Anisotropic Parameter "eta" for a Layered Material

Bill Menke, October 2017, corrected October 2018

For reference, we consider an isotropic solid, with 2 constants λ, μ. The Voight elasticity tensor is:

$$\begin{bmatrix} \{\lambda+2\mu\} & \lambda & \lambda & 0 & 0 & 0 \\ . & \{\lambda+2\mu\} & \lambda & 0 & 0 & 0 \\ . & . & \{\lambda+2\mu\} & 0 & 0 & 0 \\ . & . & . & \mu & 0 & 0 \\ . & . & . & . & \mu & 0 \\ . & . & . & . & . & \mu \end{bmatrix}$$

In the special case of Poisson solid with $\lambda = \mu$, the tensor becomes:

$$\begin{bmatrix} 3\lambda & \lambda & \lambda & 0 & 0 & 0 \\ . & 3\lambda & \lambda & 0 & 0 & 0 \\ . & . & 3\lambda & 0 & 0 & 0 \\ . & . & . & \lambda & 0 & 0 \\ . & . & . & . & \lambda & 0 \\ . & . & . & . & . & \lambda \end{bmatrix}$$

In the more general case of a transversely isotropic solid, two commonly-used forms of the tensor are (Love 1927, Dziewonski and Anderson, 1981):

$$\begin{bmatrix} a & a-2e & b & 0 & 0 & 0 \\ . & a & b & 0 & 0 & 0 \\ . & . & c & 0 & 0 & 0 \\ . & . & . & d & 0 & 0 \\ . & . & . & . & d & 0 \\ . & . & . & . & . & e \end{bmatrix} =$$

$$\rho \begin{bmatrix} v_{pH}^2 & \left(v_{pH}^2 - 2v_{SHH}^2\right) & \eta\left(v_{pH}^2 - 2v_{SVH}^2\right) & 0 & 0 & 0 \\ . & v_{pH}^2 & \eta\left(v_{pH}^2 - 2v_{SVH}^2\right) & 0 & 0 & 0 \\ . & . & v_{pV}^2 & 0 & 0 & 0 \\ . & . & . & v_{SVH}^2 & 0 & 0 \\ . & . & . & . & v_{SVH}^2 & 0 \\ . & . & . & . & . & v_{SHH}^2 \end{bmatrix} =$$

Note that the second form introduces a parameter η. Comparing forms, we derive a relationship for η:

$$\eta\left(v_{pH}^2 - 2v_{SVH}^2\right) = b \quad \text{or} \quad \eta = \frac{b}{\left(v_{pH}^2 - 2v_{SVH}^2\right)} = \frac{b}{(a-2d)}$$

We now consider a medium consisting of an alternating sequence of two layers of equal thickness, each consisting of an isotropic Poisson solid. The Lame parameters of the two layers are:

$$\lambda^{(1)} = 1 + \varepsilon \quad \text{and} \quad \lambda^{(2)} = 1 - \varepsilon$$

where $0 < \varepsilon < 1$. Backus' (1962) formulas require the volumetric average $\langle . \rangle$ of various parameters:

$$\langle \lambda \rangle = \frac{\lambda^{(1)} + \lambda^{(2)}}{2} = \frac{1+\varepsilon}{2} + \frac{1-\varepsilon}{2} = 1$$

$$\langle \lambda^{-1} \rangle = \frac{\left(\lambda^{(1)}\right)^{-1} + \left(\lambda^{(2)}\right)^{-1}}{2} = \frac{1 - \varepsilon + \varepsilon^2}{2} + \frac{1 + \varepsilon + \varepsilon^2}{2} = 1 + \varepsilon^2$$

$$\langle \lambda^{-1} \rangle^{-1} = (1 + \varepsilon^2)^{-1} = 1 - \varepsilon^2$$

The isotropic Voight tensor of the layers has parameters:

$a = 3\lambda$

$b = \lambda$

$c = 3\lambda$

$d = \lambda$

$e = \lambda$

And according to Backus (1962) the corresponding long-wavelength Voight tensor of the layered medium is:

$E = \langle e \rangle = \langle \lambda \rangle = 1$

$D = \langle d^{-1} \rangle^{-1} = \langle \lambda^{-1} \rangle^{-1} = 1 - \varepsilon^2$

$C = \langle c^{-1} \rangle^{-1} = \langle 3\lambda^{-1} \rangle^{-1} = 3(1 - \varepsilon^2)$

$B = \langle c^{-1} \rangle^{-1} \langle b c^{-1} \rangle = \langle 3\lambda^{-1} \rangle^{-1} \langle \lambda (3\lambda)^{-1} \rangle = 3 \langle \lambda^{-1} \rangle^{-1} \frac{1}{3} = 1 - \varepsilon^2$

$A = \langle a - b^2 c^{-1} \rangle + \langle c^{-1} \rangle^{-1} \langle b c^{-1} \rangle^2 = \langle 3\lambda - \lambda^2 (3\lambda)^{-1} \rangle + \langle (3\lambda)^{-1} \rangle^{-1} \langle \lambda (3\lambda)^{-1} \rangle^2$

$= 3\langle \lambda \rangle - \frac{1}{3} \langle \lambda \rangle + 3 \langle \lambda^{-1} \rangle^{-1} \frac{1}{9}$

$$= \frac{8}{3} + \frac{1}{3}\langle\lambda^{-1}\rangle^{-1} = \frac{8}{3} + \frac{1}{3}(1-\varepsilon^2) = 3 - \frac{\varepsilon^2}{3} = 3\left(1 - \frac{\varepsilon^2}{9}\right)$$

