Surface Wave Dispersion and the Half Space Cooling Model

Bill Menke, May 18, 2017
(Inspired by a talk by Celia Eddy)

1. A homogeneous half-space with thermal diffusivity κ, initial temperature θ_0 and a boundary condition $\theta(z=0,t)=0$ has a temperature evolution given by:

$$\theta(z,t) = \theta_0 \ \mathrm{erf}(az) \quad \text{with} \quad a = (4\kappa t)^{-½}$$

where z is depth and t is geologic age (e.g. Menke and Abbott, 1989, their equation 6.4.2).

2. The temperature anomaly is:

$$\Delta\theta(z,t) = \ \theta_0 - \theta_0 \ \mathrm{erf}(az) = \theta_0 \ \mathrm{erfc}(az)$$

3. The corresponding shear velocity anomaly is:

$$\Delta\beta(z,t) = \gamma \ \Delta\theta(z,t)$$

where $\gamma = d\beta/d\theta$ is assumed to be constant.

4. The perturbation in phase velocity $c(\omega,t)$ is:

$$\Delta c(\omega,t) = \int_0^\infty K(\omega,z) \ \Delta\beta(z,t) \, dz$$

where $K(\omega,z)$ is a sensitivity kernel.

5. Power series for $\Delta c(\omega,t)$. Assume that the kernel can be represented as Taylor series (or polynomial) in z:

$$K(\omega,z) = \sum_{n=0} K_n(\omega) \, z^n$$

Then:

$$\Delta c(\omega,t) = \sum_{n=0} \gamma\theta_0 K_n(\omega) \ \int_0^\infty z^n \ \mathrm{erfc}(az) \, dz \ = \sum_{n=0} \gamma\theta_0 K_n(\omega) \ I_0(n)$$

The integral is given by Ng and Geller (1969, their Equation 4.1.11):

$$I_0(n) = \int_0^\infty z^n \ \mathrm{erfc}(az) \, dz = \frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}(\ n+1) \ a^{n+1}}$$

Hence:

$$\Delta c(\omega, t) = \sum_{n=0} \gamma \theta_0 K_n(\omega) \frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}(\,n+1)} a^{-n-1}$$

$$= \gamma \theta_0 \sum_{n=0} K_n(\omega) \frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}(\,n+1)} (4\kappa t)^{(n+1)/2}$$

Note that the sequence of terms have order $(t)^{1/2}$, t, $(t)^{3/2}$, $\cdots$ that is, Δc is a power series in $\sqrt{t}$ that starts with the $\sqrt{t}$ term. The $\sqrt{t}$ term will dominate when the kernel is approximately constant over the depth range of $\Delta\beta(z, t)$, and the t will dominate when the kernel is increasing linearly from zero in this depth range.

6. Perturbation in phase velocity for an exemplary kernel: Consider the kernel:

$$K(\lambda, z) = bK_0 \exp(-bz) \quad \text{where} \quad b = 1/\lambda$$

Here λ is a characteristic wavelength. This kernel has been normalized so that its depth integral is K_0. The corresponding perturbation in phase velocity is:

$$\Delta c(\lambda, t) = \gamma \theta_0 b K_0 \int_0^\infty \exp(-bz)\ \operatorname{erfc}(az)\, dz = \gamma \theta_0 b K_0 I_1$$

Ng and Geller (1969, their Equation 4.2.3) give:

$$\int_0^\infty \exp(-bz)\ \operatorname{erf}(az)\, dz = \frac{1}{b} \exp\left\{\frac{b^2}{4a^2}\right\} \operatorname{erfc}\left\{\frac{b}{2a}\right\}$$

from which it follows:

$$I_1 = \int_0^\infty \exp(-bz)\ \operatorname{erfc}(az)\, dz = \int_0^\infty \exp(-bz)\ \{1 - \operatorname{erf}(az)\}\, dz =$$

$$\int_0^\infty \exp(-bz)\, dz - \int_0^\infty \exp(-bz)\ \operatorname{erf}(az)\ dz =$$

$$\frac{1}{b} - \frac{1}{b} \exp\left\{\frac{b^2}{4a^2}\right\} \operatorname{erfc}\left\{\frac{b}{2a}\right\}$$

Here we have used the fact that:

$$\int_0^\infty \exp(-bz)\, dz = -\frac{1}{b} \exp(-bz)\Big]_0^\infty = \frac{1}{b}$$

I have tested the formula for I_1 numerically and verified that it is correct. Thus:

$$\Delta c(\lambda, t) = \frac{\gamma \theta_0 b K_0}{b}\left[1 - \exp\left\{\frac{b^2}{4a^2}\right\} \operatorname{erfc}\left\{\frac{b}{2a}\right\}\right]$$

$$= \gamma \theta_0 K_0 \left[1 - \exp\left\{\frac{\kappa t}{\lambda^2}\right\} \operatorname{erfc}\left\{\frac{\sqrt{\kappa t}}{\lambda}\right\}\right]$$

