Moment Tensor of a Fault in a Transversely Isotropic Material

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(some typos corrected January 22, 2019)

This work was motivated by a discussion that I had with other Lamont scientists of Li et al.'s (2018) recent paper.

Summary: We consider a fault (with no volume change) within a transverse isotropic material. Fault geometry is described by the fault plane and the auxiliary plane, which intersect along the null-direction (N-axis). A moment tensor is said to contain no explosive component when its trace is identically-zero. Furthermore, if it also has one identically-zero eigenvalue, it is said to have no compensated linear-vector dipole (CLVD) component. We show that the moment tensor has zero trace and one identically-zero eigenvalue when the axis of transverse isotropic symmetry is (A) parallel to N-axis; or (B) within the fault or auxiliary planes.

(1) Definition and properties of the fault direction matrix **F.**

(1.1) Presuming that faulting produces no volume change, the fault plane normal $\mathbf{v}$ and slip direction $\mathbf{u}$ must be mutually perpendicular unit vectors satisfying $\mathbf{u}\cdot\mathbf{v}=0$. For completeness, we define a unit vector $\mathbf{w}=\mathbf{u}\times\mathbf{v}$ that is normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$; that is, $\mathbf{w}$ lies within the fault plane and is perpendicular to slip. The w-direction is called the null direction (or N-axis). The $(\mathbf{u},\mathbf{w})$ plane is called the fault plane and the $(\mathbf{v},\mathbf{w})$ plane is called the auxiliary plane.

(1.2) We define a symmetric fault-orientation matrix **F:**

$$\mathbf{F}=\mathbf{u}\mathbf{v}^{\mathrm{T}}+\mathbf{v}\mathbf{u}^{\mathbf{T}} \quad \text{or} \quad F_{ij}=u_i v_j+u_j v_i$$

The trace of **F** is zero:

$$\mathrm{tr}(\mathbf{F})=F_{ii}=u_i v_i+\ u_i v_i=2\,(\mathbf{u}\cdot\mathbf{v})=0$$

The eigenvalues $\lambda^{(\mathrm{i})}$ and eigenvectors $\mathbf{f}^{(\mathrm{i})}$ of **F** are:

$$\mathbf{f}^{(1)}=\frac{(\mathbf{u}+\mathbf{v})}{\sqrt{2}} \quad \text{and} \quad \lambda^{(1)}=+1$$

$$\mathbf{f}^{(2)}=\frac{(\mathbf{u}-\mathbf{v})}{\sqrt{2}} \quad \text{and} \quad \lambda^{(2)}=-1$$

$$\mathbf{f}^{(3)}=\mathbf{w} \quad \text{and} \quad \lambda^{(3)}=0$$

These eigenvalues and eigenvectors can be verified by substitution into the eigenvalue equation $\mathbf{F}\mathbf{f}^{(i)} = \lambda^{(i)}\mathbf{f}^{(i)}$:

$$\mathbf{F}\mathbf{f}^{(1)} = \frac{\mathbf{1}}{\sqrt{2}}\mathbf{u}\mathbf{v}^{\mathrm{T}}(\mathbf{u}+\mathbf{v}) + \frac{\mathbf{1}}{\sqrt{2}}\mathbf{v}\mathbf{u}^{\mathbf{T}}(\mathbf{u}+\mathbf{v}) = \left(\frac{\mathbf{1}}{\sqrt{2}}\mathbf{u} + \frac{\mathbf{1}}{\sqrt{2}}\mathbf{v}\right) = \lambda^{(1)}\mathbf{f}^{(1)}$$

$$\mathbf{F}\mathbf{f}^{(2)} = \frac{\mathbf{1}}{\sqrt{2}}\mathbf{u}\mathbf{v}^{\mathrm{T}}(\mathbf{u}-\mathbf{v}) + \frac{\mathbf{1}}{\sqrt{2}}\mathbf{v}\mathbf{u}^{\mathbf{T}}(\mathbf{u}-\mathbf{v}) = -\left(\frac{\mathbf{1}}{\sqrt{2}}\mathbf{u} - \frac{\mathbf{1}}{\sqrt{2}}\mathbf{v}\right) = \lambda^{(2)}\mathbf{f}^{(2)}$$

$$\mathbf{F}\mathbf{f}^{(3)} = \mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{w} + \mathbf{v}\mathbf{u}^{\mathbf{T}}\mathbf{w} = 0 = \lambda^{(3)}\mathbf{f}^{(3)}$$

The determinant of **F** is zero, since the determinant is the product of the eigenvalues and one eigenvalue is zero:

$$\det(\mathbf{F}) = 0$$

(2) Definition and properties of the moment tensor.

