also, this derivation is for the squared envelope, while theirs is for the envelope).

The squared envelope of a signal $u(t,m)$ is defined as:

$$v(t,m) = [u(t,m)]^2 + [Hu(t,m)]^2$$

where $H$ is the Hilbert transform (an anti-self-adjoint operator that phase shifts a signal by $\pi/2$ radians). Here $m$ is a model parameter. Differentiating with respect to the model parameter and evaluating at $m = m_0$ yields:

$$\left.\frac{dv}{dm}\right|_{m_0} = 2\,u_0 \left.\frac{du}{dm}\right|_{m_0} + 2[Hu_0]H \left.\frac{du}{dm}\right|_{m_0} = W_0 \left.\frac{du}{dm}\right|_{m_0}$$

$$\text{with} \quad W_0 \equiv 2\,u_0 + 2[Hu_0]H$$

Here a subscript 0 implies that a quantity is evaluated at $m_0$; for example, $u_0 = u(m_0)$. Note that the adjoint of $W_0$ is:

$$W_0^{\dagger} = 2u_0 - 2H[Hu_0]$$

where we have used the fact that the functions $2u_0$ and $[Hu_0]$ are self-adjoint. Now let the squared envelope error be:

$$E = (e,e) \quad \text{with} \quad e = v^{obs} - v$$

The derivative of the error with respect to the model is:

$$\left.\frac{dE}{dm}\right|_{m_0} = 2\left(e_0, \left.\frac{de}{dm}\right|_{m_0}\right) = -\,2\left(e_0, \left.\frac{dv}{dm}\right|_{m_0}\right) = -\,2\left(e_0, W_0 \left.\frac{du}{dm}\right|_{m_0}\right)$$

$$= 2\left(e_0, W_0 L_0^{-1} \left.\frac{dL}{dm}\right|_{m_0} u_0\right)$$

Here we have used the rule:

$$\left.\frac{du}{dm}\right|_{m_0} = -L_0^{-1} \left.\frac{dL}{dm}\right|_{m_0} u_0$$

(This rule is based on applying the Born approximation to the differential equation $Lu = f$, where $L(m)$ is a differential operator and $f$ is a source term). We now manipulate the inner product using adjoints:

$$\left.\frac{dE}{dm}\right|_{m_0} = \left( L_0^{-1\dagger} W_0^{\dagger} e_0, 2 \left.\frac{dL}{dm}\right|_{m_0} u_0 \right) = (\lambda, \xi)$$

with

$$L_0^{\dagger} \lambda = W_0^{\dagger} e_0 \quad \text{and} \quad \xi \equiv 2 \left.\frac{dL}{dm}\right|_{m_0} u_0$$

Here $\lambda$ is an adjoint field. The adjoint differential equation has source term:

$$W_0^{\dagger} e_0 = 2\, u_0\, e_0 - 2H\,\{[Hu_0]\, e_0\}$$

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% MatLab code used in the example below

