Nonuniqueness in Anisotropic Traveltime Tomography under the Radon Transform Approximation

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(This work follows up upon my 2015 BSSA paper on the same subject).

0. Summary. Given a 2D slowness function $m(x, y) =$

$$A(x, y) + B_c(x, y) \cos(2\theta) + B_s(x, y) \sin(2\theta) + C_c(x, y) \cos(4\theta) + C_s(x, y) \sin(4\theta)$$

and travel times related to it through a Radon transform, we show that:

- An $A$ can always be found that mimics a set of $(B_c, B_s)$ and/or a set of $(C_c, C_s)$;
- A pair $(B_c, B_s)$ can always be found that mimics an $A$, and furthermore, an infinite family of such pairs exist;
- A pair $(C_c, C_s)$ can always be found that mimics an $A$, and furthermore, an infinite number of such pairs exist; and
- A pair $(B_c, B_s)$ can always be found that mimics a pair $(C_c, C_s)$, and vice versa; and furthermore, an infinite number of such pairs exist.

Given a reference structure $m_0(x, y)$, a “null solution” $(B_c, B_s)$ pair can be constructed for which the travel time is identically zero (as similarly for a $(C_c, C_s)$ pair).

1. Preliminaries. Assuming straight line rays, the travel time $d(u, \theta)$ of the structure $m(x, y)$ is computed via the Radon transform:

$$d(r, \theta) = \int_{L(u, \theta)} m(x, y) \, d\ell \equiv \mathcal{R}m$$

The line integral is taken over a straight line $L$ with arc length $\ell$, parameterized by its perpendicular distance $u$ to the origin and the angle $\theta$, measured counter-clockwise, of that the perpendicular makes with the $x$-axis. The function $d(u, \theta_0)$ with $\theta_0$ held constant is called a projection.

The Fourier slice theorem shows that the 1D Fourier transform $\mathcal{F}$ of a projection, which takes $u$ into $k_u$, is the 2D Fourier Transform of the original function, evaluated on a line of angle $\theta_0$ in the wavenumber plane:

$$\tilde{d}(k_u, \theta_0) = \mathcal{F}d(u, \theta_0) = \bar{m}(k_u \cos(\theta_0), k_u \sin(\theta_0)) \quad \text{with} \quad \bar{m}(k_x, k_y) \equiv \mathcal{F}\mathcal{F}m$$
Roughly speaking, the Fourier slice theorem implies that the Radon transform is invertible under a broad range of conditions. Consequently, a Radon transform has no null space; that is, there are no non-zero functions \( m(y, x) \) for which \( \mathcal{R}m = 0 \).

Multiplying the Radon transform by a smooth function \( F(\theta) \) of angle \( \theta \) to yield \( d(r, \theta)F(\theta) \) only changes the overall scaling of a projection and, because of the Fourier slice theorem, corresponds to a fan filter in the \( (k_x, k_y) \) domain:

\[
\mathcal{R}(k_x, k_y) F\left(\tan^{-1}\left(\frac{k_x}{k_y}\right)\right)
\]

Furthermore, if \( |F(\theta)| \leq 1 \), fan-filtering does not increase the overall energy of the 2D Fourier, so by Parseval’s theorem, the energy in the \( (x, y) \)-domain image must not increase. Roughly speaking, fan filtering is a very-well behaved process that is unlikely to affect the invertibility of the Radon transform.

2. Anisotropic traveltime tomography with the Radon transform. Consider a “slowness” function \( m(x, y) \) defined as:

\[
m(x, y) = A(x, y) + B_C(x, y) \cos(2\theta) + B_S(x, y) \sin(2\theta) + C_C(x, y) \cos(4\theta) + C_S(x, y) \sin(4\theta)
\]

Here \( \theta = 0 \) has the interpretation of the fast direction. The Radon transform of \( m(x, y) \) is:

\[
d(r, \theta) = \mathcal{R}A + \mathcal{R}[B_C \cos(2\theta)] + \mathcal{R}[B_S \sin(2\theta)] + \mathcal{R}[C_C \cos(4\theta)] + \mathcal{R}[C_S \sin(4\theta)]
\]

The trigonometric functions can be moved outside the Radon transforms, since they do not vary along the transform’s straight line integration path:

\[
d(r, \theta) = \mathcal{R}A + \cos(2\theta) \mathcal{R}B_C + \sin(2\theta) \mathcal{R}B_S + \cos(4\theta) \mathcal{R}C_C + \sin(4\theta) \mathcal{R}C_S
\]

Note that the term \( \cos(2\theta) \mathcal{R}B_C \) implies fan filtering of \( B_C(x, y) \) (and similarly for the other terms containing trigonometric functions).


3.1. Isotropic structure equivalent to anisotropic structure. Consider case \( A \) where only \( A(x, y) \) is nonzero, and case \( B \) where only \( B_C(x, y) \) and \( B_S(x, y) \) are nonzero. The two cases can be made to imply the same travel times with the choice:

\[
A = \mathcal{R}^{-1}d_B \quad \text{with} \quad d_B \equiv \cos(2\theta) \mathcal{R}B_C + \sin(2\theta) \mathcal{R}B_S
\]

Since \( d_B \) is just the sum of fan-filtered versions of functions whose Radon transforms are presumed to exist, we expect its inverse Radon transform to exist, too. Hence, under a broad range of circumstances:
An $A$ can always be found that mimics a set of $(B_C, B_S)$.