(2.1) The moment tensor **m** equivalent to the fault is defined as:

$$m_{ij} = \bar{u}\Omega\, C_{ijpq} u_p v_q$$

where **C** is the Voigt elasticity tensor, $\bar{u}$ is the scalar slip and Ω is the fault area (Aki and Richards, 2009, their equation 3.19).

Because of the symmetries of the Voigt tensor ($C_{ijpq} = C_{ijqp} = C_{jipq} = C_{pqij}$), we can also write the moment tensor in terms of **F**:

$$m_{ij} = ½\bar{u}\Omega\, C_{ijpq} u_p v_q + ½\bar{u}\Omega\, C_{ijpq} u_p v_q$$

$$= ½\bar{u}\Omega\, C_{ijpq} u_p v_q + ½\bar{u}\Omega\, C_{ijqp} u_q v_p$$

$$= ½\bar{u}\Omega\, C_{ijpq}\left(u_p v_q + u_q v_p\right) = ½\bar{u}\Omega\, C_{ijpq} F_{pq}$$

(3) Fault in an isotropic medium.

(3.1) For isotropic media, the Voigt tensor is:

$$C_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu\delta_{ip}\delta_{jq} + \mu\delta_{iq}\delta_{jp}$$

The moment tensor equivalent to a fault is:

$$m_{ij} = \bar{u}\Omega\lambda\delta_{ij}\delta_{pq}u_p v_q + \bar{u}\Omega\mu\delta_{ip}\delta_{jq}u_p v_q + \bar{u}\Omega\mu\delta_{iq}\delta_{jp}u_p v_q =$$

$$= \bar{u}\Omega\lambda\delta_{ij}(\mathbf{u}\cdot\mathbf{v}) + \bar{u}\Omega\mu\left(u_i v_j + u_j v_i\right) = 0 + \bar{u}\Omega\mu F_{ij}$$

Because the moment tensor $\mathbf{m}$ is proportional to $\mathbf{F}$, it has zero trace, eigenvalues of $(+\bar{u}\Omega\mu, -\bar{u}\Omega\mu, 0)$ and zero determinant. Consequently, the moment tensor equivalent to a fault in an isotropic material has no explosive or CLVD components.

(3.2) When one contracts the first pair of indices of the isotropic Voigt tensor to form a symmetric matrix $\mathbf{Q}$, this matrix is proportional to a Kronecker delta function:

$$Q_{pq} \equiv C_{iipq} = \lambda\delta_{ii}\delta_{pq} + \mu\delta_{ip}\delta_{iq} + \mu\delta_{iq}\delta_{ip} = (3\lambda + 2\mu)\delta_{pq}$$

Therefore, another way of showing that the trace of the moment tensor is zero is:

$$\mathrm{tr}(\mathbf{m}) = ½\bar{u}\Omega\, Q_{pq}F_{pq} = ½\bar{u}\Omega\, \mathrm{tr}(\boldsymbol{QF}) = ½\bar{u}\Omega\, (3\lambda + 2\mu)\delta_{pq}F_{pq} = ½\bar{u}\Omega\, (3\lambda + 2\mu)F_{pp} =$$

$$= ½\bar{u}\Omega\, (3\lambda + 2\mu)\, \mathrm{tr}(\mathbf{F}) = 0$$

(4) Moment tensor in general isotropic media.

(4.1) In general anisotropic media, the trace of the moment tensor is:

$$\mathrm{tr}(\mathbf{m}) = m_{ii} = ½\bar{u}\Omega\, Q_{pq}F_{pq} = ½\bar{u}\Omega\, Q_{qp}F_{pq} = ½\bar{u}\Omega\, \mathrm{tr}(\mathbf{QF})$$

While $\mathrm{tr}(\mathbf{F}) = 0$, in general, $\mathrm{tr}(\mathbf{QF}) \neq 0$, implying that, in general, the moment tensor has an explosive component. A special case where $\mathrm{tr}(\mathbf{QF}) = 0$ occurs in the isotropic case, where $\mathbf{Q} = (3\lambda + 2\mu)\,\mathbf{I}$ and $\mathrm{tr}(\mathbf{QF}) \propto \mathrm{tr}(\mathbf{F}) = 0$ (implying that the explosive component is zero in this case).

(5) Trace of the moment tensor in a transverse isotropic medium aligned with the N-direction of the fault.

