Anisotropic phase velocity implies anisotropic group velocity

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The phase velocity $v(\omega) = \omega/k$ and group velocity $V(\omega) = d\omega/dk$, depend on angular frequency ω and horizontal wavenumber k. The corresponding phase and group slowness are $s(\omega) = 1/v = k/\omega$ and $S(\omega) = 1/V = dk/d\omega$. The group slowness is related to the phase velocity via:

$$S(\omega) = \frac{d}{d\omega}k = \frac{d}{d\omega}(\omega v^{-1}) = v^{-1} - \omega v^{-2}\frac{dv}{d\omega} = v^{-1}\left(1 - \omega v^{-1}\frac{dv}{d\omega}\right)$$

and the group velocity is related to the phase velocity by:

$$V = S^{-1} = v\left(1 - \frac{\omega}{v}\frac{dv}{d\omega}\right)^{-1} \equiv v\,\varphi^{-1}$$

(Eqn 1)

Here φ is an abbreviation for the factor in parenthesis. Now suppose the phase velocity is anisotropic, with the form:

$$v(\omega) = A(\omega) + B(\omega)\cos(2\theta - 2\theta_0) + C(\omega)\sin(2\theta - 2\theta_0)$$

$$\equiv A(\omega) + B(\omega)\mathrm{c}(\theta) + C(\omega)\mathrm{s}(\theta)$$

where θ is azimuth and θ_0 is a reference azimuth, and where $\mathrm{c}(\theta) \equiv \cos(2\theta - 2\theta_0)$ and $\mathrm{s}(\theta) \equiv \sin(2\theta - 2\theta_0)$. Taking the derivative with respect to frequency, we find:

$$\frac{dv}{d\omega} = \frac{dA}{d\omega} + \frac{dB}{d\omega}\mathrm{c}(\theta) + \frac{dC}{d\omega}\mathrm{s}(\theta)$$

$$\equiv A' + B'\mathrm{c}(\theta) + C'\mathrm{s}(\theta)$$

Here $A' \equiv dA/d\omega$, $B' \equiv dB/d\omega$ and $C' \equiv dC/d\omega$ are abbreviations for the derivatives. In a weakly anisotropic medium, $B \ll A$ and $C \ll A$, so that to first order:

$$\omega v^{-1} \approx \omega A^{-1}[1 - (B/A)\,\mathrm{c}(\theta) - (\mathrm{C}/A)\,\mathrm{s}(\theta)]$$

Then:

$$\varphi = 1 - (\omega v^{-1})\frac{dv}{d\omega} =$$

$$1 - (\omega A'/A)[1 - (B/A)\,\mathrm{c}(\theta) - (\mathrm{C}/A)\,\mathrm{s}(\theta)] \times [1 + (B'/A')\,\mathrm{c}(\theta) + (\mathrm{C}'/A')\,\mathrm{s}(\theta)]$$

We assume that $B' \ll A'$ and $C' \ll A'$; that is, the frequency-dependence of the anisotropy is weak. Then to first order:

$$\varphi = 1 - (\omega A'/A)[1 + (B'/A' - B/A)\ \mathrm{c}(\theta) + (\mathrm{C}'/A' - \mathrm{C}/A)\ \mathrm{s}(\theta)] =$$

$$(1 - \omega A'/A) - \omega(B'/A - BA'/A^2)\ \mathrm{c}(\theta) - \omega(\mathrm{C}'/A - \mathrm{C}A'/A^2)\ \mathrm{s}(\theta) =$$

$$(1 - \omega A'/A)\left\{1 - \frac{\omega(B'/A - BA'/A^2)}{(1 - \omega A'/A)}\ \mathrm{c}(\theta) - \frac{\omega(\mathrm{C}'/A - \mathrm{C}A'/A^2)}{(1 - \omega A'/A)}\ \mathrm{s}(\theta)\right\}$$

