Nonuniqueness in Anisotropic Traveltime Tomography under the Radon Transform Approximation

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(This work follows up upon my 2015 BSSA paper on the same subject).

0. Summary. Given a 2D slowness function $m(x, y) =$

$$A(x, y) + B_C(x, y)\cos(2\theta) + B_S(x, y)\sin(2\theta) + C_C(x, y)\cos(4\theta) + C_S(x, y)\sin(4\theta)$$

and travel times related to it through a Radon transform, we show that:

An A can always be found that mimics a set of (B_C, B_S) and/or a set of (C_C, C_S);

A pair (B_C, B_S) can always be found that mimics an A, and furthermore, an infinite family of such pairs exist;

A pair (C_C, C_S) can always be found that mimics an A, and furthermore, an infinite number of such pairs exist; and

A pair (B_C, B_S) can always be found that mimics a pair (C_C, C_S), and vice versa; and furthermore, an infinite number of such pairs exist.

Given a reference structure $m_0(x, y)$, a "null solution" (B_C, B_S) pair can be constructed for which the travel time is identically zero (as similarly for a (C_C, C_S) pair).

1. Preliminaries. Assuming straight line rays, the travel time $d(u, \theta)$ of the structure $m(x, y)$ is computed wia the Radon transform:

$$d(r, \theta) = \int_{L(u,\theta)} m(x, y)\, d\ell \equiv \mathcal{R}m$$

The line integral is taken over a straight line L with arc length ℓ, parameterized by its perpendicular distance u to the origin and the angle θ, measured counter-clockwise, of that the perpendicular makes with the x-axis. The function $d(u, \theta_0)$ with θ_0 held constant is called a projection.

The Fourier slice theorem shows that the 1D Fourier transform $\mathcal{F}$ of a projection, which takes u into k_u, is the 2D Fourier Transform of the original function, evaluated on a line of angle θ_0 in the wavenumber plane:

$$\bar{d}(k_u, \theta_0) = \mathcal{F}d(u, \theta_0) = \bar{\bar{m}}\big(k_u cos(\theta_0), k_u sin(\theta_0)\big) \quad with\ \ \bar{\bar{m}}\big(k_x, k_y\big) \equiv \mathcal{F}\mathcal{F}m$$

Roughly speaking, the Fourier slice theorem implies that the Radon transform is invertible under a broad range of conditions. Consequently, a Radon transform has no null space; that is, there are no non-zero functions $m(y,x)$ for which $\mathcal{R}m = 0$.

Multiplying the Radon transform by a smooth function $F(\theta)$ of angle θ to yield $d(r,\theta)F(\theta)$ only changes the overall scaling of a projection and, because of the Fourier slice theorem, corresponds to a fan filter in the (k_x, k_y) domain:

$$\bar{\bar{m}}(k_x, k_y)\, F\left(\tan^{-1}(k_x/k_y)\right)$$

Furthermore, if $|F(\theta)| \leq 1$, fan-filtering does not increase the overall energy of the 2D Fourier, so by Parseval's theorem, the energy in the (x,y)-domain image must not increase. Roughly speaking, fan filtering is a very-well behaved process that is unlikely to affect the invertibility of the Radon transform.

2. Anisotropic traveltime tomography with the Radon transform. Consider a "slowness" function $m(x,y)$ defined as:

$$m(x,y) =$$

$$A(x,y) + B_C(x,y)\cos(2\theta) + B_S(x,y)\sin(2\theta) + C_C(x,y)\cos(4\theta) + C_S(x,y)\sin(4\theta)$$

Here $\theta = 0$ has the interpretation of the fast direction. The Radon transform of $m(x,y)$ is:

$$d(r,\theta) = \mathcal{R}A + \mathcal{R}[B_C\cos(2\theta)] + \mathcal{R}[B_S\sin(2\theta)] + \mathcal{R}[C_C\cos(4\theta)] + \mathcal{R}[C_S\sin(4\theta)]$$

The trigonometric functions can be moved outside the Radon transforms, since they do not vary along the transform's straight line integration path:

$$d(r,\theta) = \mathcal{R}A + \cos(2\theta)\,\mathcal{R}B_C + \sin(2\theta)\,\mathcal{R}B_S + \cos(4\theta)\,\mathcal{R}C_C + \sin(4\theta)\,\mathcal{R}C_S$$

Note that the term $\cos(2\theta)\,\mathcal{R}B_C$ implies fan filtering of $B_C(x,y)$ (and similarly for the other terms containing trigonometric functions).

