Gradient of a parameter in a nonlinear differential equation: Example

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The nonlinear differential equation for the field $u(t)$:

$$u' + bu + ce^{bt}u^2 = 0 \quad \text{with} \quad u(t=0) = 1$$

(where prime denotes differentiation with respect to t) has solution:

$$u(t) = \frac{e^{-bt}}{1+ct} \quad \text{and} \quad u'(t) = \frac{-be^{-bt}}{1+ct} + \frac{-ce^{-bt}}{(1+ct)^2}$$

This solution can be verified by direct substitution into the differential equation:

$$\frac{-be^{-bt}}{1+ct} + \frac{-ce^{-bt}}{(1+ct)^2} + \frac{e^{-bt}}{1+ct} + ce^{bt}\frac{e^{-2bt}}{(1+ct)^2} \quad =$$

$$\frac{-be^{-bt}}{1+ct} + \frac{e^{-bt}}{1+ct} + \frac{-ce^{-bt}}{(1+ct)^2} + \frac{ce^{-bt}}{(1+ct)^2} \quad = 0$$

and by noting that it satisfied the boundary condition $u(t) = 1$ for all values of c.

Now consider the case $c = c_0 + \delta c$ and $u = u_0 + \delta u$. Since we know the solution, the partial derivative of the field with respect to the parameter c can be calculated directly:

$$\frac{\partial u}{\partial c} = \frac{-te^{-bt}}{(1+ct)^2} \;\text{ so that }\; u \approx u_0 + \delta u = \; u_0 + \frac{\partial u}{\partial c}\delta c = \frac{e^{-bt}}{(1+c_0 t)} + \frac{-\delta c\; te^{-bt}}{(1+c_0 t)^2}$$

The derivative can also be computed by solving the differential equation associated with the Born approximation. We insert $c = c_0 + \delta c$ and $u = u_0 + \delta u$ into the differential equation and keep only terms up to first order in small quantities:

$$u_0{}' + \delta u' + bu_0 + b\delta u + (c_0 + \delta c)\, e^{bt}(u_0 + \delta u)^2 \approx$$

$$u_0{}' + \delta u' + bu_0 + b\delta u + (c_0 + \delta c)\, e^{bt}(u_0^2 + 2u_0\delta u) =$$

$$u_0{}' + bu_0 + c_0 e^{bt}u_0^2 + \delta u' + b\delta u + \delta c\; e^{bt}u_0^2 + 2c_0 e^{bt}u_0\delta u = 0$$

Subtracting out the unperturbed equation yields a differential equation for δu:

$$\delta u' + (b + 2c_0 e^{bt}u_0)\delta u + \delta c\; e^{bt}u_0^2 = 0 \quad \text{with} \quad \delta u(t=0) = 0$$

or

$$\mathcal{L}\,\delta u = f_0 \;\; with \;\; \mathcal{L} \equiv \frac{\partial}{\partial t} + (b + 2c_0 e^{bt}u_0) \quad \text{and} \quad f_0 \equiv -\, e^{bt}u_0^2\,\delta c$$

That the solution to this equation is:

$$\delta u = \frac{-\delta c\ te^{-bt}}{(1+c_0t)^2} \quad \text{and} \quad \delta u' = \frac{-\delta c e^{-bt}}{(1+c_0t)^2} + \frac{b\ \delta c\ te^{-bt}}{(1+c_0t)^2} + \frac{2c_0\delta c\ te^{-bt}}{(1+c_0t)^3}$$

can be verified by substitution:

$$\delta u' + b\delta u + 2c_0e^{bt}u_0\delta u + \delta c\ e^{bt}u_0^2 =$$

$$\frac{-\delta c e^{-bt}}{(1+c_0t)^2} + \frac{b\ \delta c\ te^{-bt}}{(1+c_0t)^2} + \frac{2c_0\delta c\ te^{-bt}}{(1+c_0t)^3} + \frac{-b\delta c\ te^{-bt}}{(1+c_0t)^2} +$$

$$-2c_0e^{bt}\frac{e^{-bt}}{(1+c_0t)}\frac{\delta c\ te^{-bt}}{(1+c_0t)^2} + \delta c\ e^{bt}\frac{e^{-2bt}}{(1+c_0t)^2} =$$

$$\frac{-\delta c e^{-bt}}{(1+c_0t)^2} + \frac{b\ \delta c\ te^{-bt}}{(1+c_0t)^2} + \frac{2c_0\delta c\ te^{-bt}}{(1+c_0t)^3} + \frac{-b\delta c\ te^{-bt}}{(1+c_0t)^2} +$$

$$-\frac{2c_0\delta c\ te^{-bt}}{(1+c_0t)^3} + \frac{\delta c e^{-bt}}{(1+c_0t)^2} = 0$$

Now suppose that we have $u^{obs}(t)$ and define and error $E(c) = (e,e)$ with $e = u^{obs} - u$. We find,

$$\left.\frac{\partial E}{\partial c}\right|_{c_0} = \left(\frac{\partial u}{\partial c}, -2e_0\right) = \left(\frac{\partial}{\partial c}(\mathcal{L}^{-1}f_0), -2e_0\right) = \left(\mathcal{L}^{-1}\frac{\partial f_0}{\partial c}, -2e_0\right) =$$

$$(e^{bt}u_0^2, -2\mathcal{L}^{-1\dagger}e_0) = (-2e^{bt}u_0^2, \mathcal{L}^{-1\dagger}e_0) = (-2e^{bt}u_0^2, \lambda)$$

with $\mathcal{L}^\dagger\lambda = e_0$ and noting that $\partial f_0/\partial c = \partial f_0/\partial\delta c$, since c_0 is constant.