The fractional P-wave anisotropy f_P and S-wave anisotropy f_S can be quantified by:

$$f_P = \frac{½(v_{pH} - v_{pV})}{½(v_{pH} + v_{pV})} = \frac{(A^{½} - C^{½})}{(A^{½} + C^{½})} \approx \frac{\left(3\left(1-\frac{\varepsilon^2}{18}\right) - 3\left(1-\frac{\varepsilon^2}{2}\right)\right)}{\left(3\left(1-\frac{\varepsilon^2}{18}\right) + 3\left(1-\frac{\varepsilon^2}{2}\right)\right)} \approx \frac{\left(\frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{18}\right)}{2} = \frac{8\varepsilon^2}{36}$$

$$f_S = \frac{½(v_{sHH} - v_{sVH})}{½(v_{sHH} + v_{sVH})} = \frac{(E^{½} - D^{½})}{(E^{½} + D^{½})} = \frac{\left(1-\left(1-\frac{\varepsilon^2}{2}\right)\right)}{\left(1+\left(1-\frac{\varepsilon^2}{2}\right)\right)} \approx \frac{\left(\frac{\varepsilon^2}{2}\right)}{2} = \frac{\varepsilon^2}{4}$$

so that $\varepsilon = (36f_P/8)^{½} = (4f_S)^{½}$. The parameter η is:

$$\eta = \frac{B}{A-2D} = \frac{1-\varepsilon^2}{3 - \frac{\varepsilon^2}{3} - 2 + 2\varepsilon^2} = \frac{1-\varepsilon^2}{1+\frac{5}{3}\varepsilon^2} = \left(1 - \frac{5}{3}\varepsilon^2\right)(1-\varepsilon^2) = 1 - \frac{8}{3}\varepsilon^2$$

The $\varepsilon = 0$ case reproduces (as expected) the homogenous, $\eta = 1$ case. A limitation of the layered model is the ratio:

$$\frac{f_P}{f_S} = -\frac{8\varepsilon^2}{36}\frac{4}{\varepsilon^2} = \frac{32}{36} \approx 0.8421 \text{ and } \quad \frac{f_P}{\eta - 1} = -\frac{8\varepsilon^2}{36}\frac{3}{8\varepsilon^2} = -\frac{1}{12} \approx 0.0833$$

are fixed “un-tunable” constants.

For 100 km depth, the Dziewonski and Anderson (1981) PREM model gives $v_{pV} = 7.86732$, $v_{pH} = 8.06410$, $v_{sVH} = 4.32041$, $v_{sHH} = 4.44818$ and $\eta = 0.92987$, implying that $f_P = 0.0124$, $f_S = 0.0146$ $f_P/f_S = 0.8477$ and $f_P/(\eta - 1) = -0.1761$. The first ratio is similar to the layered prediction, but the second is far off. Furthermore, PREM’s f_P value implies $\varepsilon = 0.24$, corresponding to a layered model with $\eta \approx 0.85$ (figure 1), a prediction that is far from PREM’s $\eta \approx 0.93$. The layered model does cannot match PREM. Consequently, fine-scale layering cannot be a major source of anisotropy in the Earth’s mantle.

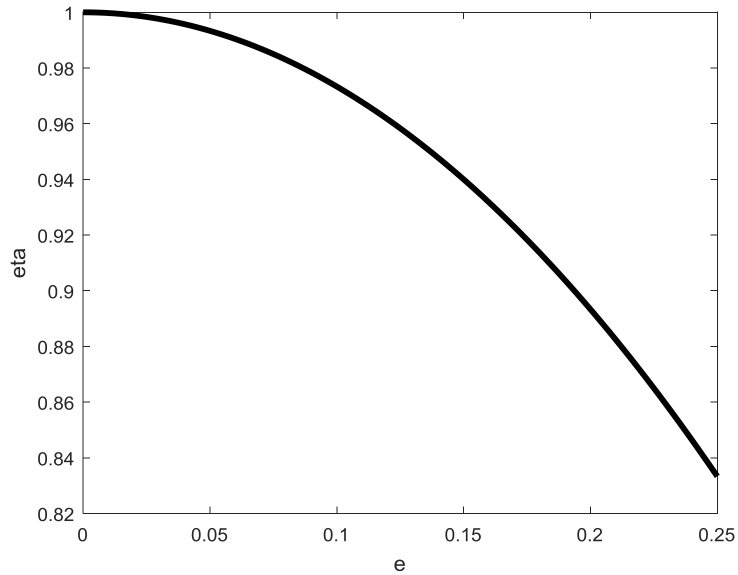

Figure 1. The function $\eta(\varepsilon)$ for the layered model.

Note: I have numerically checked the small number approximations against the exact expressions.

References:

Backus, George E, Long-wave elastic anisotropy produced by horizontal layering, J. Geophys. Res. 67, 4427–4440, 1962.

Dziewonski and Anderson, Preliminary reference Earth Model, Phys. Earth and Planet. Interior 25, 297-356, 1981.

Love, AEH, A treatise on the mathematical theory of elasticity, 4ed, Cambridge U. Press, 643pp, 1927.