Heat Flow in the Lithosphere

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After discussions with Dallas Abbott

We consider a two-layer problem in depth z. The top layer has $0 \leq z \leq L$, top boundary condition with temperature $T(0) = 0$ and has radioactive heat production $H = A/L$. Note that H scales inversely with L, so that the total heat production A is independent of layer thickness. This models a process that moves heat producing radioactive elements up and down, without changing the total amount of them. The bottom layer has $L < z \leq Z$, bottom boundary condition with temperature $T(Z) = T_0$ and no heat production. The thermal conductivity k is presumed constant.

Derivation

In the top layer, the steady-state heat flow equation with thermal conductivity k is:

$$\frac{d^2}{dx^2}T + \frac{H}{k} = \frac{d^2}{dx^2}T + \frac{A}{Lk} = 0$$

Integrating once yields:

$$\frac{d}{dx}T = -\frac{A}{Lk}z + a$$

Where a is an integration constant. Integrating a second time yields:

$$T^{top} = -\frac{A}{2Lk}z^2 + az + b$$

Where b is an integration constant. The top boundary condition implies $b = 0$. The heat flow is

$$q^{top} = -k\frac{d}{dx}T = \frac{A}{L}z - ak$$

And the surface heat flow is

$$q_0 = -ak$$

In the bottom layer, the steady-state heat flow equation with thermal conductivity k is:

$$\frac{d^2}{dx^2}T = 0$$

Hence T varies linearly with z. The solution that satisfies $T(Z) = T_0$ is:

$$T^{bot} = -c(Z - z) + T_0$$

Where the constant c is yet to be determined. The corresponding heat flow is

$$q^{bot} = -k\frac{d}{dx}T = -ck$$

The continuity condition $q^{top}(L) = q^{bot}(L)$ implies:

$$\frac{A}{L}L - ak = -ck \quad \text{or} \quad c = a - \frac{A}{k}$$

The continuity condition $T^{top}(L) = T^{bot}(L)$ implies:

$$-\frac{A}{2Lk}L^2 + aL = -c(Z - L) + T_0$$

$$\frac{A}{2k}L + aL = -\left(a - \frac{A}{k}\right)(Z - L) + T_0$$

$$aL = -\left(a - \frac{A}{k}\right)(Z - L) + \frac{A}{2k}L + T_0$$

$$aL = -aZ + aL + \frac{A}{k}\ Z - \frac{A}{k}L + \frac{A}{2k}L + T_0$$

$$aZ = -aL + aL + \frac{A}{k}\ Z - \frac{A}{k}L + \frac{A}{2k}L + T_0$$

$$aZ = \frac{A}{2k}\ 2Z - \ 2\frac{A}{k}2L + \frac{A}{2k}L + T_0$$

$$a = \frac{A}{2k}\left(2 - \frac{L}{Z}\right) + \frac{T_0}{Z}$$

$$c = a - \frac{A}{k} = \frac{A}{2k}\left(2 - \frac{L}{Z}\right) - \ 2\frac{A}{2k} + \frac{T_0}{Z} = -\frac{A}{2k}\ \frac{L}{Z} + \frac{T_0}{Z}$$

The solution

$$T^{top} = -\frac{A}{2Lk}z^2 + \left\{\frac{A}{2k}\left(2 - \frac{L}{Z}\right) + \frac{T_0}{Z}\right\}z$$

$$T^{bot} = \left\{\frac{A}{2k}\ \frac{L}{Z} - T_0\right\}(Z - z) + T_0$$

$$q^{top} = \frac{A}{L}z - k\left\{\frac{A}{2k}\left(2 - \frac{L}{Z}\right) + \frac{T_0}{Z}\right\}$$

$$q^{bot} = -k\left\{-\frac{A}{2k}\ \frac{L}{Z} + \frac{T_0}{Z}\right\}$$

The negative of the surface heat flow is:

$$-q_0 = \frac{A}{2}\left(2 - \frac{L}{Z}\right) + k\frac{T_0}{Z}$$

Verifying the solution

Check top b.c.:

$$T^{top}(0) = -\frac{A}{2Lk}0^2 + \left\{\frac{A}{2k}\left(2 - \frac{L}{Z}\right) + \frac{T_0}{Z}\right\}0 = 0$$

