Fréchet Derivative for Optical Recording

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The displacement $u(t, A)$ varies with time t and depends on a parameter A. The displacement is optically recorded on photographic film, producing an image $I(x, y)$. We seek to calculate the Fréchet derivative $\partial I/\partial A$ (presuming that $\partial u/\partial A$ is known).

Position on the photographic film is given by Cartesian coordinates (x, y). The displacement traces out a line on the photographic material parameterized by:

$$x_u(t) = a_t + b_t\, t$$

$$y_u(t) = a_u + b_u\, u(t, A)$$

Here a_t, b_t, a_u and b_u are constants.

The light beam that exposes the film has a finite cross-sectional area of varying brightness and is described by a stylus $S(x, y)$. It exposes the film according to the rule:

$$I(x, y) = \int S[x - x_u(t), y - y_u(t)]\ \mathrm{d}t$$

where $I(x, y)$ is the exposure level of the film. That is, a point on the film accumulates exposure as the stylus passes over it.

We employ discrete approximation to all quantities. The displacement is a time series $u_p = u(t_p, A)$ with $t_p = p\Delta t$. The film has discrete coordinates (x_i, y_j) with $x_i = i\Delta x$ and $y_j = j\Delta y$. The displacement u_p corresponds to sequence of points (x_q, y_r) on the film with:

$$q(p) = \text{floor}\left(\frac{a_t + b_t\, p\Delta t}{\Delta x}\right) + 1 \quad \text{and} \quad r(p) = \text{floor}\left(\frac{a_u + b_u\, u_p}{\Delta y}\right) + 1$$

The discrete versions of the image and the stylus are $I_{ij} = I(x_i, y_j)$ and $S_{ij} = S(x_i, y_j)$, respectively. The equation for the image is then:

$$I_{ij} = \sum_p \sum_q \sum_r S_{i-q(p), j-r(p)}$$

That is, the image is obtained by stacking a series of styli, each offset accord to (t, u).

Now suppose that the parameter A is perturbed slightly from A_0 to $A_0 + \delta A$. The displacement is perturbed from u_0 to $u_0 + \delta u$, with $\delta u = (\partial u/\partial A)\, \delta A$. The y coordinate is perturbed from $y_{u0} = a_u + b_u\, u_0$ to $y_{u0} + \delta y_u$ with $\delta y_u = b_u \delta u$. For fixed time and for fixed position on the film, the stylus intensity changes by:

$$\delta S = \frac{\partial S}{\partial A}\delta A = \left(-\frac{\partial S}{\partial y}\right)\frac{\partial y}{\partial u}\frac{\partial u}{\partial A}\delta A = \left(-\frac{\partial S}{\partial y}\right) b_u \frac{\partial u}{\partial A}\delta A$$

The minus sign is introduced because, when the stylus is deflected in the positive y direction, the local beam intensity at a fixed point (x, y) *decreases* when $\partial S/\partial y$ is positive. The perturbation in the image is:

$$\delta I_{ij} = \frac{\partial I_{ij}}{\partial A}\delta A = \left(-b_u \sum_p \sum_q \sum_r \left[\frac{\partial S}{\partial y}\right]_{i-q(p),j-r(p)} \left[\frac{\partial u}{\partial A}\right]_p\right)\delta A$$

Hence the term in parenthesis is the Fréchet derivative $\partial I_{ij}/\partial A$. In the following example, we employ a stylus with a Gaussian cross-section.

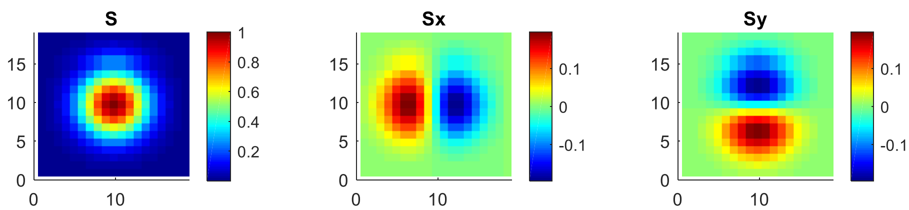

Figure 1. Stylus and its spatial gradients.

The displacement has the form $u(t) = [A \sin \omega t + B \cos \omega t] \exp(-\omega t/5)$ with $\omega = 0.02$, $A_0 = 0.9$, $B_0 = 0.1$, and $\delta A = 0.001$. The image is:

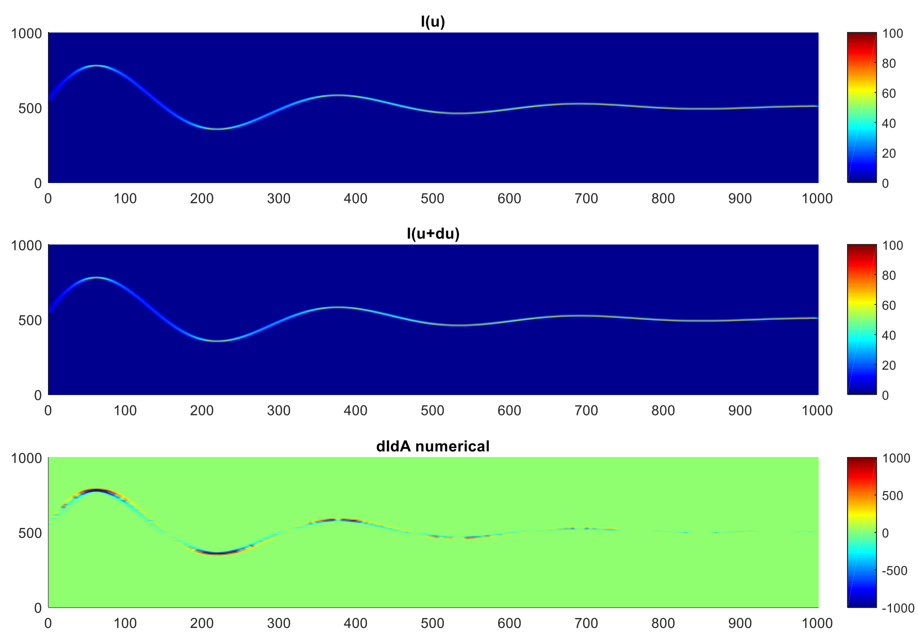

Figure 2. The images (A) $I(A)$ and (B) $I(A + \delta A)$ and (C) a numerical version of the Fréchet derivative, $\partial I/\partial A \approx [I(A + \delta A) - I(A)]/\delta A$.

The numerical derivative, shown in figure 2, compares well with the theoretical formula developed above.

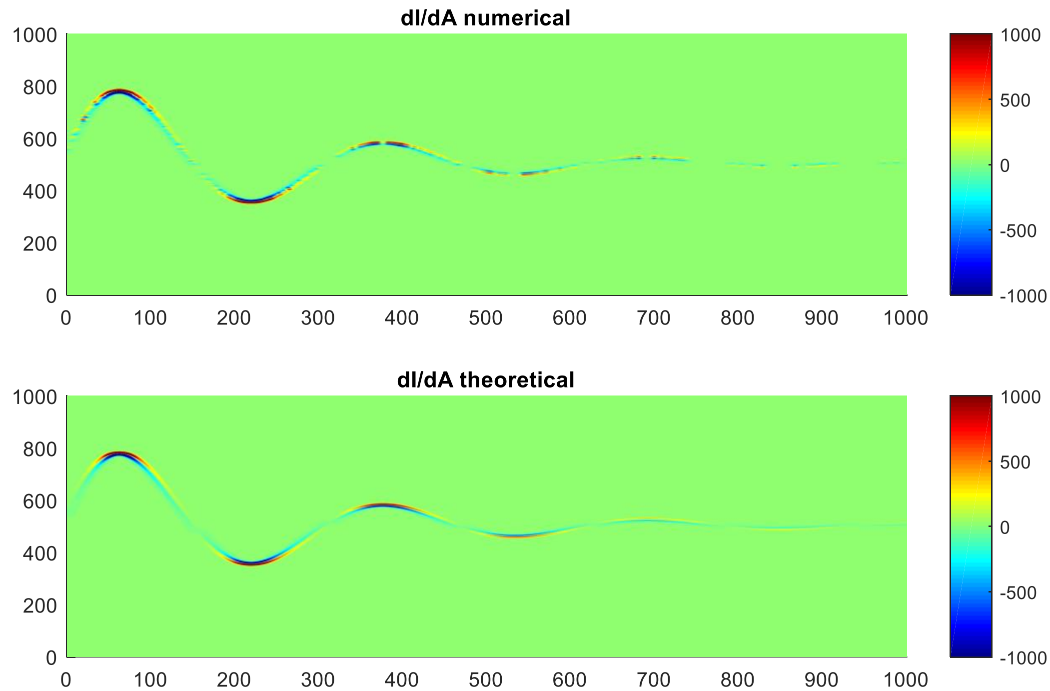

Figure 3. The (A) numerical and (B) theoretical versions of the Fréchet derivative, $\partial I/\partial A$.

Finally, we preform the first iteration of a non-linear least-squares inversion to estimate δA, starting with the guess $\delta A \approx 0.0$ and obtaining $\delta A^{est} \approx 0.00127$ (the exact value is $\delta A = 0.001$).

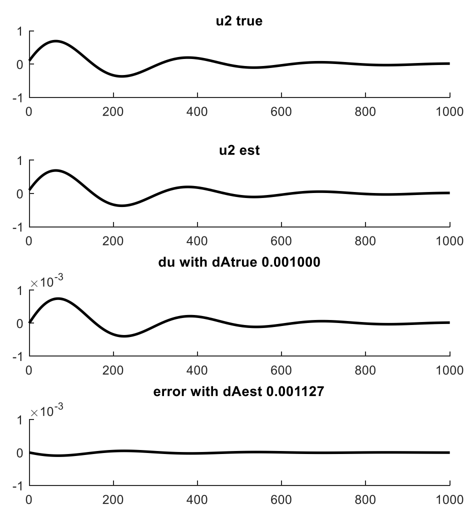

Figure 4. First iteration of the non-linear least-squares inversion f a non-linear least-squares inversion. (A) u^{true}, (B) u^{est}, (C)$\delta u = u^{true} - u(0)$ and (D) the error $e = u^{true} - u(\delta A^{est})$.