Consider a ‘toy’ non-linear inverse problem with data $\mathbf{d}^{\text{obs}}$ and the model parameters $\mathbf{m}$, of the form:

\[
\begin{align*}
  d_1 &= G_{11} m_1 + G_{12} m_2 + Q_{111} m_1^2 + (Q_{112} + Q_{121}) m_1 m_2 + Q_{122} m_2^2 \\
  d_2 &= G_{21} m_1 + G_{22} m_2 + Q_{211} m_1^2 + (Q_{212} + Q_{221}) m_1 m_2 + Q_{222} m_2^2
\end{align*}
\]

(1)

with

\[
G = \begin{bmatrix}
  1 & -1 \\
  1 & 1
\end{bmatrix} \quad \text{and} \quad Q_{ijk} = -\delta_{i1}\delta_{j1}\delta_{k1}
\]

(2)

where $\delta_{ij}$ is the Kronecker Delta. I set $\mathbf{m}^{\text{true}} = [0.452, 0.690]^{\text{T}}$ and $\mathbf{d}^{\text{obs}} = \mathbf{d}^{\text{true}}$. The true p.d.f. $p(\mathbf{m} | \mathbf{d}^{\text{obs}})$ is taken to be Normal in the error $\mathbf{e} = \mathbf{d}^{\text{obs}} - \mathbf{d}^{\text{pre}}(\mathbf{m})$ with covariance:

\[
\mathbf{C}_d = \begin{bmatrix}
  (0.02)^2 & 0 \\
  0 & (0.1)^2
\end{bmatrix}
\]

(3)

This p.d.f. is shown in Figure 1. Its maximum likelihood point is at $\mathbf{m}^{\text{true}}$. Its mean (calculated numerically) is not coincident with the maximum likelihood point, but rather is displaced to the right of it at $\mathbf{m}^{\text{mean}} = [0.485, 0.678]^{\text{T}}$, because the p.d.f. is skewed. The covariance (calculated numerically) is:

\[
\mathbf{C}_m = \begin{bmatrix}
  (0.1133)^2 & -0.0014 \\
  -0.0014 & (0.0311)^2
\end{bmatrix}
\]

(4)

The inverse problem is solved with interactive linearized Generalized Least Squares (e.g. Menke, 2018, Equation 9.27 with no prior information). The gradient:

\[
G_{ij}^{(n)} = \left. \frac{\partial d_i}{\partial m_j} \right|_{\mathbf{m}^{(n)}}
\]

(5)

(where $n$ indexes iteration number) is calculated by analytically differentiating Equation (1). The solution rapidly converges on $\mathbf{m}^{\text{true}}$ (Figure 1) in about $N = 20$ iterations. The solution $\mathbf{m}^{(N)}$ closely approximates the maximum likelihood point $\mathbf{m}^{\text{true}}$. The covariance is estimated as:
\[
c_m^{(N)} = [G^{(N)T}C_d^{-1}G^{(N)}]^{-1} = \begin{bmatrix}
(0.0924)^2 & 0.0004 \\
0.0004 & (0.0204)^2
\end{bmatrix}
\]

(6)

The values are somewhat smaller than the true covariance (but of the same order of magnitude). A Normal p.d.f. based on \( m^{(N)} \) and \( C_m^{(N)} \) approximates the true non-Normal p.d.f. reasonably well (Figure 2).

Reference


Figure 1. The non-Normal p.d.f. \( p(m|d^{\text{obs}}) \) associated with the inverse problem (colors). The maximum likelihood point (red) and the true mean and standard deviation (green) are shown. The trajectory of a gradient-based iterative inversion (white) converges on the maximum likelihood point in a few iterations. The estimated maximum likelihood point (blue) and estimated standard deviation (blue) of the inversion are shown; they match the true values reasonably well
Figure 2. (A) The non-Normal p.d.f. $p(m|d^{obs})$ associated with the inverse problem (colors). (B) The Normal approximation for $p(m|d^{obs})$ computed from the maximum likelihood point and covariance matrix associated with the gradient-based iterative inversion.