Eigenvalue Notes

Bill Menke, October 31, 2019

(These note were done in preparation for working on a difficult tensor analysis problem in which I am interested but have so far found intractable).

Part 1. Relations between Levi-Civita ϵ_{ijk} and Kronecker Delta δ_{ij} for three dimensions.

(Source: Wikipedia articles "Levi-Civita Symbol" and "Dyadics").

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$$

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$$

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

(1.1 a,b,c,d)

Note also that:

$$\delta_{ij} = v_i^A v_j^A + v_i^B v_j^B + v_i^C v_j^C$$

(1.2)

for any three mutually perpendicular unit vectors v_i^A, v_i^B, v_i^C.

Part 2. Exploration of the properties of the fourth order tensor $d_{jkmn} = M_{il}\epsilon_{ijk}\epsilon_{lmn}$, where **M** is a symmetric second order tensor, $M_{ij} = M_{ji}$.

$$d_{jkmn} = M_{il}\epsilon_{ijk}\epsilon_{lmn}$$

(2.1)

$$d_{jkmn} = M_{il}\delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - M_{il}\delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + M_{il}\delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) =$$

$$= M_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - M_{lm}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + M_{ln}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) =$$

$$= M_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - (M_{lm}\delta_{jl}\delta_{kn} - M_{lm}\delta_{jn}\delta_{kl}) + (M_{ln}\delta_{jl}\delta_{km} - M_{ln}\delta_{jm}\delta_{kl}) =$$

$$= M_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - (M_{jm}\delta_{kn} - M_{km}\delta_{jn}) + (M_{jn}\delta_{km} - M_{kn}\delta_{jm}) \text{ so}$$

$$d_{jkmn} = M_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + M_{jn}\delta_{km} + M_{km}\delta_{jn} - M_{jm}\delta_{kn} - M_{kn}\delta_{jm}$$

(2.2)

The fourth order tensor can also be written:

$$d_{pqkl} = M_{\overline{pq},\overline{kl}}\epsilon_{\overline{pq},p,q}\epsilon_{\overline{kl},k,l}$$

$$\overline{pq} = \begin{cases} \text{excluded index if } p \neq q \\ 0 \text{ if } p = q \end{cases} \quad \text{and} \quad M_{ij} = 0 \quad \text{if } i = 0 \text{ or } j = 0$$

(2.3)

Properties:

$$d_{aamn} = 0 \quad \text{with no sum over } a$$

$$d_{aamn} = M_{il}\epsilon_{iaa}\epsilon_{lmn} = 0$$

$$d_{aamn} = M_{ii}(\delta_{am}\delta_{an} - \delta_{an}\delta_{am}) + M_{an}\delta_{am} + M_{am}\delta_{an} - M_{am}\delta_{an} - M_{an}\delta_{am} =$$

$$= M_{ii}(0) + M_{an}\delta_{am} - M_{an}\delta_{am} + M_{am}\delta_{an} - M_{am}\delta_{an} = 0$$

(2.3)

$$d_{pqaa} = 0 \quad \text{with no sum over } a$$

$$d_{jkaa} = M_{ij}\epsilon_{ijk}\epsilon_{jaa} = 0$$

$$d_{jkaa} = M_{ii}\left(\delta_{ja}\delta_{ka} - \delta_{ja}\delta_{ka}\right) + M_{ja}\delta_{ka} + M_{ka}\delta_{ja} - M_{ja}\delta_{ka} - M_{ka}\delta_{ja} =$$

$$= M_{ii}(0) + M_{ja}\delta_{ka} - M_{ja}\delta_{ka} + M_{ka}\delta_{ja} - M_{ka}\delta_{ja} = 0$$

(2.4)

$$d_{jkmn} = d_{mnjk}$$

$$d_{mnjk} = M_{li}\epsilon_{imn}\epsilon_{ljk} = M_{il}\epsilon_{lmn}\epsilon_{ijk} = M_{il}\epsilon_{ijk}\epsilon_{lmn} = d_{jkmn}$$

$$d_{mnjk} = M_{ii}\left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right) + M_{mk}\delta_{nj} + M_{nj}\delta_{mk} - M_{mj}\delta_{nk} - M_{nk}\delta_{mj} =$$

$$= M_{ii}\left(\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}\right) + M_{jn}\delta_{km} + M_{mk}\delta_{jn} - M_{jm}\delta_{kn} - M_{kn}\delta_{jm} = d_{jkmn}$$

