The Effect of Resampling on the Trace-Off of Resolution and Variance

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These ideas are based on a discussion that Dan Blatter and I had on the effect of resampling on the trade-off of resolution and variance, as computed by the method we recently developed. The discussion here imagines that a coarsely-sampled time series $\mathbf{m}$ is the fundamental one (e.g. computed via an MCMC method) but that a finely-sampled time series $\mathbf{y}$, computed from $\mathbf{m}$ by interpolation, is the one upon which the resolution/variance analysis is based. The question is how much bias is introduced by having done the resolution/variance analysis in $\mathbf{y}$ rather than $\mathbf{m}$. I analyze this question below and provide the answer "not much".

My identification of $\mathbf{m}$ as a coarsely-sampled time series representing some underlying continuous function $m(x)$ is a simplification. It is meant to stand-in for the more interesting – and more complicated - case, not treated here, in which $\mathbf{m}$ is the set of coefficients of a spline representation of $m(x)$.

1. Eigenvalues and Eigenvectors of a "Modulated Repeating Pattern" Matrix. The $N \times N$ matrix $\mathbf{B}$ is constructed by repeating a $P \times P$ block $\mathbf{W}$ on an $M \times M$ grid, so that that $N = PM$. Each instance of $\mathbf{W}$ is multiplied by a factor given by the elements of an $M \times M$ symmetric matrix $\mathbf{A}$, so that:

$$\mathrm{B}_{i,j} = \mathrm{A}_{(i/P),(j/P)} W_{(i\backslash P),(j\backslash P)} \tag{1}$$

Here matrix indices start at zero, "/" is integer division and "\" is remainder. We assume that $\mathbf{A}$ and $\mathbf{W}$ satisfy the algebraic eigenvalue equations, $\mathbf{A}\mathbf{a}^{(s)} = \alpha_k \mathbf{a}^{(s)}$ and $\mathbf{W}\mathbf{w}^{(s)} = \omega_k \mathbf{w}^{(s)}$, respectively. We can show that the quantities:

$$\beta_s = \alpha_{(s/P)} \omega_{(s\backslash P)} \quad \text{and} \quad b_k^{(s)} = a_{(k/P)}^{(s/P)} w_{(k\backslash P)}^{(s\backslash P)} \tag{2}$$

are the eigenvalues and eigenvector of $\mathbf{B}$ by considering the product $\mathbf{B}\mathbf{b}^{(k)}$:

$$\left[\mathbf{B}\mathbf{b}^{(s)}\right]_q = \sum_{k=0}^{N-1} B_{q,k} b_k^{(s)} = \sum_{k=0}^{N-1} \mathrm{A}_{(q/P),(k/P)} W_{(q\backslash P),(k\backslash P)} a_{(k/P)}^{(s/P)} w_{(k\backslash P)}^{(s\backslash P)} \tag{3}$$

With the substitution, $k = iP + j$, the single sum $k = 1 \cdots N$ can be replaced by two sums, $i = 1 \cdots M$ and $j = 1 \cdots P$. Then, noting that $k/P = i$ and $k\backslash P = j$, we have:

$$\left[\mathbf{B}\mathbf{b}^{(s)}\right]_q = \sum_{i=0}^{M-1} \sum_{j=0}^{P-1} \mathrm{A}_{(q/P),i} W_{(q\backslash P),j} a_i^{(s/P)} w_j^{(s\backslash P)} =$$

$$= \sum_{i=0}^{M-1} \mathrm{A}_{(q/P)i} a_i^{(s/P)} \sum_{j=0}^{P-1} W_{(q\backslash P)j} w_j^{(s\backslash P)} =$$

$$= \alpha_{(s/P)} \omega_{(s\backslash P)} \; a_{(q/p)}^{(s/P)} w_{(q\backslash p)}^{(s/P)} = \beta_s b_q^{(s)}$$

(4)

Thus, the choices above satisfy the algebraic eigenvalue equation $\mathbf{Bb}^{(s)} = \beta_s \mathbf{b}^{(s)}$.

2. **Eigenvalues and Eigenvectors of a "Block-Constant" Matrix.** Now consider the special case where **W** is a constant matrix with $W_{i,j} = 1$, so that **B** is block-constant. One eigenvector of **W** is $\mathbf{w}^{(1)} = [1,1,\cdots,1]^{\mathrm{T}}$ with eigenvalue $\omega_1 = P$. A choice for the other $P-1$ linearly-independent eigenvectors is $\mathbf{w}^{(2)} = [1,-1,0,\cdots,1]^{\mathrm{T}}$, $\mathbf{w}^{(3)} = [1,0,-1,0,\cdots,1]^{\mathrm{T}}$, etc., all with identically-zero eigenvalue. Equation (2) implies that the eigenvalues of **B** consist of the M eigenvalues of **A**, all multiplied by P, together with $M-P$ identically-zero eigenvalues. Equation (2) also implies that the corresponding eigenvectors of **B** consist of "interpolated" versions of the M eigenvectors of **A** (with constant interpolation), together with eigenvectors $b_j^{(s)}$ (with $M < s \leq N$) that oscillate rapidly with j.