(5.1) When a medium has a rotational symmetry, the Voigt tensor is invariant under a rotation $\mathbf{R}$ that expresses that symmetry. Here $\mathbf{R}$ is an orthogonal matrix that satisfies $\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$ (or $R_{ip}R_{iq} = \delta_{pq}$). The symmetry conditions is:

$$C_{ijkl} = R_{ip}R_{jp}R_{kr}R_{ls}C_{pqrs}$$

Contracting the first two indices yields a constraint on $\mathbf{Q}$:

$$Q_{kl} = C_{iikl} = R_{ip}R_{ip}R_{kr}R_{ls}C_{pqrs} = \delta_{pq}R_{kr}R_{ls}C_{pqrs} = R_{kr}R_{ls}C_{pprs} = R_{kr}R_{ls}Q_{rs}$$

Thus, the contracted Voigt tensor $\mathbf{Q}$ has the same symmetry as the Voigt tensor, itself.

(5.2) In the special case of transverse isotropy, $\mathbf{R} = \mathbf{R}(\theta)$ corresponds to a rotation by an arbitrary angle θ around the symmetry axis. For traverse isotropic material with the symmetry axis parallel to the z-axis, the nonzero elements of the Voigt tensor are:

$$C_{1111} = C_{2222} = A \quad \text{and} \quad C_{3333} = C$$

$$C_{2323} = C_{1313} = L \quad \text{and} \quad C_{1212} = N$$

$$C_{1122} = A - 2N \quad \text{and} \quad C_{1133} = C_{2233} = F = \eta(A - 2N)$$

(and their allowable permutations). Here (A, C, L, N, F) are Love's (1927) constants and η is the ratio $F/(A - 2N)$. Note that any element with an "unrepeated" index is zero.

Invariance under this **R** implies that, in a coordinate system with the z-axis aligned with the symmetry axis, **Q** has the form:

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{11} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}$$

This form can be derived by requiring that the equation $\mathbf{Q} = \mathbf{RQR}^{\mathrm{T}}$ hold for all rotations about the z-axis, irrespective of rotation angle θ. Defining $c \equiv \cos(\theta)$ and $s = \sin(\theta)$, this equation is:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} (cQ_{11} - sQ_{12}) & (cQ_{12} - sQ_{22}) & (cQ_{13} - sQ_{23}) \\ (sQ_{11} + cQ_{12}) & (sQ_{12} + cQ_{22}) & (sQ_{13} + cQ_{23}) \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} \mathrm{c} & \mathrm{s} & 0 \\ -\mathrm{s} & \mathrm{c} & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} (c^2 Q_{11} + s^2 Q_{22}) & csQ_{11} - s^2 Q_{12} + c^2 Q_{12} - csQ_{22} & (cQ_{13} - sQ_{23}) \\ . & (s^2 Q_{11} + c^2 Q_{22}) & (sQ_{13} + cQ_{23}) \\ . & . & Q_{33} \end{bmatrix}$$

The (1,1) and (2,2) elements of the equation are:

$$Q_{11} = c^2 Q_{11} + s^2 Q_{22} \quad \text{and} \quad Q_{22} = s^2 Q_{11} - c^2 Q_{22}$$

Subtracting them, we find:

$$Q_{11} - Q_{22} = (c^2 - s^2) Q_{11} - (c^2 - s^2) Q_{22} = \cos(2\theta)\,(Q_{11} - Q_{22})$$

This equation can only be satisfied when $Q_{11} - Q_{22} = 0$; that is, $Q_{11} = Q_{22}$. The (1,2) element of the equation is then:

$$Q_{12} = csQ_{11} - s^2 Q_{12} + c^2 Q_{12} - csQ_{22} = -s^2 Q_{12} + c^2 Q_{12} = \cos(2\theta) Q_{12}$$

Which implies that $Q_{12} = 0$. The (1,3) and (2,3) elements are:

$$Q_{13} = cQ_{13} - sQ_{23} \quad \text{and} \quad Q_{23} = sQ_{13} + cQ_{23}$$

Or

$$\begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix}$$

This equation implies that the two-vector $[Q_{13}, Q_{23}]^{\mathrm{T}}$ is unchanged by a rotation, and that can only occur when the vector is zero; that is, $Q_{13} = 0$ and $Q_{23} = 0$.

(5.3) Now suppose that the symmetry axis of anisotropy aligns with the z-axis and that the fault is in its principal coordinate system. As shown previously, **Q** and **F** are both diagonal matrices in this coordinate system. Furthermore, **Q** has eigenvalues $\Lambda^{(i)} = Q_{ii}$. The trace of the moment tensor is then:

$$\mathrm{tr}(\mathbf{m}) = \mathrm{tr}(\mathbf{QF}) = \mathrm{tr}\left(\begin{bmatrix} \Lambda^{(1)} & 0 & 0 \\ 0 & \Lambda^{(1)} & 0 \\ 0 & 0 & \Lambda^{(3)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) =$$

$$= \mathrm{tr}\left(\begin{bmatrix} \Lambda^{(1)} & 0 & 0 \\ 0 & -\Lambda^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$$

Furthermore, the trace remains zero as we rotate the fault about the z-axis:

$$\mathbf{F}' = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ -s & -c & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} (c^2 - s^s) & -2cs & 0 \\ -2cs & -(c^2 - s^s) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{tr}(\mathbf{m}) = \mathrm{tr}(\mathbf{QF}') = \begin{bmatrix} \Lambda^{(1)}(c^2 - s^s) & -2\Lambda^{(1)}cs & 0 \\ -2\Lambda^{(1)}cs & -\Lambda^{(1)}(c^2 - s^s) & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(6) Eigenvalues of the moment tensor in a transverse isotropic medium aligned with the N-direction of the fault.

