Notes on RC Circuits

Bill Menke, November 9-13, 2017

These notes analyze Huebner and Dillenburg's (1995) method of determining the resistivity of a rock sample by measuring the frequency-dependent impedance and plotting its real and imaginary parts on the complex plane.

1. Review of the RLC Series Circuit:

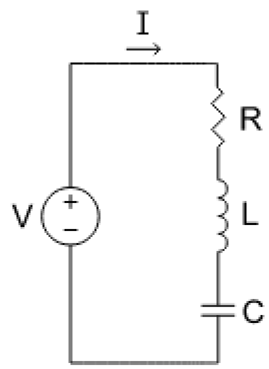


Figure 1. RLC series circuit with resistor R, inductor L and capacitor C in series. The source supplies an alternating voltage V.

Circuit impedance:

$$Z = \frac{V}{I}$$

Kirchoff Voltage Law: The directed sum of the electrical potential differences (voltage) around any closed network is zero

$$V_L + V_R + V_C = V_0 \ \exp(i\omega t)$$

Resistor; voltage proportional to current:

$$V_R = RI$$

Inductor; voltage proportional to rate of change of current

$$V_L = L\frac{dI}{dt}$$

Capacitor; voltage proportional to total charge:

$$V_C = \frac{1}{C}\int_0^t I(t')\,dt'$$

Substitute into Kirchoff Voltage Law:

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(t')\,dt' = V_0\ \exp(i\omega t)$$

Take derivative:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = i\omega V_0\ \exp(i\omega t)$$

Assume solution $I = I_0\ \exp(i\omega t)$:

$$-\omega^2 L I_0 \exp(i\omega t) + i\omega R I_0 \exp(i\omega t) + \frac{1}{C} I_0 \exp(i\omega t) = i\omega V_0\ \exp(i\omega t)$$

Solve for impedance

$$Z = \frac{V_0}{I_0} = \frac{-\omega^2 L + i\omega R + \frac{1}{C}}{i\omega} = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

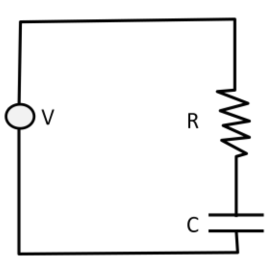
$$\text{real}(Z) = R$$

$$\text{imag}(Z) = \omega L - \frac{1}{\omega C}$$

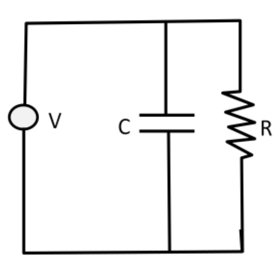
$$|Z|^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2$$

$$\varphi = \text{atan}\left(\frac{\omega L}{R} - \frac{1}{\omega R C}\right)$$

2. Review of the RC Series and Parallel Circuits



(A) Series



(B) Parallel

Figure 2. (A) RC series circuit and (B) RC parallel circuit, with resistor R and capacitor C. The source supplies an alternating voltage V.

2.1 Series case. This is just the $L = 0$ case of the RCL series circuit:

$$Z = \frac{V_0}{I_0} = R - i\left(\frac{1}{\omega C}\right)$$

2.2. Parallel case. When the voltage at the source $V = V_0 \exp(i\omega t)$ is known, the current flowing through the resistor is $I_R = V/R$. By Kirchoff Current Law, which states that the net current in any junction of wires equals the net current out, the current flowing through the capacitor is:

$$I_C = I_0 \exp(i\omega t) - V_0 \frac{\exp(i\omega t)}{R}$$

where $I_0 \exp(i\omega t)$ is the current at the source. The voltage across the capacitor must equal that across the voltage source, so:

$$V_C = \frac{1}{C}\int_0^t I_C(t')\,dt' \quad \text{or} \quad C\frac{dV_C}{dt} = i\omega C I_0 \exp(i\omega t) = I_0 \exp(i\omega t) - V_0 \frac{\exp(i\omega t)}{R}$$

Rearranging, we find:

$$i\omega C Z = 1 - \frac{Z}{R} \quad \text{or} \quad Z = \frac{R}{1 + i\omega RC}$$