3. **Interpolation of a Timeseries.** Suppose that the time series **m** has sampling interval Δt and length M. Suppose also that the time series **y** has sampling interval $\Delta t/P$ and length $N = PM$. Consider a $N \times M$ interpolation operator **T** that takes **m** into **y** (i.e. $\mathbf{y} = \mathbf{Tm}$) and that preserves the values of **m**. That is, $T_{iP,j} = \delta_{i,j}$ so that $y_{kP} = m_k$, for $0 \leq k < M$. The $P = 2$ case corresponding to linear interpolation is:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ ½ & 0 & ½ & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & ½ & 0 & ½ & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(5)

A symmetric $N \times N$ matrix **B** can be formed from a symmetric $M \times M$ matrix **A** via:

$$\mathbf{B} = \mathbf{TAT^T}$$

(6)

This matrix preserves the values of **A** in **B**:

$$B_{kP,kP} = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} T_{kP,i} T_{kP,j} A_{i,j} = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \delta_{k,i} \delta_{k,j} A_{i,j} = A_{i,j}$$

(7)

and interpolates the other values. Furthermore, the interpolation process can be view as acting on the eigenvectors of $\mathbf{A}$, since it follows from $\mathbf{A} = \sum_s \alpha_s \, \mathbf{a}^{(s)} \mathbf{a}^{(s)\mathrm{T}}$ that:

$$\mathbf{B} = \mathbf{TAT^T} = \sum_{s=0}^{M-1} \alpha_s \, \mathbf{Ta}^{(s)} \mathbf{a}^{(s)\mathrm{T}} \mathbf{T^T} = \sum_{s=0}^{M-1} \alpha_s \left(\mathbf{Ta}^{(s)}\right)\left(\mathbf{Ta}^{(s)}\right)^{\mathrm{T}}$$

Note, however that $\mathbf{Ta}^{(s)}$ is not, in general, an eigenvector of $\mathbf{B}$.

Now consider forming the covariance $\mathbf{C}_y$ of $\mathbf{y}$ from the covariance matrix $\mathbf{C}_m$ of $\mathbf{m}$ using the usual rules of error propagation:

$$\mathbf{C}_y = \mathbf{TC}_m\mathbf{T}^{\mathrm{T}} \tag{8}$$

Comparing Equations (6) and (8), we conclude that $\mathbf{C}_y$ is an interpolated version of $\mathbf{C}_m$. We expect that $\mathbf{C}_y$ has at least $N - M$ identically-zero eigenvalues, because the interpolation has created $N - M$ linear elements $\mathbf{y}$ that are completely dependent on the other M elements and that, consequently, have no error.

4. **Effect of Interpolation on Eigenvalues and Eigenvectors.** We now address the question of how the eigenvalues and eigenvectors of $\mathbf{C}_y$ differ from those of $\mathbf{C}_m$. Since we have established that $\mathbf{C}_y$ is an interpolated version of $\mathbf{C}_m$, we start by writing;

$$\mathbf{C}_y = \mathbf{C}_y^{(0)} + \delta\mathbf{C}_y \tag{9}$$

We choose $\mathbf{C}_y^{(0)}$ to be a block-constant matrix built from $\mathbf{C}_m$ (with block size P). We have previously established that this block-constant matrix shares the eigenvectors and (up to a multiplicative constant) eigenvalues of $\mathbf{C}_m$, and additionally has $N - M$ highly oscillatory eigenvectors with identically-zero eigenvalues. The perturbation $\delta\mathbf{C}_y$ does not necessarily have identically-zero mean, either overall or within individual $P \times P$ blocks. However, since $\mathbf{C}_y$ shares every Pth value with $\mathbf{C}_y^{(0)}$ (the others being determined by interpolation), we expect that these means are very much smaller than $\|\delta\mathbf{C}_y\|$.

