Derivatives of Wavefield Error with Respect to Source Parameters using Adjoint Methods
Bill Menke, 01/01/2017

1. Wave Equation. Consider a wave equation with wavefield $\mathbf{u}(\mathbf{x},t)$, differential operator $\boldsymbol{\mathcal{L}}(\mathbf{x},t)$ and source $\mathbf{s}(\mathbf{x},t;\mathbf{m})$:

$$\boldsymbol{\mathcal{L}}\mathbf{u} = \mathbf{s} \quad \text{so} \quad \boldsymbol{\mathcal{L}}\delta\mathbf{u} = \delta\mathbf{s} \;\; \text{and} \;\; \delta\mathbf{u} = \boldsymbol{\mathcal{L}}^{-1}\delta\mathbf{s} \tag{1}$$

Note that source depends upon model parameters $\mathbf{m}$ but the differential operator is independent of them. A source perturbation $\delta\mathbf{m}$ in model parameters leads to a perturbation $\delta\mathbf{s}$ in the source, which in turn cases a perturbation in the field:

$$\delta\mathbf{u} = \boldsymbol{\mathcal{L}}^{-1}\delta\mathbf{s} \tag{2}$$

2. Frechet derivative of observational error with respect to the source. The observational error is $E = \left(\mathbf{u}^{obs} - \mathbf{u}^{(0)}, \mathbf{u}^{obs} - \mathbf{u}^{(0)}\right) = \left(\mathbf{e}^{(0)}, \mathbf{e}^{(0)}\right)$, where $\mathbf{u}^{(0)}$ is the unperturbed field and $\mathbf{e}^{(0)} = \mathbf{u}^{obs} - \mathbf{u}^{(0)}$. The inner product $(.,.)$ is over time, space and component. The perturbation δE in error due to a perturbation in wavefield $\delta\mathbf{u}$ is:

$$\delta E = -2\left(\mathbf{e}^{(0)}, \delta\mathbf{u}\right) \tag{3}$$

Inserting (2) into (3) and rearranging using adjoint manipulations yields:

$$\delta E = -2\left(\mathbf{e}^{(0)}, \boldsymbol{\mathcal{L}}^{-1}\delta\mathbf{s}\right) = -2\left(\boldsymbol{\mathcal{L}}^{-1\dagger}\mathbf{e}^{(\mathbf{0})}, \delta\mathbf{s}\right) \equiv (\mathbf{g}, \delta\mathbf{s}) \tag{4}$$

where the Frechet derivative $\mathbf{g} \equiv \delta E/\delta\mathbf{s}$ of error with respect to source is an adjoint field that satisfies:

$$\boldsymbol{\mathcal{L}}^{\dagger}\mathbf{g} = -2\mathbf{e}^{(0)} \tag{5}$$

(3) Parameterized source. Now suppose that the source is parameterized:

$$\delta \mathbf{s}(\mathbf{m}) = \sum_{i=1}^{M} \left.\frac{\partial \mathbf{s}(\mathbf{x}, \mathbf{t})}{\partial m_i}\right|_{\mathbf{m}^{(0)}} \delta m_i \equiv \mathbf{S}\, \delta \mathbf{m}$$

(6)

where M is the number of model parameters. The partial derivative of error with respect to source parameter becomes:

$$\delta E = (\mathbf{g}, \delta \mathbf{s}) = (\mathbf{g}, \mathbf{S}\, \delta \mathbf{m}) = (\mathbf{g}, \mathbf{S})\, \delta \mathbf{m}$$

(7)

By inspection, we find that:

$$\frac{\partial E}{\partial \mathbf{m}} = (\mathbf{g}, \mathbf{S})$$

(8)

(4) Impulsive source. Consider an impulsive point source of unknown amplitude m_1 that occurs at known location $\mathbf{x}_s$, time t_s and direction $\mathbf{t}$:

$$\mathbf{s}(\mathbf{m}) = m_1 \delta(\mathbf{x} - \mathbf{x}_s\,)\delta(t - t_s\,)\mathbf{t}$$

(9)

The derivative with respect to the model parameter is:

$$\frac{\partial \mathbf{s}}{\partial m_1} = \delta(\mathbf{x} - x_s\,)\delta(t - t_s\,)\mathbf{t}$$