(2.5)

$$d_{kjmn} = -d_{jkmn}$$

$$d_{kjmn} = M_{il}\epsilon_{ikj}\epsilon_{lmn} = -M_{il}\epsilon_{ijk}\epsilon_{jmn} = -d_{jkmn}$$

$$d_{kjmn} = M_{ii}\left(\delta_{km}\delta_{jn} - \delta_{kn}\delta_{jm}\right) + M_{kn}\delta_{jm} + M_{jm}\delta_{kn} - M_{km}\delta_{jn} - M_{jn}\delta_{km} =$$

$$= -M_{ii}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - M_{jn}\delta_{km} - M_{km}\delta_{jn} + M_{jm}\delta_{kn} + M_{kn}\delta_{jm} = -d_{jkmn}$$

(2.6)

$$d_{jknm} = -d_{jkmn}$$

$$d_{jkmn} = M_{il}\epsilon_{ijk}\epsilon_{lnm} = -M_{il}\epsilon_{ijk}\epsilon_{lmn} = -d_{jkmn}$$

$$d_{jknm} = M_{ii}\left(\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}\right) + M_{jm}\delta_{kn} + M_{kn}\delta_{jm} - M_{jn}\delta_{km} - M_{km}\delta_{jn} =$$

$$= -M_{ii}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - M_{jn}\delta_{km} - M_{km}\delta_{jn} + M_{jm}\delta_{kn} + M_{kn}\delta_{jm} = -d_{jkmn}$$

(2.7)

$$d_{2323} = M_{11}$$

$$d_{2323} = M_{ij}\epsilon_{i23}\epsilon_{j23} = M_{11}\epsilon_{123}\epsilon_{123} = (1)(1)M_{11} = M_{11}$$

$$d_{2323} = M_{ii}(\delta_{22}\delta_{33} - \delta_{23}\delta_{32}) + M_{2n}\delta_{32} + M_{32}\delta_{23} - M_{22}\delta_{33} - M_{33}\delta_{22} =$$
$$= M_{ii} - M_{22} - M_{33} = M_{11}$$

(2.8)

$$d_{1313} = M_{22}$$

$$d_{1313} = M_{ij}\epsilon_{i13}\epsilon_{j13} = M_{22}\epsilon_{213}\epsilon_{213} = (-1)(-1)M_{13} = M_{22}$$

$$d_{131n} = M_{ii}(\delta_{11}\delta_{33} - \delta_{13}\delta_{31}) + M_{13}\delta_{31} + M_{31}\delta_{13} - M_{11}\delta_{33} - M_{33}\delta_{11} =$$
$$= M_{ii} + M_{13}\delta_{31} - M_{11}\delta_{33} - M_{33}\delta_{11} = M_{22}$$

(2.9)

$$d_{1212} = M_{33}$$

$$d_{1212} = M_{ij}\epsilon_{i12}\epsilon_{j12} = M_{33}\epsilon_{312}\epsilon_{312} = (1)(1)M_{33} = M_{33}$$

$$d_{1212} = M_{ii}(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) + M_{12}\delta_{21} + M_{21}\delta_{12} - M_{11}\delta_{22} - M_{22}\delta_{11} =$$
$$= M_{ii} - M_{11} - M_{22} = M_{33}$$

(2.10)

$$d_{1323} = M_{12}$$

$$d_{1323} = M_{ij}\epsilon_{i13}\epsilon_{j23} = M_{21}\epsilon_{213}\epsilon_{123} = (-1)(1)M_{12} = -M_{12}$$

$$d_{1323} = M_{ii}(\delta_{12}\delta_{33} - \delta_{13}\delta_{32}) + M_{13}\delta_{32} + M_{32}\delta_{13} - M_{12}\delta_{33} - M_{33}\delta_{12} = -M_{12}$$

(2.11)

$$d_{1223} = M_{13}$$

$$d_{1223} = M_{ij}\epsilon_{i12}\epsilon_{j23} = M_{31}\epsilon_{312}\epsilon_{123} = (1)(1)M_{13} = M_{13}$$

$$d_{1223} = M_{ii}(\delta_{12}\delta_{23} - \delta_{13}\delta_{22}) + M_{13}\delta_{22} + M_{22}\delta_{13} - M_{12}\delta_{23} - M_{23}\delta_{12} = M_{13}$$

(2.12)

$$d_{1213} = -M_{23}$$

$$d_{1213} = M_{ij}\epsilon_{i12}\epsilon_{j13} = M_{32}\epsilon_{312}\epsilon_{213} = (1)(-1)M_{23} = -M_{23}$$

$$d_{1213} = M_{ii}(\delta_{11}\delta_{23} - \delta_{13}\delta_{21}) + M_{13}\delta_{21} + M_{21}\delta_{13} - M_{11}\delta_{23} - M_{23}\delta_{11} = -M_{23}$$

(2.13)