(6.1) When coordinate system is aligned with the principal directions of **F**, so that:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the axis of transverse isotropy is aligned with the z-axis, the diagonal elements of the moment tensor can be easily calculated:

$$\frac{m_{33}}{½\bar{u}\Omega} = C_{3311}F_{11} + C_{3322}F_{22} = C_{3311} - C_{3322}$$

Since the Voigt tensor is invariant with respect to interchanging the x and y axes, $C_{3311} = C_{3322}$ and $m_{33} = 0$. The other diagonal elements are:

$$\frac{m_{11}}{½\bar{u}\Omega} = C_{1111}F_{11} + C_{1122}F_{22} = (C_{1111} - C_{1122})$$

$$\frac{m_{22}}{½\bar{u}\Omega} = C_{2211}F_{11} + C_{2222}F_{22} = C_{2211} - C_{2222} = -(C_{1111} - C_{1122}) = -\frac{m_{11}}{½\bar{u}\Omega}$$

Here we have used the rotational symmetry $C_{2222} = C_{1111}$ and the symmetry $C_{1122} = C_{2211}$. As expected, $m_{22} = -m_{11}$. The off-diagonal components are all zero:

$$\frac{m_{12}}{½\bar{u}\Omega} = C_{1211}F_{11} + C_{1222}F_{22} = 0$$

$$\frac{m_{13}}{½\bar{u}\Omega} = C_{1311}F_{11} + C_{1322}F_{22} = 0$$

$$\frac{m_{23}}{½\bar{u}\Omega} = C_{2311}F_{11} + C_{2322}F_{22} = 0$$

since in a traverse isotropic material, $C_{1211} = C_{1222} = C_{1311} = C_{1322} = C_{2311} = C_{2322} = 0$. Thus **m** has one identically-zero eigenvalue and no CLVD component. Rotating **u** and **v** about the z-axis does not change the eigenvalues, because due to the rotational symmetry of the Voigt tensor, it corresponds to a coordinate rotation, and the eigenvalues of a matrix are invariant under coordinate rotations.

(6.2) In detail, the above argument is as follows: After a coordinate rotation about the z-axis, the moment tensor becomes

$$m'_{ij} = ½\bar{u}\Omega\,\left[R_{ip}R_{jp}R_{kr}R_{ls}C_{pqrs}\right][R_{km}R_{ln}F_{mn}]$$

The eigenvalue of $\mathbf{m}'$ are invariant under this rotation. Since the Voigt tensor is invariant under the rotation, we can rewrite the equation as:

$$½sA\,\left[C_{ijkl}\right][R_{km}R_{ln}F_{mn}]$$

This equation can be interpreted as counter-rotating the fault, as contrasted to rotating the coordinate system.

(7.0) Here we show that the moment tensor has no explosive or CLVD component when the axis of symmetry is in (A) the fault plane; or (B) in the auxiliary plane.

(7.1) Starting with the **F** matrix is in its principal coordinate system, we rotate it about the z-axis by $\pm 45°$, so that either the fault plane or the auxiliary plane aligns with the x-axis:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm 1 & 0 \\ \mp 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mp 1 & 0 \\ \pm 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \pm\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and tilt it by an angle θ in the (x, z) plane. Defining $c \equiv \cos(\theta)$ and $s \equiv \sin(\theta)$, we have:

$$\pm\begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} = \mp\begin{bmatrix} 0 & c & 0 \\ c & 0 & s \\ 0 & s & 0 \end{bmatrix}$$

The matrix **QF** is then:

$$\mathbf{QF} = \mp\begin{bmatrix} \Lambda^{(1)} & 0 & 0 \\ 0 & \Lambda^{(1)} & 0 \\ 0 & 0 & \Lambda^{(3)} \end{bmatrix}\begin{bmatrix} 0 & c & 0 \\ c & 0 & s \\ 0 & s & 0 \end{bmatrix} = \mp\begin{bmatrix} 0 & \Lambda^{(1)}c & 0 \\ \Lambda^{(1)}c & 0 & \Lambda^{(1)}s \\ 0 & \Lambda^{(3)}s & 0 \end{bmatrix}$$

Note that $\mathrm{tr}(\mathbf{QF}) = 0$. The moment tensor has no explosive component.