Finally, we assume that the factors multiplying $\mathrm{c}(\theta)$ and $\mathrm{s}(\theta)$ are small, so that we can employ a first order approximation:

$$V = v\ \varphi^{-1} =$$

$$v(\omega) = A[1 + (B/A)\ \mathrm{c}(\theta) + (C/A)\ \mathrm{s}(\theta)] \times$$

$$\left(1 - \frac{\omega A'}{A}\right)^{-1}\left\{1 + \frac{\omega(B'/A - BA'/A^2)}{(1 - \omega A'/A)}\ \mathrm{c}(\theta) + \frac{\omega(\mathrm{C}'/A - \mathrm{C}A'/A^2)}{(1 - \omega A'/A)}\ \mathrm{s}(\theta)\right\} \approx$$

$$\mathrm{A}\left(1 - \frac{\omega A'}{A}\right)^{-1}\left\{1 + \left[\frac{B}{A} + \frac{\omega(B'/A - BA'/A^2)}{(1 - \omega A'/A)}\right]\ \mathrm{c}(\theta) + \left[\frac{C}{A} + \frac{\omega(\mathrm{C}'/A - \mathrm{C}A'/A^2)}{(1 - \omega A'/A)}\right]\ \mathrm{s}(\theta)\right\}$$

(Eqn 2)

I have checked these formulas numerically (Figure 1).

Thus, a phase velocity with a weak 2θ anisotropy implies a group velocity with a weak 2θ anisotropy.

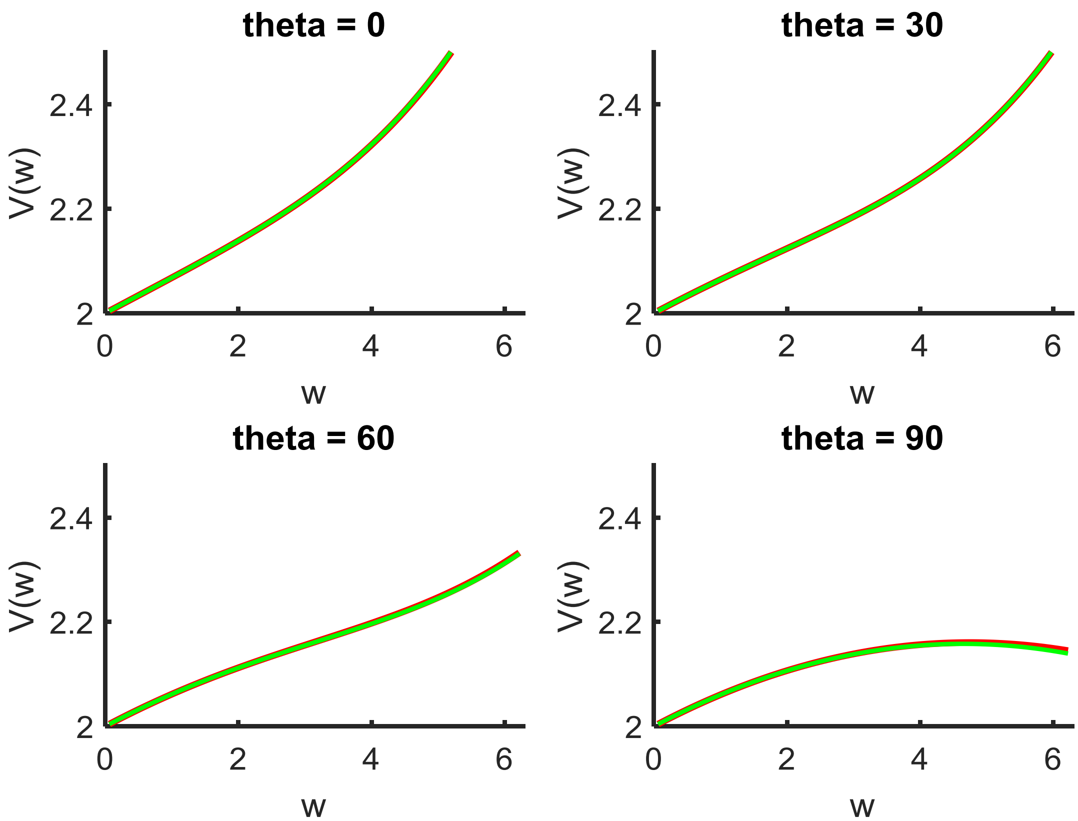

Fig. 1. Exact (using Eqn 1, red) and approximate (using Eqn 2, green) phase velocity $V(\omega)$ for the case $0 \leq \omega \leq 2\pi$ for four azimuths $\theta = \{0°,\ 30°,\ 60°,\ 90°\}$ when $A = 2 + 0.4 \times (\omega/2\pi)$, $B = 0.25 \times (\omega/2\pi)^4$ and $C = -0.25 \times (\omega/2\pi)^2$.