3. Analysis of Huebner and Dillenburg (1995). The formula, above, matches the one in Huebner and Dillenburg (1995); that is, the rock is equivalent to a resistor and capacitor in parallel. The real and imaginary parts are:

$$Z_r = \mathrm{real}(Z) = \frac{R}{1 + (RC\omega)^2}$$

$$Z_i = \mathrm{imag}(Z) = \frac{-R(RC\omega)}{1 + (RC\omega)^2}$$

Recall that the formula for a circle of radius r and centered at (x_c, y_c) in the (x, y) is $(x - x_c)^2 + (y - y_c)^2 = r^2$. Consider now the complex impedance plane (Z_r, Z_i). The real and imaginary parts of the impedance satisfies:

$$(Z_r - ½R)^2 + {Z_i}^2 = \left[\frac{R}{1 + (RC\omega)^2} - \frac{(½R) + (½R)(RC\omega)^2}{1 + (RC\omega)^2}\right]^2 + \left[\frac{-R(RC\omega)}{1 + (RC\omega)^2}\right]^2$$

$$= ¼R^2 \left[\frac{1-(RC\omega)^2}{1+(RC\omega)^2}\right]^2 + ¼R^2 \left[\frac{4(RC\omega)^2}{1+(RC\omega)^2}\right]^2$$

$$= ¼R^2 \frac{1-2(RC\omega)^2+(RC\omega)^4+4(RC\omega)^2}{[1+(RC\omega)^2]^2}$$

$$= ¼R^2 \frac{1+2(RC\omega)^2+(RC\omega)^4}{1+2(RC\omega)^2+(RC\omega)^4} = ¼R^2 = (½R)^2$$

That is, when plotted on the complex (Z_r, Z_i) plane, the real and imaginary parts of the impedance fall on a circle of radius $½R$ centered at $(½R, 0)$. This is demonstrated in the following plot, where the dots are equally spaced in frequency.

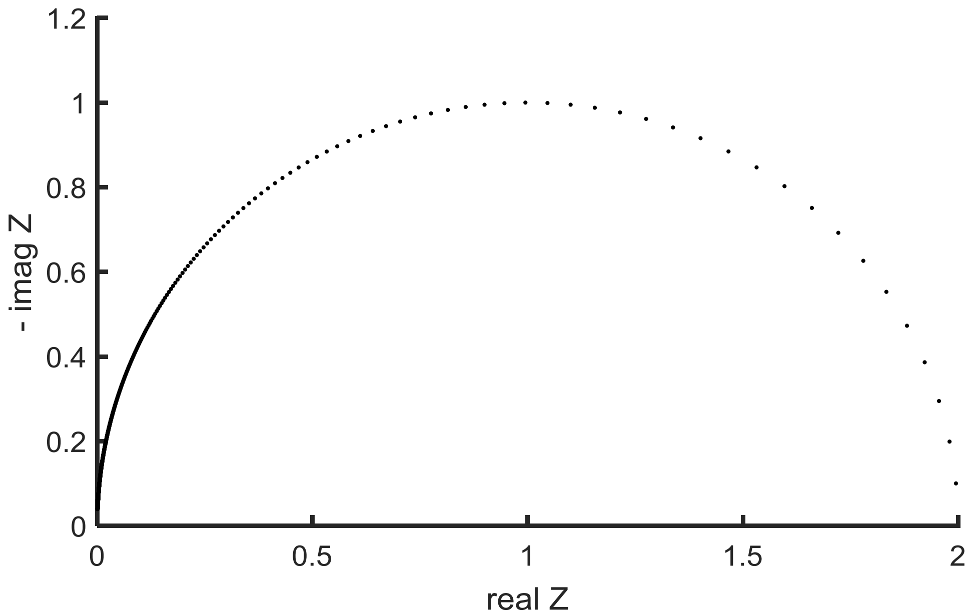


Figure 3. Impedance of an RC parallel circuit plotted in the complex plane for $R = 2, C = 0.5$ and a suite of equally-spaced frequencies $0 < \omega < 16\pi$ (dots).

Reference:

Huebner, J.S. and R.G. Dillenburg, Impedance spectra of hot, dry silicate minerals and rock: qualitative interpretation of spectra, American Mineralogist 80, 46-64, 1995.