Exemplary solution to the advection-diffusion equation  
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Exemplary Solution

Consider a problem with cylindrical symmetry and only radial variation. Suppose \( w \) obeys \( \nabla \cdot w = 0 \) (except at the origin). Then \( \Delta w = 0 \) and \( \nabla w = \text{constant} \).

The equation for temperature \( T \) is:

\[
\Delta^2 T + w \cdot \Delta T = 0 = \nabla \cdot (\nabla T) + w \cdot \nabla T
\]

Assume \( \Delta T \) radial (\( \Delta T \)c = \( \partial T \)/\( \partial r \)) is:

\[
\nabla \cdot g_r + \frac{w_0}{r} g_r = 0 = \frac{d}{dr}(rg_r) + \frac{w_0}{r} g_r
\]

\[
r \frac{dg_r}{dr} = -(w_0 + 1) g_r
\]

\[
\frac{dg_r}{g_r} = \left( \frac{w_0 + 1}{r} \right) dr
\]

\[
\ln g_r = -(w_0 + 1) \ln r + \ln C
\]

\[
g_r = C r^{-(w_0 + 1)}
\]

\[
T(r) = -\int g_r dr = \frac{C}{w_0} r^{-(w_0 + 1)}
\]

Check:

\[
\Delta^2 T = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left( -rC r^{-(w_0 + 1)} \right)
\]

\[
= -C \frac{1}{r^2} \frac{d}{dr} \left( r^{-(w_0 + 1)} \right) = -C \frac{1}{r^{-(w_0 + 1)}} r^{-(w_0 + 1)}
\]

\[
= C w_0 r^{-(w_0 + 2)}
\]

\[
\frac{\partial}{\partial r} \left( r w_0 \frac{\partial T}{\partial r} \right) = \frac{w_0}{r} \left( -w_0 r^{-(w_0 + 1)} \right)
\]

\[
= -C w_0 r^{-2}
\]

\[
\Delta^2 T + w \frac{\partial^2 T}{\partial r^2} = 0
\]
Thoughts about the uniqueness of \( w \) (given \( T \))

\[
given \ (T, w) \ that \ solves \ \nabla^2 T + w \cdot \nabla T = 0
\]

We find that \((T, w + \alpha p)\) also solves this equation, where

\[
p = \begin{bmatrix} \frac{2T}{3y} & -\frac{2T}{3x} & 0 \end{bmatrix}^T, \quad \alpha = \text{function}
\]

since

\[
\alpha \left[ \frac{\partial T}{\partial y} - \frac{\partial T}{\partial x} \right] = 0
\]

now in general \( \nabla \cdot (\alpha p) \neq 0 \)

\[
\nabla \cdot (\alpha p) = \alpha \left[ \frac{\partial^2 T}{\partial y^2} - \frac{\partial^2 T}{\partial x^2} \right] + \alpha \left[ \frac{\partial T}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial T}{\partial x} \frac{\partial T}{\partial y} \right] \neq 0
\]

but choosing \( \alpha = c_0 + c_1 T \) where \( c_0, c_1 \) constant

\[
c_1 \frac{\partial^2 T}{\partial x^2} - c_1 \frac{\partial^2 T}{\partial y^2} = 0
\]

so

\[
w \rightarrow w + (c_0 + c_1 T) \begin{bmatrix} \frac{\partial T}{\partial y} \\ -\frac{\partial T}{\partial x} \\ 0 \end{bmatrix}
\]

both preserves \( \nabla \cdot w = 0 \)

and satisfies \( \nabla^2 T + w \cdot \nabla T = 0 \)

hence it represents a non-uniqueness of the problem.