(7.2) The elements of the moment tensor are:

$$\frac{m_{11}}{½sA} = C_{1112}F_{12} + C_{1121}F_{21} + C_{1123}F_{23} + C_{1132}F_{32} = 0$$

$$\frac{m_{22}}{½sA} = C_{2212}F_{12} + C_{2221}F_{21} + C_{2223}F_{23} + C_{2232}F_{32} = 0$$

$$\frac{m_{33}}{½sA} = C_{3312}F_{12} + C_{3321}F_{21} + C_{3323}F_{23} + C_{3332}F_{32} = 0$$

$$\frac{m_{12}}{½sA} = C_{1212}F_{12} + C_{1221}F_{21} + C_{1223}F_{23} + C_{1232}F_{32} =$$

$$= C_{1212}F_{12} + C_{1221}F_{21} = \mp 2Nc$$

$$\frac{m_{13}}{½sA} = C_{1312}F_{12} + C_{1321}F_{21} + C_{1323}F_{23} + C_{1332}F_{32} = 0$$

$$\frac{m_{23}}{½sA} = C_{2312}F_{12} + C_{2321}F_{21} + C_{2323}F_{23} + C_{2332}F_{32} =$$

$$= C_{2323}F_{23} + C_{2332}F_{32} = \mp 2sL$$

$$\frac{\mathbf{m}}{½sA} = \mp 2\begin{bmatrix} 0 & cN & 0 \\ cN & 0 & sL \\ 0 & sL & 0 \end{bmatrix}$$

Note that the matrix on the r.h.s. of this equation has the form:

$$\begin{bmatrix} 0 & A & 0 \\ A & 0 & B \\ 0 & B & 0 \end{bmatrix}$$

where $A \equiv -cN$ and $B \equiv sL$. The characteristic equation for its eigenvalues is:

$$\det \begin{bmatrix} -\lambda & A & 0 \\ A & -\lambda & B \\ 0 & B & -\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + A^2\lambda + B^2\lambda = -\lambda[\lambda^2 - (A^2 + B^2)] = 0$$

The characteristic equation has the solutions $\lambda^{(1)} = (A^2 + B^2)^{½}$, $\lambda^{(2)} = -(A^2 + B^2)^{½}$, and $\lambda^{(3)} = 0$. The moment tensor has one identically-zero eigenvalue and, consequently, no CLVD component.

8.0 Numerical Test: We test these ideas using a transverse isotropic tensor built from Love parameters:

$$A = \rho v_{pH}^2 \;\text{ and }\; C = \rho v_{pV}^2 \;\text{ and }\; N = \rho v_{sH}^2 \;\text{ and }\; L = \rho v_{sV}^2 \;\text{ and }\; F = \eta(A - 2L)$$

where the parameters are given by Dziewonski and Anderson's (1981) PREM model, evaluated at 100 km depth: $v_{pV} = 7.86732$ km/s,$v_{pH} = 8.06410$ km/s, $v_{sV} = 4.32041$ km/s, $v_{sH} = 4.44818$ km/s, $\eta = 0.92987$ and $\rho = 3372.54$ kg/m^3. The reference tensor has a symmetry axis that is parallel to the z-axis.

The reference fault is a vertical strike-slip fault, with the u-direction parallel to the x-axis, the N-direction parallel to the z-axis, and $\bar{u}\Omega = 1$.

The reference fault is held fixed during the simulation, but the reference Voigt tensor is tilted from the z-axis by an angle θ and then rotated around the z-axis by an angle ψ. The rotated tensor, evaluated for many (θ, ψ), is used to calculate the moment tensor **m**.

We now need to develop proxies that quantify the amount of explosive and CLVD components in a given moment tensor. The overall size of the moment tensor is given by $\|\mathbf{m}\|_2 = \left(m_{ij}m_{ij}\right)^{½} = (\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\lambda})^{½}$, where $\boldsymbol{\lambda} = \left[\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\right]^{\mathrm{T}}$ are its eigenvalues. The normalized amplitude X of the explosive component is then defined as:

$$X \equiv \frac{⅓\,\mathrm{tr}(\mathbf{m})}{\|\mathbf{m}\|_2}$$

The deviatoric part of the moment tensor:

$$\Delta\mathbf{m} = \mathbf{m} - ⅓\,\mathrm{tr}(\mathbf{m})\mathbf{I} \quad \text{with eigenvalues} \quad \Delta\lambda^{(i)} = \lambda^{(i)} - ⅓\,\mathrm{tr}(\mathbf{m})$$

The deviatoric part has zero trace and, therefore, no explosive component.