3. Nonuniqueness.

3.1. Isotropic structure equivalent to anisotropic structure. Consider case A where only $A(x,y)$ is nonzero, and case B where only $B_c(x,y)$ and $B_s(x,y)$ are nonzero. The two cases can be made to imply the same travel times with the choice:

$$A = \mathcal{R}^{-1}d_B \quad \text{with} \quad d_B \equiv \cos(2\theta)\,\mathcal{R}B_C + \sin(2\theta)\,\mathcal{R}B_S$$

Since d_B is just the sum of fan-filtered versions of functions whose Radon transforms are presumed to exist, we expect its inverse Radon transform to exist, too. Hence, under a broad range of circumstances:

$$\text{An } A \text{ can always be found that mimics a set of } (B_C, B_S).$$

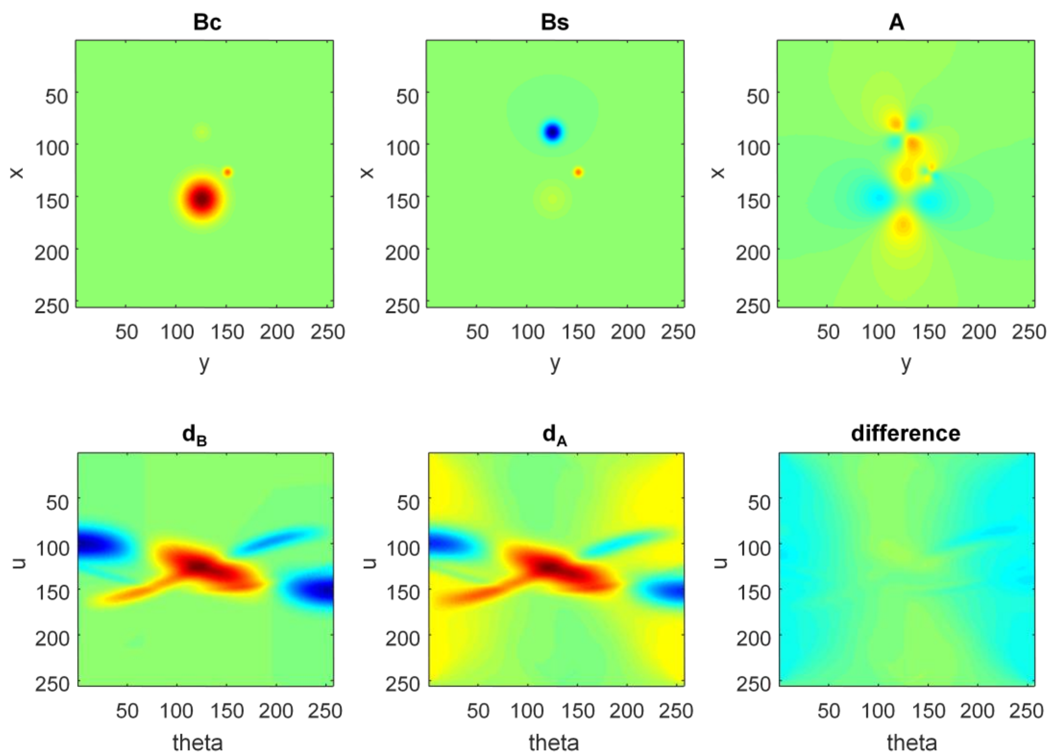
Similarly, for a case C where only $C_c(x,y)$ and $C_s(x,y)$ are nonzero, the equivalent A is:

$$A = \mathcal{R}^{-1} d_C \quad \text{with} \quad d_C \equiv \cos(4\theta)\, \mathcal{R} C_C + \sin(4\theta)\, \mathcal{R} C_S$$

Hence, under a broad range of circumstances:

$$\text{An } A \text{ can always be found that mimics a set of } (C_C, C_S).$$

In the example, we find the A equivalent to a hypothetical (B_C, B_S) amd show that they have the same traveltime d:



3.2. Anisotropic structure equivalent to isotropic structure. Note that, we can write:

$$A(x,y) = A(x,y)\cos^2(2\theta) + A(x,y)\sin^2(2\theta)$$

Consider case A where only $A(x,y)$ is nonzero:

$$m(x,y) = A(x,y)\cos^2(2\theta) + A(x,y)\sin^2(2\theta)$$

and case B where only $B_c(x,y)$ and $B_s(x,y)$ are nonzero:

$$m(x,y) = B_C(x,y)\cos(2\theta) + B_S(x,y)\sin(2\theta)$$

By matching terms, two cases can be made to have the same travel time:

$$\mathcal{R}B_C = \left(\frac{\cos^2(2\theta)}{\cos(2\theta)}\mathcal{R}A\right) \quad \text{or} \quad B_C = \mathcal{R}^{-1}[\cos(2\theta)\,\mathcal{R}A]$$

$$\mathcal{R}B_S = \left(\frac{\sin^2(2\theta)}{\sin(2\theta)}\mathcal{R}A\right) \quad \text{or} \quad B_S = \mathcal{R}^{-1}[\sin(2\theta)\,\mathcal{R}A]$$