Part 3. Proof that, in three dimensions, if two eigenvectors v_i^A and v_i^B, together with corresponsing eigenvalues λ^A and λ^B, are known for a symmetric tensor M_{ij}, then the remaining eigenvector is $\mathbf{v}^C = \mathbf{v}^A \times \mathbf{v}^B$ and the corresponding eigenvalue is $\lambda^C = \mathrm{tr}(\mathbf{M}) - \lambda^A - \lambda^B$. We start with:

$$M_{ij}v_j^A = \lambda^A\, v_j^A \quad \text{with} \quad \lambda^A = M_{ij}v_i^A v_j^A \quad \text{and} \quad v_i^A v_j^A = 1$$

$$M_{ij}v_j^B = \lambda^B\, v_j^B \quad \text{with} \quad \lambda^B = M_{ij}v_i^B v_j^B \quad \text{and} \quad v_i^B v_j^B = 1$$

$$M_{ij}\left(\epsilon_{jkl}v_k^A v_l^B\right) = \lambda^C v_i^C \quad \text{with} \quad \lambda^C = M_{ij}v_i^C v_j^C \quad \text{and} \quad v_i^C = \epsilon_{ijk}v_j^A v_k^B$$

$$v_i^X v_i^Y = \delta_{XY} \tag{3.1}$$

First, note v_i^C is a unit vector:

$$v_i^C v_i^C = \epsilon_{ijk}\epsilon_{ipq}v_j^A v_k^B v_p^A v_q^B = \left(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\right)v_j^A v_k^B v_p^A v_q^B = v_j^A v_j^A v_q^B v_q^B - v_j^A v_j^B v_k^B v_k^A = 1 + 0 = 1 \tag{3.2}$$

Now let us solve for λ^C:

$$\lambda^C = v_i^C M_{ij} v_j^C = M_{il}\left(\epsilon_{ijk}v_j^A v_k^B\right)\left(\epsilon_{lmn}v_m^A v_n^B\right) = M_{il}\epsilon_{ijk}\epsilon_{lmn}v_j^A v_k^B v_m^A v_n^B =$$

$$= M_{il}\epsilon_{ijk}\epsilon_{lmn}v_j^A v_k^B v_m^A v_n^B = d_{jkmn}v_j^A v_k^B v_m^A v_n^B =$$

$$= \left\{M_{ii}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) + M_{jn}\delta_{km} + M_{km}\delta_{jn} - M_{jm}\delta_{kn} - M_{kn}\delta_{jm}\right\}v_j^A v_k^B v_m^A v_n^B =$$

$$= \left\{M_{ii}\left(\delta_{jm}\delta_{kn}v_j^A v_k^B v_m^A v_n^B - \delta_{jn}\delta_{km}v_j^A v_k^B v_m^A v_n^B\right) + M_{jn}\delta_{km}v_j^A v_k^B v_m^A v_n^B + M_{km}\delta_{jn}v_j^A v_k^B v_m^A v_n^B \right.$$
$$\left. - M_{jm}\delta_{kn}v_j^A v_k^B v_m^A v_n^B - M_{kn}\delta_{jm}v_j^A v_k^B v_m^A v_n^B\right\} =$$

$$= \left\{M_{ii}\left(\left(v_j^A v_j^A\right)\left(v_k^B v_k^B\right) - \left(v_j^A v_j^B\right)\left(v_k^A v_k^B\right)\right) + \left(v_n^B M_{nj} v_j^A\right)\left(v_k^B v_k^A\right) + \left(v_k^B M_{km} v_m^A\right)\left(v_j^A v_j^B\right) \right.$$
$$\left. - \left(v_j^A M_{jm} v_m^A\right)\left(v_k^B v_k^B\right) - \left(v_k^B M_{kn} v_n^B\right)\left(v_j^A v_j^A\right)\right\} =$$

$$= \left\{M_{ii} - \left(v_j^A M_{jm} v_m^A\right) - \left(v_k^B M_{kn} v_n^B\right)\right\} = \mathrm{tr}(\mathbf{M}) - \lambda_A - \lambda_B \tag{3.4}$$

Thus, as expected, $\lambda_A + \lambda_B + \lambda_C = \mathrm{tr}(\mathbf{M})$. Note that result alternately can be arrived using Equation (1.2):

$$\mathrm{tr}(\mathbf{M}) = M_{ij}\delta_{ij} = M_{ij}v_i^A v_j^A + M_{ij}v_i^B v_j^B + M_{ij}v_i^C v_j^C = \lambda_A + \lambda_B + \lambda_C \tag{3.5}$$