A CLVD pattern of eigenvalues is of the form $\lambda^{(p)} = \lambda^{(q)} = -½\lambda^{(r)}$, where (p, q, r) are unique indices (i.e. (p, q, r) is a permutation of $(1,2,3)$). The smallest CVLD pattern that can be subtracted from $\Delta\mathbf{m}$ to produce one identically-zero eigenvalue is the pattern $\lambda^{(p)} = \lambda^{(q)} = -½\lambda^{(r)} = \Delta\lambda^{(min)}$, were $\Delta\lambda^{(min)}$ is eigenvalue of $\Delta\mathbf{m}$ with the smallest absolute value. The normalized amplitude V of the CLVD component is, therefore, defined as:

$$V \equiv \frac{\min_i \left|\Delta\lambda^{(i)}\right|}{\|\mathbf{m}\|_2}$$

We calculate X, V and their ratio for the PREM model described above (Figure 1) for a large suite of randomly chosen rotations and tilts. As expected, both $X$and V are zero along the lines of constant tilt, $\theta = 0$, $\theta = \pi$ and $\theta = 2\pi$, and along the lines of constant rotation, $\psi = 0$, $\psi = \pi/2$, $\psi = 2\pi/2$, $\psi = 3\pi/4$ and $\psi = 4\pi/2$. The calculation verifies that these are the only loci of points at which either the explosive or CLVD components are zero. In this example, the ratio V/X seems to be a function of θ, only, but in general it can be a function of both (θ, ψ) (Figure 2).

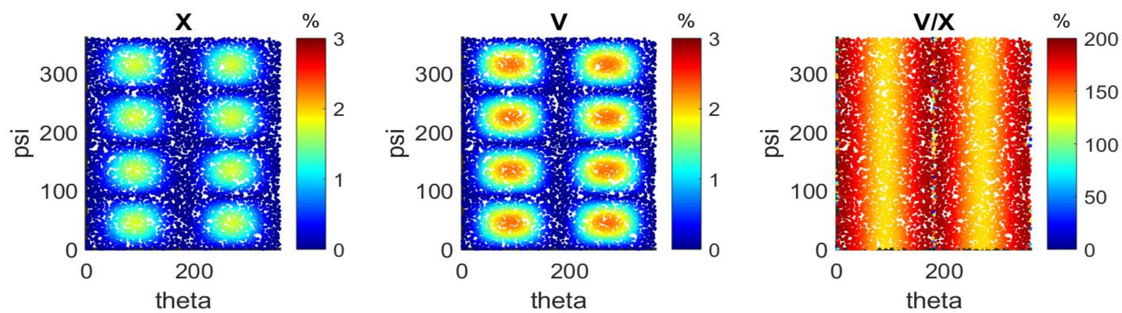

Figure 1. The explosion proxy X, CLVD proxy V, and the ratio V/X as a function of tilt θ and rotation ψ, using the PREM anisotropy.

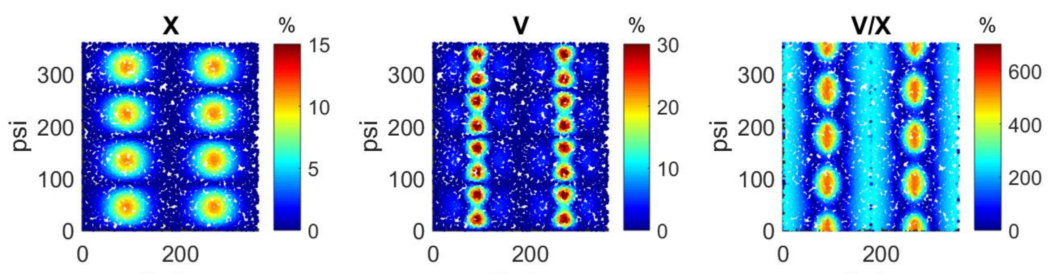

Figure 2. The explosion proxy X, CLVD proxy V, and the ratio V/X as a function of tilt θ and rotation ψ, for a Voigt tensor in which anisotropy is a factor of ten greater than PREM's.

9. I have checked most of the equations in this note numerically.

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Dziewonski, A. and D. Anderson, 1981. Preliminary reference Earth Model, Phys. Earth Planet. Int. 25, 297-256, 1981.

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Love, A.E.H., 1927. A Treatise on the Mathematical Theory of Elasticity, 4 ed., Cambridge University Press, Cambridge (UK), 643 pp.