Note that this approach relies upon the zeros in the $\cos^2(2\theta)$ and $\sin^2(2\theta)$ in the numerators cancelling the poles in the $\cos(2\theta)$ and $\sin(2\theta)$ in the denominators, so that the fractions are finite. If we could find a pair of functions $c(2\theta)$and $s(2\theta)$ with the property $c(2\theta) + s(2\theta) = 1$ and with zeros in the appropriate positions, we could write:

$$A(x,y) = A(x,y)\,c(2\theta) + A(x,y)\,s(2\theta)$$

$$B_C = \mathcal{R}^{-1}\left(\frac{c(2\theta)}{\cos(2\theta)}\mathcal{R}A\right) \quad \text{and} \quad B_S = \mathcal{R}^{-1}\left(\frac{s(2\theta)}{\sin(2\theta)}\mathcal{R}A\right)$$

Evidentially, many such pairs of function exist, including:

$$c(2\theta) = \cos^{2n}(2\theta) \quad \text{and} \quad s(2\theta) = [\cos^2(2\theta) + \sin^2(2\theta)]^n - \cos^{2n}(2\theta)$$

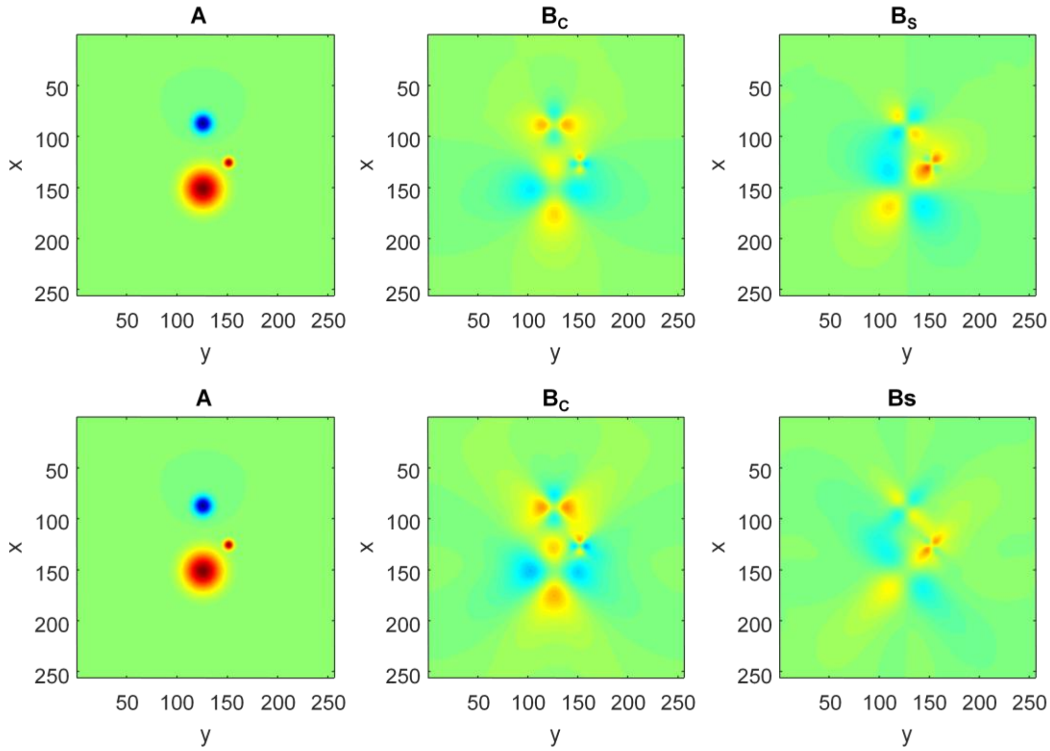
with $n \geq 1$. Hence we conclude:

A pair (B_C, B_S) can always be found that mimics an A, and
furthermore, an infinite family of such pairs exist.

By replacing 2θ with 4θ in the above argument, we conclude:

A pair (C_C, C_S) can always be found that mimics an A, and
furthermore, an infinite family of such pairs exist.

