Exemplary solution to the advection-diffusion equation

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Exemplary Solution

Consider a problem with cylindrical symmetry and only radial variation. Suppose $\underline{w}$ obeys $\nabla \cdot \underline{w} = 0$ (except at the origin). Then $2\pi r w_r$ = constant and $\boxed{w_r(r) = w_0 r^{-1}}$ $(w_0 > 0)$

The equation for temperature T is

$$\nabla^2 T + \underline{w} \cdot \nabla T = 0 = \nabla \cdot (\nabla T) + \underline{w} \cdot \nabla T$$

Assume ∇T radial $(\nabla T)_r = \frac{\partial T}{\partial r} = -q_r$ $(q_r > 0)$

$$\nabla \cdot q_r + \frac{w_0}{r} q_r = 0 = \frac{1}{r}\frac{d}{dr}(r q_r) + \frac{w_0}{r} q_r$$

$$r \frac{dq_r}{dr} = -(w_0 + 1) q_r$$

$$\frac{dq_r}{q_r} = (w_0 + 1) \frac{dr}{r}$$

integration const

$$\ln q_r = -(w_0 + 1) \ln r + \ln C$$

$$q_r = C r^{-(w_0+1)}$$

$$\boxed{T(r) = -\int q_r \, dr = \frac{C}{w_0} r^{-w_0}} \qquad (w_0 > 0)$$

check $\nabla^2 T = \frac{1}{r}\frac{d}{dr}\left(r \frac{dT}{dr}\right) = \frac{1}{r}\frac{d}{dr}\left(-r C r^{-w_0 - 1}\right)$

$$= -C \frac{1}{r} \frac{d}{dr}\left(r^{-w_0}\right) = -C \frac{1}{r}(-w_0) r^{-w_0 - 1}$$

$$= +C w_0 r^{-w_0 - 2}$$

$$w_r \frac{\partial T}{\partial r} = \frac{w_0}{r} \frac{C}{w_0}\left(-w_0 r^{-w_0 - 1}\right)$$

$$= -C w_0 r^{-w_0 - 2}$$

$$\nabla^2 T + w_r \frac{\partial T}{\partial r} = 0$$

Thoughts about the uniqueness of **w** (given T)

given $(T, \underline{w})$ that solves $\nabla^2 T + \underline{w}\cdot\nabla T = 0$

we find that $(T, \underline{w} + \alpha \underline{p})$ also

solves this equation, where

$$p = \left[\frac{\partial T}{\partial y}, -\frac{\partial T}{\partial x}, 0 \right], \quad \alpha = \text{function}$$

since $\alpha \left[\partial T/\partial y, -\frac{\partial T}{\partial x}, 0 \right]^T \left[\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, 0 \right] = 0$

now in general $\nabla\cdot(\alpha p) \neq 0$

$$\nabla\cdot(\alpha p) = \underbrace{\left[\alpha \frac{\partial^2 T}{\partial x \partial y} - \alpha \frac{\partial^2 T}{\partial x \partial y} \right]}_{0} + \frac{\partial \alpha}{\partial x}\frac{\partial T}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial T}{\partial x} \neq 0$$

but choosing $\alpha = c_0 + c_1 T$ where c_0, c_1 const

$$c_1 \frac{\partial T}{\partial x}\frac{\partial T}{\partial y} - c_1 \frac{\partial T}{\partial y}\frac{\partial T}{\partial x} = 0$$

so $$w \to w + (c_0 + c_1 T) \begin{bmatrix} \partial T/\partial y \\ -\partial T/\partial x \\ 0 \end{bmatrix}$$

both preserves $\nabla\cdot w = 0$

and satisfies $\nabla^2 T + \underline{w}\cdot\nabla T = 0$

hence it represents a non-uniqueness of the problem.