Matlab script that implements the plot.

```
clear all;


EXPLscale = 0.03;
CLVDscale = 0.03;
RATIOscale = 2;

NITER = 10000;

% PREM ar 100 km
VpV_PREM = 7.86732;
VpH_PREM = 8.06410;
Vp_PREM = (VpH_PREM+VpV_PREM)/2;
Dvp_PREM = (VpH_PREM-VpV_PREM)/2;
VsVH_PREM = 4.32041;
VsHH_PREM = 4.44818;
Vs_PREM = (VsHH_PREM+VsVH_PREM)/2;
Dvs_PREM = (VsHH_PREM-VsVH_PREM)/2;
eta_PREM = 0.92987;

% Model is PREM-like, but with amplified anisotropy
aniP = 1.00;
aniS = 1.00;
VpV = Vp_PREM - aniP*Dvp_PREM;
VpH = Vp_PREM + aniP*Dvp_PREM;
VsVH = Vs_PREM - aniS*Dvs_PREM;
VsHH = Vs_PREM + aniS*Dvs_PREM;
eta = eta_PREM;
rho = 3.27254;
rho=1;  % override density, since absolute moment not of interest.

v = [1,0,0]';
u = [0,1,0]';
w = [0,0,1]';

c1 = aniso(VpH,VpV,VsHH,VsVH,eta,rho);
C1 = c2C(c1);
checksym(C1);

figure(1);
clf;
cm = colormap('jet');
Ncm = length(cm);

subplot(1,3,1);
set(gca,'LineWidth',2);
set(gca,'FontSize',14);
hold on;
axis( [0, 360, 0, 360] );
xlabel('theta');
ylabel('psi');
title('X');
caxis( [0, 100*EXPLscale] );
cb = colorbar();
set(get(cb,'title'),'string','%');

subplot(1,3,2);
set(gca,'LineWidth',2);
```

```
set(gca,'FontSize',14);
hold on;
axis( [0, 360, 0, 360] );
xlabel('theta');
ylabel('psi');
title('V');
caxis( [0, 100*CLVDscale] );
colorbar();
cb = colorbar();
set(get(cb,'title'),'string','%');

subplot(1,3,3);
set(gca,'LineWidth',2);
set(gca,'FontSize',14);
hold on;
axis( [0, 360, 0, 360] );
xlabel('theta');
ylabel('psi');
title('V/X');
caxis( [0, 100*RATIOscale] );
cb = colorbar();
set(get(cb,'title'),'string','%');

for itt = [1:NITER]

% note: not clear to me whether I am uniformly samplig the sphere
phi = 0;
theta = random('Uniform',0,2*pi,1,1);
psi = random('Uniform',0,2*pi,1,1);
R = eulermatrix( phi, theta, psi );
CR = rotateC(C1,R);
checksym(CR);

mt1 = makemt(CR,u,v);
d1 = eig(mt1);
AMP = sqrt(d1'*d1);
tr = sum(d1);
EXPL = abs(tr/3)/AMP;

d1d = d1 - tr/3;
d1dmin = min(abs(d1d));
CLVD = d1dmin/AMP;

RATIO = abs(CLVD)/(abs(EXPL)+1e-6);

k = 1+floor( (Ncm-1)*EXPL/EXPLscale );
if( k<1 )
    k=1;
elseif (k>Ncm )
    k=Ncm;
end
subplot(1,3,1);
plot( (180*theta/pi), (180*psi/pi), '.', 'Color', cm(k,:), 'LineWidth', 3 );

k = 1+floor( (Ncm-1)*CLVD/CLVDscale );
if( k<1 )
    k=1;
elseif (k>Ncm )
    k=Ncm;
end
subplot(1,3,2);
plot( (180*theta/pi), (180*psi/pi), '.', 'Color', cm(k,:), 'LineWidth', 3 );
```

```
k = 1+floor( (Ncm-1)*RATIO/RATIOscale );
if( k<1 )
    k=1;
elseif (k>Ncm )
    k=Ncm;
end
subplot(1,3,3);
plot( (180*theta/pi), (180*psi/pi), '.', 'Color', cm(k,:), 'LineWidth', 3 );

end


function c = aniso(VpH,VpV,VsHH,VsVH,eta,rho)
VpH2 = VpH^2;
VpV2 = VpV^2;
VsHH2 = VsHH^2;
VsVH2 = VsVH^2;
c = zeros(6,6);
c(1,1) = VpH2;
c(2,2) = VpH2;
c(3,3) = VpV2;
c(4,4) = VsVH2;
c(5,5) = VsVH2;
c(6,6) = VsHH2;
c(1,2) = VpH2-2*VsHH2;
c(2,1) = c(1,2);
c(1,3) = eta*(VpH2-2*VsVH2);
c(3,1) = c(1,3);
c(2,3) = eta*(VpH2-2*VsVH2);
c(3,2) = c(2,3);
c = rho*c;
end



function [C] = c2C(c)
% 6x6 to 3x3x3x3 Hook's Law tensor conversion
% from Fuchs, K, Phys. Earth Planet. Int. 31, 93-118, 1983
C = zeros(3,3,3,3);
    for m=[0:2]