Similarly, for a case where only $C_c(x,y)$ and $C_s(x,y)$ are nonzero, the equivalent $A$ is:

$$A = R^{-1}d_C \quad \text{with} \quad d_C \equiv \cos(4\theta) R C_C + \sin(4\theta) R C_S$$

Hence, under a broad range of circumstances:

An $A$ can always be found that mimics a set of $(C_C, C_S)$.

In the example, we find the $A$ equivalent to a hypothetical $(B_C, B_S)$ and show that they have the same traveltime $d$:

3.2. Anisotropic structure equivalent to isotropic structure. Note that, we can write:

$$A(x, y) = A(x, y) \cos^2(2\theta) + A(x, y) \sin^2(2\theta)$$

Consider case $A$ where only $A(x, y)$ is nonzero:

$$m(x, y) = A(x, y) \cos^2(2\theta) + A(x, y) \sin^2(2\theta)$$

and case $B$ where only $B_c(x, y)$ and $B_s(x, y)$ are nonzero:

$$m(x, y) = B_c(x, y) \cos(2\theta) + B_s(x, y) \sin(2\theta)$$

By matching terms, two cases can be made to have the same travel time:
\[\mathcal{R}B_c = \left(\frac{\cos^2(2\theta)}{\cos(2\theta)} \mathcal{R}A\right) \quad \text{or} \quad B_c = \mathcal{R}^{-1}[\cos(2\theta) \mathcal{R}A]\]

\[\mathcal{R}B_s = \left(\frac{\sin^2(2\theta)}{\sin(2\theta)} \mathcal{R}A\right) \quad \text{or} \quad B_s = \mathcal{R}^{-1}[\sin(2\theta) \mathcal{R}A]\]

Note that this approach relies upon the zeros in the \(\cos^2(2\theta)\) and \(\sin^2(2\theta)\) in the numerators cancelling the poles in the \(\cos(2\theta)\) and \(\sin(2\theta)\) in the denominators, so that the fractions are finite. If we could find a pair of functions \(c(2\theta)\) and \(s(2\theta)\) with the property \(c(2\theta) + s(2\theta) = 1\) and with zeros in the appropriate positions, we could write:

\[A(x, y) = A(x, y) c(2\theta) + A(x, y) s(2\theta)\]

\[B_c = \mathcal{R}^{-1}\left(\frac{c(2\theta)}{\cos(2\theta)} \mathcal{R}A\right) \quad \text{and} \quad B_s = \mathcal{R}^{-1}\left(\frac{s(2\theta)}{\sin(2\theta)} \mathcal{R}A\right)\]

Evidentially, many such pairs of function exist, including:

\[c(2\theta) = \cos^{2n}(2\theta) \quad \text{and} \quad s(2\theta) = [\cos^2(2\theta) + \sin^2(2\theta)]^n - \cos^{2n}(2\theta)\]

with \(n \geq 1\). Hence we conclude:

A pair \((B_c, B_s)\) can always be found that mimics an \(A\), and furthermore, an infinite family of such pairs exist.

By replacing \(2\theta\) with \(4\theta\) in the above argument, we conclude:

A pair \((C_c, C_s)\) can always be found that mimics an \(A\), and furthermore, an infinite family of such pairs exist.

In the example, we find two examples of \((B_c, B_s)\)'s that are equivalent to an \(A\):
We have verified that they have the same traveltime $d$ (not shown).

3.3. Two-theta anisotropic structure equivalent to four-theta anisotropic structure, and vice-versa. We can find the $(C_C, C_S)$ equivalent to a $(B_C, B_S)$ by using the method of Section 3.1 to find the $A$ equivalent to $(B_C, B_S)$ and then using the method of Section 3.2 to find the $(C_C, C_S)$ equivalent to that $A$. Similarly, we can find the $(B_C, B_S)$ equivalent to an $(C_C, C_S)$ by using the method of Section 3.1 to find the $A$ equivalent to $(C_C, C_S)$ and then using the method of Section 3.2 to find the $(B_C, B_S)$ equivalent to that $A$. Recalling that the method of Section 3.2 is non-unique, we conclude:

A pair $(B_C, B_S)$ can always be found that mimics a pair $(C_C, C_S)$, and vice versa; and furthermore, an infinite family of such pairs exist.

In the example, we find a $(C_C, C_S)$ equivalent to a $(B_C, B_S)$:
and a \((B_C, B_S)\) equivalent to a \((C_C, C_S)\):

4.0 Anisotropic Null Solution. Consider two-theta anisotropic travel time functions \((B_C, B_S)\) constructed from a reference solution \(m_0(x, y)\) in the following way:

\[
B_C = \mathcal{R}^{-1}(\sin(2\theta) d_0) \quad \text{and} \quad B_S = -\mathcal{R}^{-1}(\cos(2\theta) d_0) \quad \text{with} \quad d_0 = \mathcal{R} m_0
\]
The travel time $d_B$ associated with this $(B_C, B_S)$ pair is identically zero:

$$d_B = \mathcal{R}[\cos(2\theta)B_C + \sin(2\theta)B_S] = \cos(2\theta)\mathcal{R}B_C + \sin(2\theta)\mathcal{R}B_S = \cos(2\theta) \cdot \sin(2\theta) \cdot d_0 - \sin(2\theta) \cdot \cos(2\theta) \cdot d_0 = 0$$

Thus, a two-theta null solution can be constructed from any reference solution $m_0$. The above argument can be extended to four-theta anisotropy simply by replacing $2\theta$ with $4\theta$. Hence:

Given a reference structure $m_0(x, y)$, a “null solution” $(B_C, B_S)$ pair can be constructed for which the travel time is identically zero (as similarly for a $(C_C, C_S)$ pair).

In the example below, we construct a $(B_C, B_S)$ null solution verify that it has zero travel time.