N = 1024;
Dt = 0.8;
t = Dt*[0:N-1]';
dm = 0.2;
t0 = t(end)/2;
t2 = t(end)/2+t(end)/20;
s0 = t(end)/20;
u0 = exp( -((t-t0).^2) / (2*s0*s0) ) - 0.5*exp( -((t-t2).^2) / (2*s0*s0) );
Hu0 = imag(hilbert(u0));
v0 = u0.^2 + Hu0.^2;
t1 = t(end)/2+t(end)/40;
s1 = t(end)/30;
dudm = exp( -((t-t1).^2) / (2*s1*s1) );
Hdudm = imag(hilbert(dudm));
du = dudm*dm;
Hdu = imag(hilbert(du));
u = u0+du;
Hu = imag(hilbert(u));
v = u.^2 + Hu.^2;
vobs = zeros(N,1);
e0 = vobs - v0;
e = vobs - v;
dEdm1 = Dt*(e'*e-e0'*e0)/dm;
dEdm2 = -2*Dt*(e0)'*(2*u0.*dudm+2*Hu0.*Hdudm);
dEdm3 = -2*Dt*(2*u0.*e0 - 2*imag(hilbert(Hu0.*e0)))'*(dudm);
```

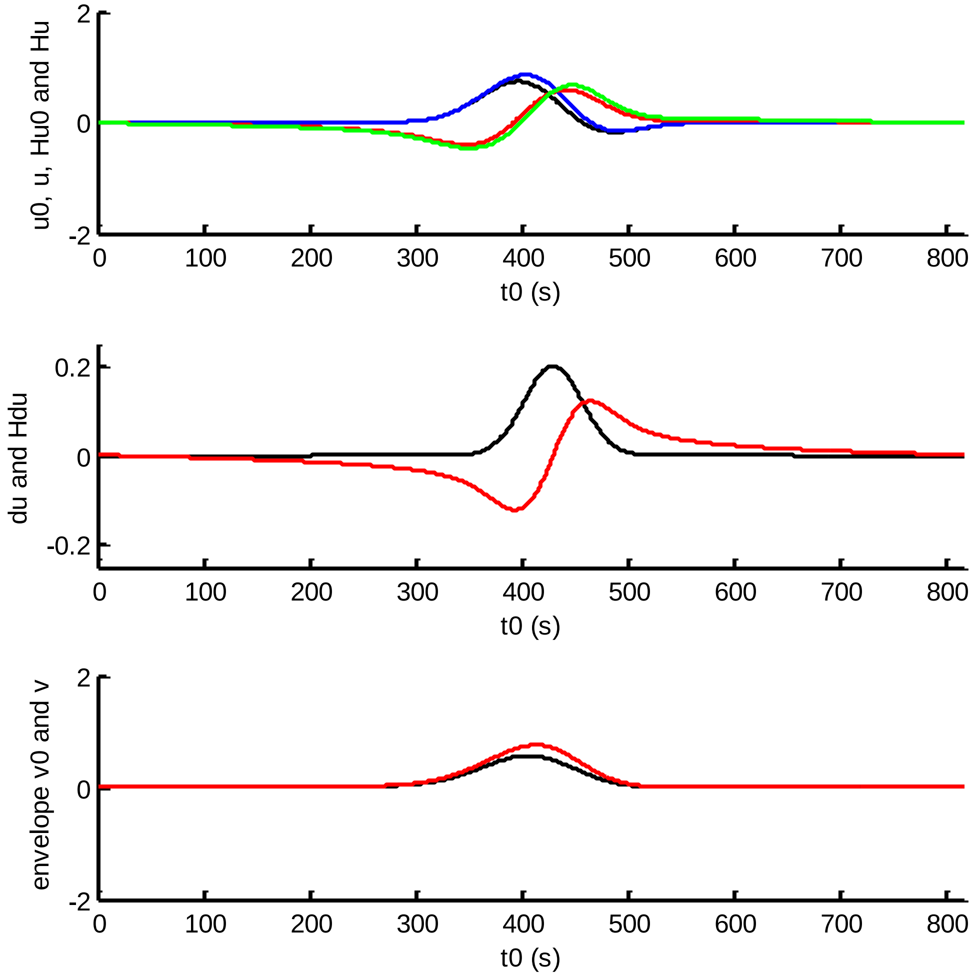


Figure 1. (Top) Sample signal $u_0(t)$ (black) and $u(t) = u_0(t) + m\delta u(t)$ (blue) and their Hilbert transforms $\mathrm{H}u_0(t)$ (red) and $Hu(t)$ (green). (Middle) Perturbation $\delta u(t)$ and its Hilbert transform $H\delta u(t)$. (bottom) Squared envelope functions $v_0(t)$ (black) and $v(t)$ (red).

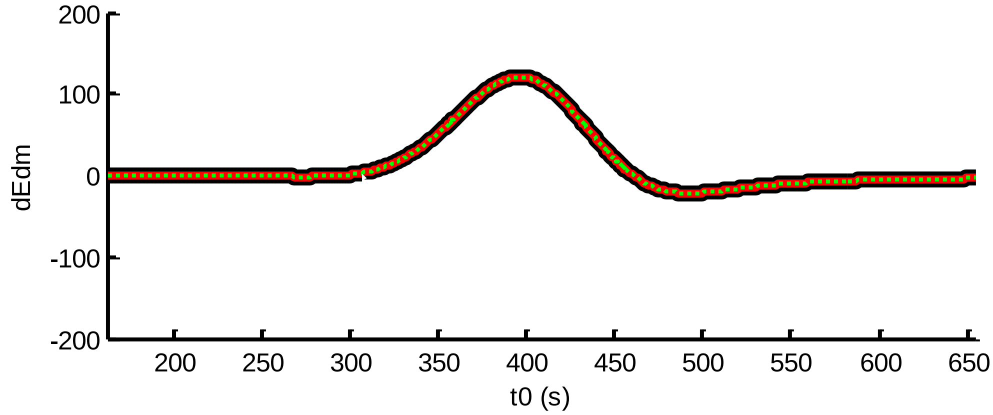


Figure 2. Derivative $dE/dm|_{m_0}$ for $m_0 = 0$ and $v^{obs} = 0$, calculated in three ways: finite differences (black), inner product without adjoint manipulation (red) and inner product with adjoint manipulation (green). The calculation starts with $du/dm|_{m_0}$, treated as known, and does not substitute in the Born approximation. The perturbation $\delta u(t)$ is the same as in Figure 1, except that it is centered at variable position $t_0$.