    for n=[0:2]
    for p=[0:2]
    for q=[0:2]

    if( m==n )
        i=(m+n+2)/2;
    else
        i=9-m-n-2;
    end

    if( p==q )
        j=(p+q+2)/2;
    else
        j=9-p-q-2;
    end
    % C-language to MATLAB index conversion here
    C(m+1,n+1,p+1,q+1) = c(i-1+1,j-1+1);
    end
    end
    end
    end

end
```

```
function status = checksym(C)
    % check c(ijkl)=c(ijlk) */
    for i=[1:3]
    for j=[1:3]
    for k=[1:3]
    for l=[1:3]
        a = 0.5*(C(i,j,k,l) + C(i,j,l,k));
        d = 0.5*(C(i,j,k,l) - C(i,j,l,k));
        if( abs(d/a)>1e-6 )
            fprintf('test1 failed C%d%d%d%d ~= C%d%d%d%d\n',i,j,k,l,i,j,l,k);
            fprintf('%f ~= %f\n',  C(i,j,k,l), C(i,j,l,k) );
            status=0;
            return;
        end
    end
    end
    end
    end

    % check c(ijkl)=c(jikl)
    for i=[1:3]
    for j=[1:3]
    for k=[1:3]
    for l=[1:3]
        a = 0.5*(C(i,j,k,l) + C(j,i,k,l));
        d = 0.5*(C(i,j,k,l) - C(j,i,k,l));
        if( abs(d/a) > 1e-6 )
            fprintf('test2 failed C%d%d%d%d ~= C%d%d%d%d\n',i,j,k,l,j,i,k,l);
            fprintf('%f ~= %f\n',  C(i,j,k,l), C(j,i,k,l) );
            status=0;
            return;
        end
    end
    end
    end
    end

    % check c(ijkl)=c(klij)
    for i=[1:3]
    for j=[1:3]
    for k=[1:3]
    for l=[1:3]
        a = 0.5*(C(i,j,k,l) + C(k,l,i,j));
        d = 0.5*(C(i,j,k,l) - C(k,l,i,j));
        if( abs(d/a) > 1e-6 )
            fprintf('test3 failed C%d%d%d%d ~= C%d%d%d%d\n',i,j,k,l,k,l,i,j);
            fprintf('%f ~= %f\n',  C(i,j,k,l), C(k,l,i,j) );
            status=0;
            return;
        end
    end
    end
    end
    end

    status=1;
end


function s = eulermatrix( phi, theta, psi )
```

```
% euler angles, phi, theta, psi in degrees
% see Corbin & Stehle (1960)
% rotation thru phi about 3-axis
% thru theta about new 1-axis;
% thru psi about new 3-axis
% note that these angles rotate the coordinate system, not;
% the object! So a phi of 30 deg rotates am object in that
% coordinate system by -30 deg

s = zeros(3,3);

s(1,1) = cos(psi)*cos(phi) - sin(psi)*cos(theta)*sin(phi);
s(1,2) = cos(psi)*sin(phi) + sin(psi)*cos(theta)*cos(phi);
s(1,3) = sin(psi)*sin(theta);

s(2,1) = (-sin(psi)*cos(phi)-cos(psi)*cos(theta)*sin(phi));
s(2,2) = (-sin(psi)*sin(phi)+cos(psi)*cos(theta)*cos(phi));
s(2,3) = cos(psi)*sin(theta);

s(3,1) = sin(theta)*sin(phi);
s(3,2) = (-sin(theta)*cos(phi));
s(3,3) = cos(theta);

end



function [Cout] = rotateC(Cin,R)
Cout = zeros(3,3,3,3);
for i=[1:3]
for j=[1:3]
for k=[1:3]
for l=[1:3]
    Cout(i,j,k,l)=0;
    for p=[1:3]
    for q=[1:3]
    for r=[1:3]
    for s=[1:3]
        Cout(i,j,k,l) = Cout(i,j,k,l) + R(i,p)*R(j,q)*R(k,r)*R(l,s)*Cin(p,q,r,s);
    end
    end
    end
    end
end
end
end
end
end


function mt = makemt(c,u,v)
% makesthe moment tensor
% see Aki and Richards (2009) eqn 3.19
% u slip discontinuity, v fault normal
mt=zeros(3,3);
for p=[1:3]
for q=[1:3]
    for i=[1:3]
    for j=[1:3]
        mt(p,q) = mt(p,q) + u(i)*v(j)*c(i,j,p,q);
    end
    end
end
```

```
    end
    end
```