Check heat flow continuity condition:

$$q^{top} = q^{bot}$$

$$\frac{A}{2}2 - \frac{A}{2}\left(2 - \frac{L}{Z}\right) - T_0 = \frac{A}{2}\frac{L}{Z} - T_0$$

$$\frac{A}{2}\frac{L}{Z} - \frac{T_0}{Z} = \frac{A}{2}\frac{L}{Z} - \frac{T_0}{Z}$$

$$0 = 0$$

Check bottom b.c.:

$$T^{bot}(Z) = \left\{\frac{A}{2k}\frac{L}{Z} - \frac{T_0}{Z}\right\}0 + T_0 = T_0$$

Check temperature continuity condition:

$$T^{top}(L) = T^{bot}(L)$$

$$-\frac{A}{2Lk}L^2 + \left\{\frac{A}{2k}\left(2 - \frac{L}{Z}\right) + \frac{T_0}{Z}\right\}L = \left\{\frac{A}{2k}\frac{L}{Z} - \frac{T_0}{Z}\right\}(Z - L) + T_0$$

$$-\frac{A}{2k}L - \frac{A}{2k}\frac{L^2}{Z} + \frac{A}{2k}2L + T_0\frac{L}{Z} = \frac{A}{2k}L - \frac{A}{2k}\frac{L^2}{Z} + T_0\frac{L}{Z} - T_0 + T_0$$

$$\frac{A}{2k}L - \frac{A}{2k}\frac{L^2}{Z} + T_0\frac{L}{Z} = \frac{A}{2k}L - \frac{A}{2k}\frac{L^2}{Z} + T_0\frac{L}{Z}$$

$$0 = 0$$

Thus, all four conditions are satisfied. (I have checked all these formulas numerically).

Interpretation of the surface heat flow:

$$-q_0 = \frac{A}{2}\left(2 - \frac{L}{Z}\right) + k\frac{T_0}{Z}$$

The second term kT_0/Z is the *basal heat flow*, that is, the heat flow associated with overall temperature difference between the bottom and top of the layer. It does not depend upon the thickness L of the heat-producing layer.

The first term ½A $(2 - L/Z)$ is the *radioactive heat flow*, that is, the heat flow associated with overall heat producing layer. It does depend upon the thickness L, and varies from ½A when $L = Z$; that is, the heat producing layer is at its maximum thickness, to A when $L = 0$; that is, the heat producing layer is very thin. The total heal production in a column of height L is A, so at steady state the heat flowing from it is also A. Thus, one interpretation of the term is that all the heat goes up then when $L = 0$, but only half of it goes up when $L = 0$.

Application to the lithosphere:

Let's assume that the lithosphere is $Z = 100$ km thick, with a basal temperature of $T_0 = 1350 + 0.5Z$ and a thermal conductivity such that $kT_0/Z = 38$ mW/m^2 (the heat flow associated with 200 Ma oceanic lithosphere). That is, in the absence of heat production, $A = 0$ and the surface heat flow is 38 mW/m^2. We assume that the crust is $100/3 \approx 33$ km thick and that top L km of it is heat producing, so that $0 \leq L \leq 100/3$ km. The first term in the heat flow equation therefore varies between A and $5A/6$, which is a relatively small range of variation.

If we assume that the heat producing part of the crust has the same total heat production as a $100/3$ km thick layer of basalt with $H = 1$ μW/m^3, then $A = 100/3 \approx 33$ mW/m^2 and the crustal heat flow is in the range:

$$65.7 \leq -q_0 \leq 71.3 \text{ mW/m}^2 \quad \text{for} \quad 33 \geq L \geq 0 \text{ km}$$