The effect of the small perturbation $\delta\mathbf{C}_y$ on the non-zero eigenvalues can be determined using perturbation theory. The eigenvalues and eigenvectors of $\mathbf{C}_m$ are denoted as α_s and $\mathbf{a}^{(s)}$, respectively, and the eigenvalues and eigenvectors of $\mathbf{C}_y$ as $\beta_s = \beta_s^0 + \delta\beta_s$ and $\mathbf{b}^{(s)} = \mathbf{b}^{(0s)} + \delta\mathbf{b}^{(s)}$, respectively. The M non-zero eigenvalues are non-degenerate, so the first-order perturbations can be shown to be:

$$\delta\beta_s = \mathbf{b}^{(0s)\mathrm{T}} \delta\mathbf{C}_y \, \mathbf{b}^{(0s)} \quad \text{and} \quad \delta\mathbf{b}^{(s)} = \sum_{\substack{j=0 \\ j \neq s}}^{N-1} \frac{\left(\mathbf{b}^{(0s)\mathrm{T}} \delta\mathbf{C}_y \mathbf{b}^{(0j)}\right)}{\beta_s^0 - \beta_j^0} \mathbf{b}^{(0j)}$$

(10)

We consider these first-order perturbations to be "small". In fact, $\delta\beta_s$ is especially small: Because $\mathbf{b}^{(0s)}$ is block-constant, the $\delta\beta_s$ would be zero if each block of $\delta\mathbf{C}_y$ has zero mean:

$$\delta\beta_s = \mathbf{b}^{(0s)\mathrm{T}}\delta\mathbf{C}_y\,\mathbf{b}^{(0s)} = \sum_{p=0}^{N-1}\sum_{q=0}^{N-1} b_p^{(0s)}\left[\delta\mathbf{C}_y\right]_{p,q} b_q^{(0s)}$$

$$= \sum_{p=0}^{N-1}\sum_{q=0}^{N-1} a_{(p/P)}^{(s/P)} w_{(p\backslash P)}^{(s\backslash P)}\left[\delta\mathbf{C}_y\right]_{p,q} a_{(q/P)}^{(s/P)} w_{(q\backslash P)}^{(s\backslash P)}$$

$$= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i^{(s/P)} a_j^{(s/P)} \sum_{u=0}^{P-1}\sum_{v=0}^{P-1} w_u^{(s\backslash P)}\left[\delta\mathbf{C}_y\right]_{iP+u,jP+v} w_v^{(s\backslash P)}$$

$$= P^2 \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i^{(s/P)} a_j^{(s/P)} \left(\sum_{u=0}^{P-1}\sum_{v=0}^{P-1}\left[\delta\mathbf{C}_y\right]_{iP+u,jP+v}\right) = 0$$

$$\text{when}\quad \left(\sum_{u=0}^{P-1}\sum_{v=0}^{P-1}\left[\delta\mathbf{C}_y\right]_{iP+u,jP+v}\right) = 0 \quad \text{for all}\quad i,j$$

The term in parenthesis is the block-mean. Here we have used the fact that, for the non-zero eigenvalues, $w_v^{(s\backslash P)} = w_v^{(s\backslash P)} = P$. In actuality, the block-means are almost but not quite zero, so that $\delta\beta_s$ is non-zero but much smaller than $\left\|\mathbf{b}^{(0s)}\right\|^2\left\|\delta\mathbf{C}_y\right\|$;

The $N-M$ zero eigenvalues are degenerate. Their perturbations also be calculated using degenerate perturbation theory, but we omit discussion of it here.

5. Effect on Menke & Blatter Style Trade-Off Curves. The eigenvalue spectrum of $\mathbf{C}_y$ differs from that of $\mathbf{C}_m$ in three ways: an overall scaling factor of P is introduce that represents the decrease of the sampling interval from Δt to $\Delta t/P$; the M non-zero eigenvalues of $\mathbf{C}_y$ are slightly perturbed with respect to those of $\mathbf{C}_m$; and $N-M$ zero eigenvalues are added. Superficially, the introduction of the zero eigenvalues might seem to be a boon, since they imply that zero-variance features have been added to the problem. However, these features arise from the interpolation and are associated with highly-oscillatory eigenvectors. Thus, given $[y_1, y_2, y_3, y_4, y_5, \cdots]^T$ with the linear interpolation $y_2 = ½y_1 + ½y_3$ and $y_4 = ½y_3 + ½y_5$, the linear combinations $½y_1 - y_2 + ½y_3 = 0$ and $½y_3 - y_4 + ½y_5 = 0$ have identically-zero variance. However, they are also oscillatory and highly unlocalized and cannot be used to form a localized weighted average. For instance, although the sum of these two linear combinations, say $\langle y_3\rangle = ½y_1 - y_2 + y_3 - y_4 + ½y_5$ is centered about y_3, it is not usefully localized around y_3. Consequently, the $N-M$ zero eigenvalues merely add a long tail of small-variance, large-spread values to the trade-off curve. The part of the trade-off curve with small-spread is controlled by the M non-zero eigenvalues, and since these are only slightly perturbed with respect to those of $\mathbf{C}_m$, this part of the trade-off curve for $\mathbf{y}$ is very similar to that for $\mathbf{m}$ (up to an overall scaling).

The upshot is that a sound interpretation of variance and resolution can be made from $\mathbf{y}$.