We now verify that $l_p \equiv \lambda_C^{-1}M_{pi}\left(\epsilon_{ijk}v_j^A v_k^B\right)$ (with $\lambda_C \neq 0$) is a unit vector:

$$\lambda_C^2 l_p l_p = M_{pi}\left(\epsilon_{ijk}v_j^A v_k^B\right)M_{pl}\left(\epsilon_{lmn}v_l^A v_m^B\right) = \left(M_{pi}M_{pl}\right)\epsilon_{ijk}\epsilon_{lmn}v_j^A v_k^B v_l^A v_m^B = W_{il}\epsilon_{ijk}\epsilon_{lmn}v_j^A v_k^B v_l^A v_m^B \tag{3.6}$$

Where $W_{il} \equiv M_{pi}M_{pl}$ is a symmetric tensor with eigenvalues λ_A^2, λ_B^2 and λ_C^2. Since equation has the same form as the one previously considered, we can write:

$$\lambda_C^2 l_p l_p = \mathrm{tr}(\mathbf{S}) - \lambda_A^2 - \lambda_B^2 = \lambda_C^2$$

$$(3.7)$$

Hence, we conclude that $l_p l_p = 1$ as long as $\lambda_C \neq 0$.

Finally, let us write the left and right hand sides of the eigenvalue equation as:

$$l_p \equiv \lambda_C^{-1} M_{pi}\left(\epsilon_{ijk} v_j^A v_k^B\right) \text{ and } v_p \equiv \left(\epsilon_{pqr} v_q^A v_r^B\right) \tag{3.8}$$

We have already established in (3.2) and (3.7) that v_p and l_p are unit vectors (as long as $\lambda_C \neq 0$). Furthermore, we have also established that $l_p v_p = 1$. The only way that the dot product between two unit vectors can be unity is if the two vectors are equal. Hence, $l_p = v_p$ and the eigenvalue equation is satisfied. For the $\lambda_C = 0$ case, the eigenvalue equation becomes $M_{ij}\left(\epsilon_{jkl} v_k^A v_l^B\right) = (0)\left(\epsilon_{ipq} v_p^A v_q^B\right) = [0]_i$. We can trivially modify the result in (3.7) above to show that $\left|M_{ij}\left(\epsilon_{jkl} v_k^A v_l^B\right)\right| = \lambda_C = 0$ and then, noting that a vector is equal to the zero vector when it has zero length, conclude that the eigenvalue equation is satisfied.

I think that the above proof, "the cross product between any two eigenvectors is also an eigenvector" could also be used to prove that, when the eigenvalues are distinct, there can be only three mutually-perpendicular eigenvectors. But I am not pursuing the matter.

Part 4. An alternate proof. Another way to prove that the cross-product satisfies the eigenvalue equation is to use the relationship:

$$\epsilon_{ijk} M_{iw} M_{jp} M_{kq} = \det(\mathbf{M})\, \epsilon_{wpq} \tag{3.9}$$

(Source: Wikipedia article "Determinant"). Presuming $\lambda_A \neq 0$ and $\lambda_B \neq 0$:

$$\begin{gathered}
\lambda_A^{-1} M_{ij} v_j^A = v_j^A \\
\lambda_B^{-1} M_{ij} v_j^B = v_j^B \\
v_i^C = \epsilon_{ijk} v_j^A v_k^B = \lambda_A^{-1} \lambda_B^{-1} \left(\epsilon_{ijk} M_{jp} M_{kq}\right)\left(v_p^A\, v_q^B\right) \\
M_{wi} v_i^C = \epsilon_{ijk} v_j^A v_k^B = \lambda_A^{-1} \lambda_B^{-1} \left(\epsilon_{ijk} M_{iw} M_{jp} M_{kq}\right)\left(v_p^A\, v_q^B\right) \\
M_{wi} v_i^C = \lambda_A^{-1} \lambda_B^{-1} \det(\mathbf{M}) \left(\epsilon_{wpq} v_p^A\, v_q^B\right) \\
M_{wi} v_i^C = \lambda_C v_i^C \text{ with } \lambda_C = \lambda_A^{-1} \lambda_B^{-1} \det(\mathbf{M})
\end{gathered} \tag{3.10}$$

Note that this derivation yields the well-known rule, $\det(\mathbf{M}) = \lambda_A \lambda_B \lambda_C$. I have not tried to work out a patch for the case $\lambda_A = 0$ and/or $\lambda_B = 0$.

Part 5. Proof that if $\lambda_A = \lambda_B = \lambda$ any linear combination $v_i = c_A v_i^A + c_B v_i^B$ is an eigenvector:

$$\lambda v_i = \lambda c_A v_i^A + \lambda c_B v_i^B = c_A M_{ij} v_j^A + c_B M_{ij} v_j^B = M_{ij}\left(c_A v_j^A + c_B v_j^B\right) = M_{ij} v_j \tag{3.11}$$