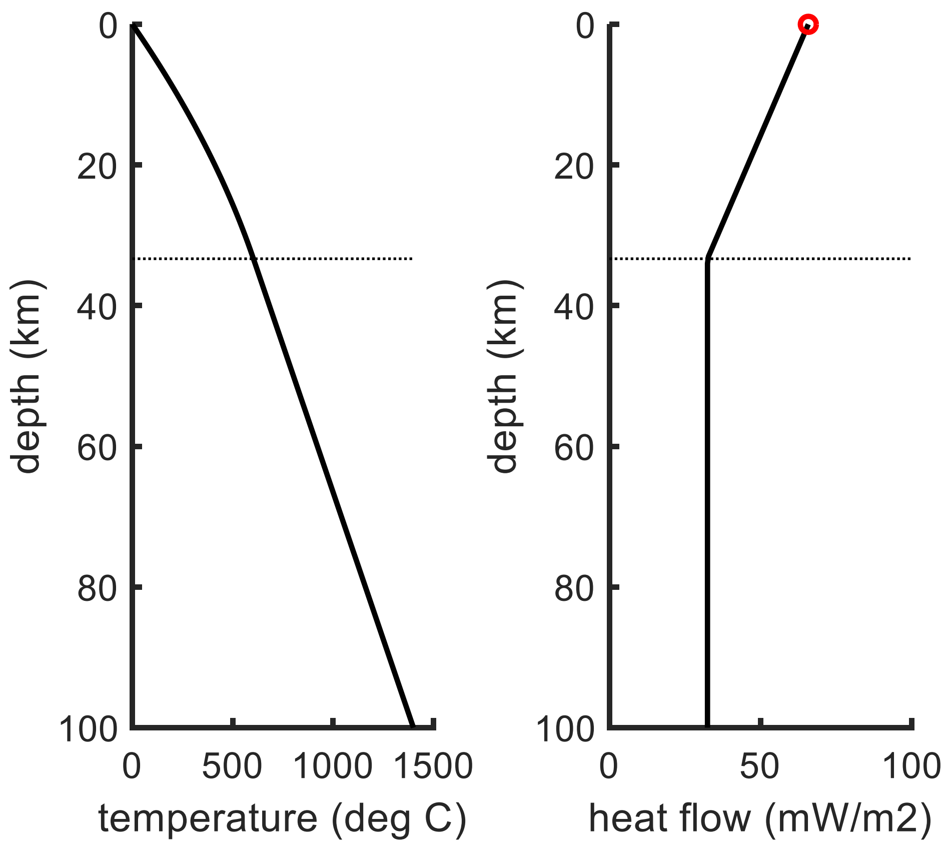

Figure 1. Example in text with $L = 100/3$ km. The surface heat flow is 65.7 mW/m^2 (red circle).

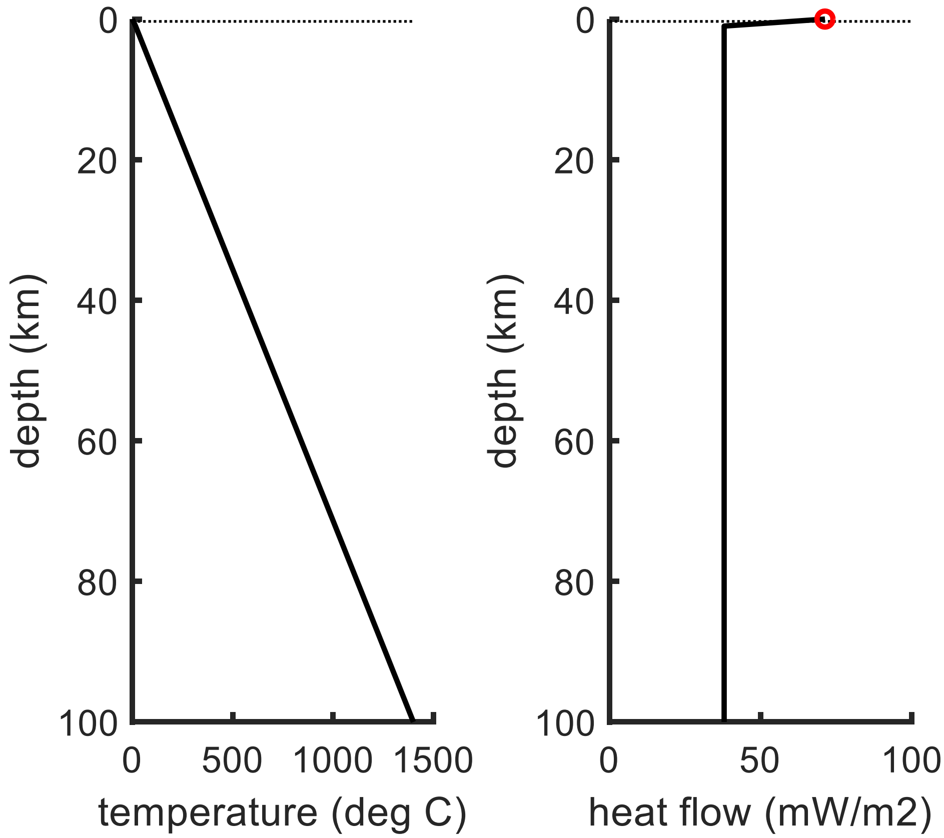

Figure 2 Example in text with $L = 1/3$ km. The surface heat flow is 71.3 mW/m^2 (red circle).