In the example, we find two examples of $(B_C, B_S)'s$ that are equivalent to an A:

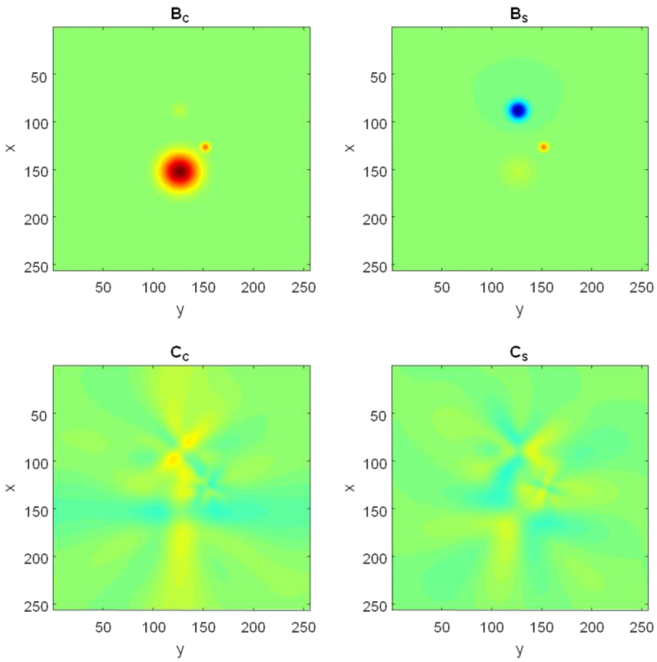


We have verified that they have the same traveltime d (not shown).

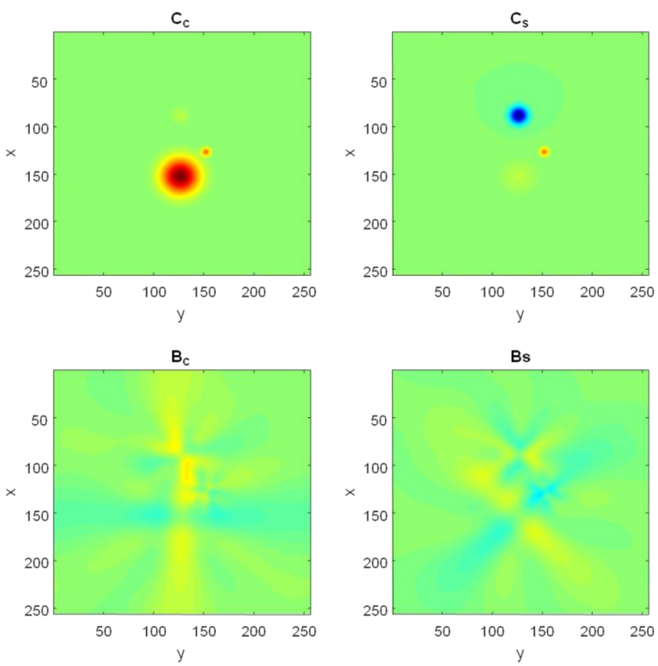
3.3. Two-theta anisotropic structure equivalent to four-theta anisotropic structure, and vice-versa. We can find the (C_C, C_S) equivalent to a (B_C, B_S) by using the method of Section 3.1 to find the A equivalent to (B_C, B_S) and then using the method of Section 3.2 to find the (C_C, C_S) equivalent to that A. Similarly, we can find the (B_C, B_S) equivalent to an (C_C, C_S) by using the method of Section 3.1 to find the A equivalent to (C_C, C_S) and then using the method of Section 3.2 to find the (B_C, B_S) equivalent to that A. Recalling that the method of Section 3.2 is non-unique, we conclude:

A pair (B_C, B_S) can always be found that mimics a pair (C_C, C_S), and vice versa; and furthermore, an infinite family of such pairs exist.

In the example, we find a (C_C, C_S) equivalent to a (B_C, B_S):



and a (B_C, B_S) equivalent to a (C_C, C_S):



4.0 Anisotropic Null Solution. Consider two-theta anisotropic travel time functions (B_C, B_S) constructed from a reference solution $m_0(x, y)$ in the following way:

$$B_C = \mathcal{R}^{-1}(\sin(2\theta)\, d_0) \quad \text{and} \quad B_S = -\mathcal{R}^{-1}(\cos(2\theta) d_0) \quad \text{with} \quad d_0 = \mathcal{R} m_0$$

The travel time d_B associated with this (B_C, B_S) pair is identically zero:

$$d_B = \mathcal{R}[\cos(2\theta)B_C + \sin(2\theta)B_s] \ = \ \cos(2\theta)\mathcal{R}B_C + \sin(2\theta)\mathcal{R}B_s =$$

$$\cos(2\theta)\sin(2\theta)\, d_0 - \sin(2\theta)\cos(2\theta)\, d_0 = 0$$

Thus, a two-theta null solution can be constructed from any reference solution m_0. The above argument can be extended to four-theta anisotropy simply by replacing 2θ with 4θ. Hence:

> Given a reference structure $m_0(x, y)$, a "null solution" (B_C, B_S) pair can be constructed for which the travel time is identically zero (as similarly for a (C_C, C_S) pair).

In the example below, we construct a (B_C, B_S) null solution verify that it has zero travel time.