(10)

and

$$\delta E = \left(\mathbf{g}, \frac{\partial s}{\partial m_1}\right) \delta m_1 = (\mathbf{g}, \delta(\mathbf{x} - \mathbf{x}_s\,)\delta(t - t_s\,)\mathbf{t})\, \delta m_1 = \ [\mathbf{g}(\mathbf{x}_s, t_s) \cdot \mathbf{t}]\, \delta m_1$$

(11)

By inspection, the partial derivative of error with respect to source amplitude is:

$$\frac{\partial E}{\partial m_1} = \mathbf{g}(\mathbf{x}_s, t_s) \cdot \mathbf{t}$$

(12)

(4) Double couple source. Consider an impulsive double-couple source of unknown moment **M** that occurs at an unknown location $\mathbf{x}_s$ and an unknown time t_s (that is, $\mathbf{M}, \mathbf{x}_s$ and t_s are all model parameters):

$$s_i = -\sum_{j=1}^{3} M_{ij} \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ )$$

(13)

The partial derivatives of the source with respect to components of the moment tensor are:

$$\frac{\partial s_i}{\partial M_{pq}} = -\sum_{j=1}^{3} \frac{\partial M_{ij}}{\partial M_{pq}} \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ ) =$$

$$= -\delta_{ip} \sum_{j=1}^{3} \delta_{jq}\ \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ ) = -\delta_{ip} \frac{\partial}{\partial x_q} \delta(\mathbf{x} - \mathbf{x}_s)\, \delta(t - t_s\ )$$

(14)

And the partial derivatives of the error with respect to components of the moment tensor are:

$$\frac{\partial E}{\partial M_{pq}} = \left(\mathbf{g}, \frac{\partial \mathbf{s}}{\partial M_{pq}}\right) =$$

$$= -\sum_{i=1}^{3} \iiint \int g_i \delta_{ip} \frac{\partial}{\partial x_q} \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ )\ dt\ d^3x$$

$$= \iiint \int \frac{\partial}{\partial x_q} g_p\ \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ )\ dt\ d^3x$$

$$= \left.\frac{\partial g_p}{\partial x_q}\right|_{\mathbf{x}_s, t_s}$$

(15)

The partial derivatives of the source with respect to components of the source location are:

$$\frac{\partial s_i}{\partial x_{sp}} = -\sum_{j=1}^{3} M_{ij} \frac{\partial}{\partial x_{sp}} \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ ) = \sum_{j=1}^{3} M_{ij} \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ )$$

(16)

And the partial derivatives of the error with respect to components of the source coordinates are:

$$\frac{\partial E}{\partial x_{sp}} = \left(\mathbf{g}, \frac{\partial \mathbf{s}}{\partial x_{sp}}\right) =$$

$$= \sum_{i=1}^{3} \iiint \int g_i \sum_{j=1}^{3} M_{ij} \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \delta(t - t_s\ )\, dt\, d^3x$$

$$= \sum_{j=1}^{3} M_{ij}\ \frac{\partial^2 g_i}{\partial x_j\, \partial x_p}\bigg|_{\mathbf{x}_s, t_s}$$

(17)

Finally, the partial derivatives of the source with respect to the origin time are:

$$\frac{\partial s_i}{\partial t_s} = -\sum_{j=1}^{3} M_{ij}\ \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \frac{\partial}{\partial t_s} \delta(t - t_s\ ) = \sum_{j=1}^{3} M_{ij}\ \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \frac{\partial}{\partial t} \delta(t - t_s\ )$$

(17)

And the partial derivative of the error with respect to source time is:

$$\frac{\partial E}{\partial t_s} = \left(\mathbf{g}, \frac{\partial \mathbf{s}}{\partial x_{sp}}\right) =$$

$$= \sum_{i=1}^{3} \iiint \int g_i \sum_{j=1}^{3} M_{ij}\ \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{x}_s)\ \frac{\partial}{\partial t} \delta(t - t_s\ )\, dt\, d^3x$$

$$= \sum_{i=1}^{3}\ \sum_{j=1}^{3} M_{ij}\ \frac{\partial^2 g_i}{\partial x_j\, \partial t}\bigg|_{\mathbf{x}_s, t_s}$$

(18)