Time Needed for the Amplitude of an Upper Mantle Thermal Anomaly to Decay by 50%

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Consider a three dimensional diffusion equation with temperature θ and diffusivity D:

$$D^{-1}\frac{\partial\theta}{\partial t}-\frac{\partial^2\theta}{\partial x^2}-\frac{\partial^2\theta}{\partial y^2}-\frac{\partial^2\theta}{\partial z^2}=0$$

with boundary conditions: $\lim_{\to 0}\theta(x,y,z,t)\to 0$ and $\int\theta(x,y,z,t)\,dx\,dy\,dz=Q$. We conjecture that the solution is separable:

$$\theta(x,y,z)=X(x,t)\,Y(y,t)\,Z(z,t)$$

Substituting this form of the solution into the diffusion equation yields:

$$\left(D^{-1}\frac{1}{X}\frac{\partial X}{\partial t}-\frac{1}{X}\frac{\partial^2 X}{\partial x^2}\right)+\left(D^{-1}\frac{1}{Y}\frac{\partial Y}{\partial t}-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}\right)+\left(D^{-1}\frac{1}{Z}\frac{\partial Z}{\partial t}-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right)=0$$

The first term in parenthesis is not a function of (y,z); the second not a function of (x,z); the third not a function of (x,y). In order for their sum to equal zero, they must individually equal zero. Hence:

$$D^{-1}\frac{\partial X}{\partial t}=\frac{\partial^2 X}{\partial x^2}\quad\text{and etc.}$$

Each equation is a one-dimensional diffusion equation, whose solution is well known to be a Gaussian with variance $\sigma(t)=2Dt$:

$$X(x,t)\propto(2\pi)^{-1/2}\,(2Dt)^{-1/2}\,\exp\left(-½\frac{x^2}{2Dt}\right)\quad\text{and etc.}$$

With amplitude normalized so that:

$$\int X(x,t)\,dx=1\quad\text{and etc.}$$

The overall solution is therefore:

$$\theta(r,t)=Q\,(2\pi)^{-3/2}\,(2Dt)^{-3/2}\exp\left(-½\frac{r^2}{2Dt}\right)\quad\text{with}\quad r^2=x^2+y^2+z^2$$

Note that this equation has the same spatial variance as the one-dimensional solution. Now suppose we use $\sqrt{\sigma}$ the as a measure of half width w of the temperature distribution:

$$w^2=\sigma=2Dt$$

The time τ_1 need to reach half width w is $\tau_1 = w^2/2D$, the time τ_2 need to reach half width $2w$ is $\tau_1 = 4w^2/2D$, and the doubling time is $\tau = \tau_2 - \tau_2 = 4w^2/2D - w^2/2D = 3w^2/2D$. From the point of view of amplitudes, a doubling of width implies a decrease in amplitude by a factor of $2^3 = 8$, since the heat spreads out in three dimensions. The change in half-width associated with a halving of amplitude is $\sqrt[3]{2} \approx 1.26$ and the characteristic time is $\tau = 2^{2/3} w^2/2D - w^2/2D = \left(2^{2/3} - 1\right) w^2/2D \approx 0.59\, w^2/2D$. For a hot (>1000 K) upper mantle, the thermal diffusivity is in the range $D \approx 5 - 12 \times 10^{-7}\ \mathrm{m^2/s}$ (Gibert & Seipold 2003), leading to a prediction of time needed for 50% decay as shown in the figure.

Gibert, B. and U. Seipold, Thermal diffusivity of upper mantle rocks: Influence of temperature, pressure, and the deformation fabric, JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 108, NO. B8, 2359, doi:10.1029/2002JB002108, 2003

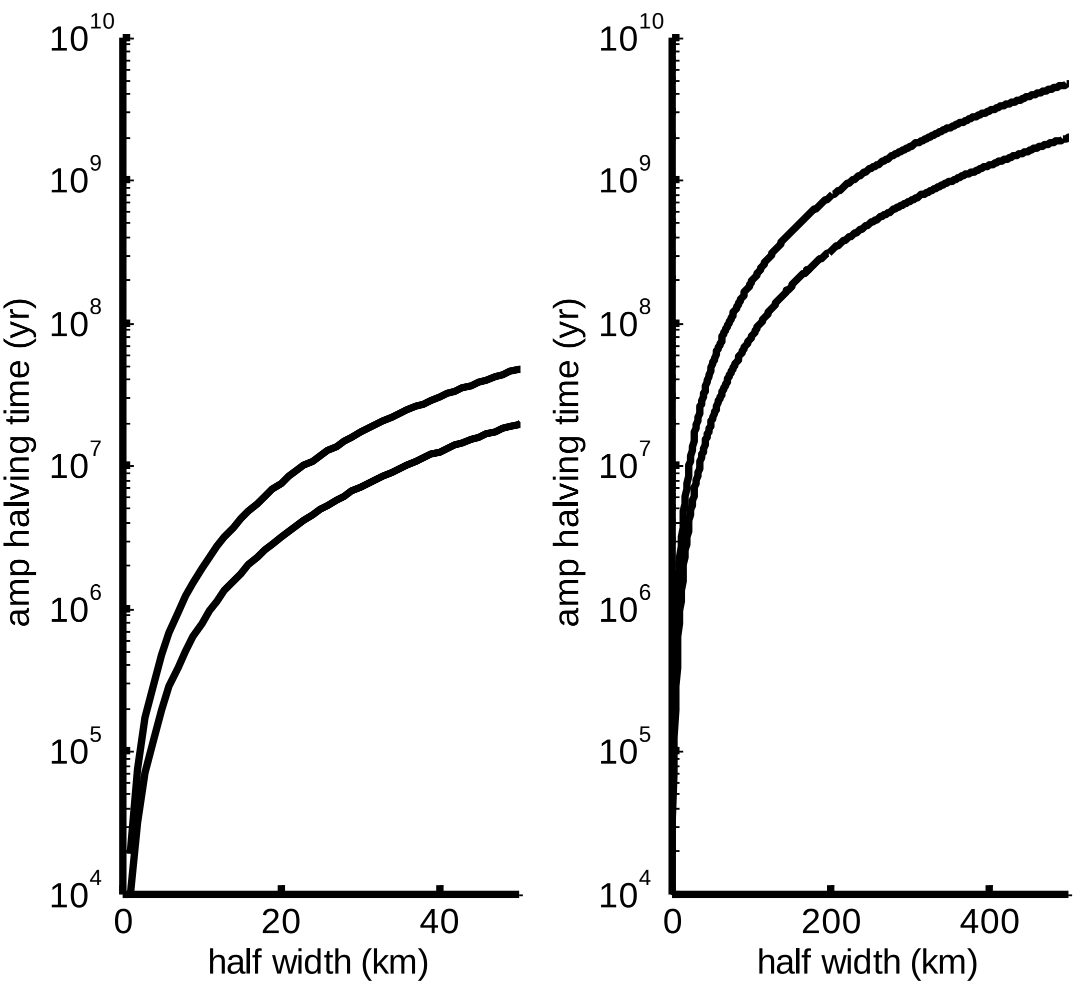

Figure. Decay times for $D \approx 5 \times 10^{-7}\ \mathrm{m^2/s}$ (uppe curve) and $D \approx 12 \times 10^{-7}\ \mathrm{m^2/s}$ (lower curve).