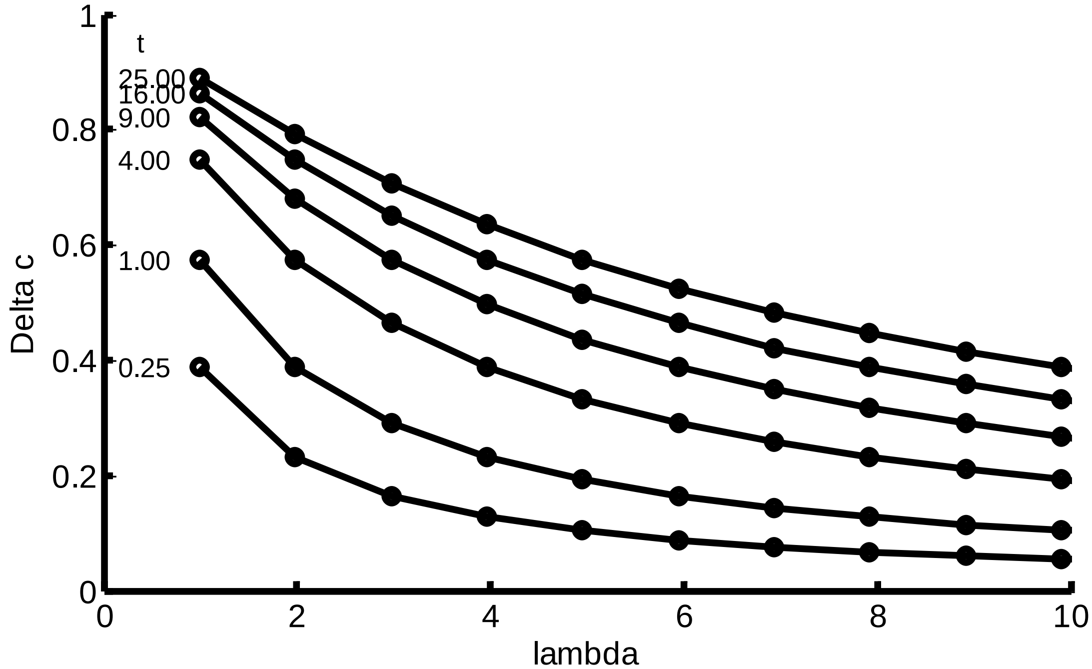

Figure. $\Delta c(\lambda, t)$ for the case $\kappa = \gamma = \theta_0 = K_0 = 1$. At fixed time t, the perturbation in phase velocity Δc decreases with wavelength λ because the kernel is averaging over increasingly deeper intervals, where the temperature perturbation is small. At fixed wavelength λ, the perturbation in phase velocity Δc increased with time t because the temperature perturbation is extending deeper and deeper into the earth.

6. Perturbation in phase velocity for another exemplary kernel: Consider a kernel:

$$K(\lambda, z) = K_0\, b^{\,2} z\ \exp(-bz) \quad \text{where} \quad b = 1/\lambda$$

Here λ is a characteriztic wavelength. The kernel has been normalized so that its depth integral is K_0. The corresponding perturbation in phase velocity is:

The corresponding perturbation in phase velocity is:

$$\Delta c(\lambda, t) = \gamma \theta_0 b^2 K_0 \int_0^\infty z \exp(-bz) \operatorname{erfc}(az)\, dz \ = \gamma \theta_0 b^2 K_0 I_2$$

The integral follows from Ng and Geller (1969, their Equation 4.2.7):

$$I_2 = \int_0^\infty z \exp(-bz) \operatorname{erfc}(az)\, dz = \left\{\frac{1}{b^2} - \frac{1}{b} \exp\left\{\frac{b^2}{4a^2}\right\} \left[\frac{1}{b} - \frac{b}{2a^2}\right] \operatorname{erfc}\left\{\frac{b}{2a}\right\} - \frac{1}{ab\sqrt{\pi}}\right\}$$

together with:

$$\int_0^\infty \mathrm{z} \exp(-bz)\, dz = \frac{1}{b^2}$$

I have tested the formula for I_2 numerically and verified that it is correct. Thus:

$$\Delta c(\lambda, t) = \gamma \theta_0 b^2 K_0 \left\{\frac{1}{b^2} - \frac{1}{b} \exp\left\{\frac{b^2}{4a^2}\right\} \left[\frac{1}{b} - \frac{b}{2a^2}\right] \operatorname{erfc}\left\{\frac{b}{2a}\right\} - \frac{1}{ab\sqrt{\pi}}\right\}$$

$$= \gamma \theta_0 K_0 \left\{1 - b \exp\left\{\frac{b^2}{4a^2}\right\} \left[\frac{1}{b} - \frac{b}{2a^2}\right] \operatorname{erfc}\left\{\frac{b}{2a}\right\} - \frac{b}{a\sqrt{\pi}}\right\}$$

$$= \gamma \theta_0 K_0 \left\{1 - \frac{1}{\lambda} \exp\left\{\frac{\kappa t}{\lambda^2}\right\} \left[\lambda - \frac{2\kappa t}{\lambda}\right] \operatorname{erfc}\left\{\frac{\sqrt{\kappa t}}{\lambda}\right\} - \frac{\sqrt{4\kappa t}}{\lambda\sqrt{\pi}}\right\}$$

References:

Menke, W. and D. Abbott, Geophysical Theory (Textbook), Columbia University Press, 458p, 1989.

Ng, E. and M. Geller, Table of Integrals of the Error Funtion, Journal of Research of the National Bureau of Standards – B. Mathematical Sciences 738, 1969.