Non-uniqueness of Anisotropy Inversions in the Weak-Anisotropy Limit

Bill Menke, November 7, 2019

I am concerned here with the inverse problem of using observations of the non-double-couple component of earthquakes to determine the anisotropy of the Earth in the source region.

In the weak-anisotropy limit, we can think of the elastic tensor c_{ijpq} as consisting of the sum of an isotropic part $c_{ijpq}^{(0)}$ and an anisotropic perturbation Δc_{ijpq}. Furthermore, the perturbation can be constructed by summing perturbations associated with each of the of twenty-one independent ways that Δc_{ijpq}can be varied: $\Delta c_{ijpq} = \sum_{s=1}^{21} m_s \Delta c_{ijpq}^{(s)}$. Here $\Delta c_{ijpq}^{(s)}$ represents the unit perturbation of one independent set of elements of Δc_{ijpq} and m_s is an amplitude.

The unit moment tensor M_{ij} of an earthquake is related to the fault matrix D_{ij} by $M_{ij} = c_{ijpq} D_{pq}$. The fault matrix $\mathbf{D}$ can be parameterized by the strike, dip and rake $(\varphi, \lambda, \delta)$ of the fault and a formula for it can be obtained from the expression for the moment tensor on page 112 of Aki and Rickards (2009) by setting $\mu A = 1$. The explosive component of $\mathbf{M}$ is $X = ⅓\,\mathrm{tr}(\mathbf{M})$ and the CLVD component V is equal to the eigenvalue of $\mathbf{M}$ with least absolute value.

Now consider a suite of $(k = 1 \cdots M)$ earthquakes on faults with strike, dip and rake $(\varphi_k, \lambda_k, \delta_k)$, explosive components $X_k \equiv X(\varphi_k, \lambda_k, \delta_k)$ and CLVD components $V_k \equiv V(\varphi_k, \lambda_k, \delta_k)$. Because the anisotropy is weak, X_k and V_k can be consider linear functions of the perturbations:

$$X_k \equiv X(\varphi_k, \lambda_k, \delta_k) = X_k^{(0)} + \sum_{i=1}^{21} G_{ks}^X m_s \quad \text{with} \quad G_{ks}^X = \frac{\partial X_k}{\partial \Delta c_{ijpq}^{(j)}}$$

$$V_k \equiv V(\varphi_k, \lambda_k, \delta_k) = V_k^{(0)} + \sum_{i=1}^{21} G_{ks}^V m_s \quad \text{with} \quad G_{ks}^V = \frac{\partial V_k}{\partial \Delta c_{ijpq}^{(s)}}$$

The $X^{(0)}$and $V^{(0)}$ terms are zero, since the unperturbed medium is isotopic. The explosive component $X = ⅓ c_{iipq} D_{pq}$ is a linear function of $\Delta c_{ijpq}^{(s)}$. Hence, the kernel G_{ks}^X exists and is trivial to calculate. Since V involves extracting an eigenvalue, it is a non-linear function of c_{iipq}. However, for small m_s G_{ks}^V can easily be calculated using perturbation theory. In the large m_s case, G_{ks}^V can be subject to the "eigenvalue switching" problem, and can have discontinuities.

We now view these equations as an inverse problem for unknown m_i given observed X_k and V_k. A key question is whether the problem is unique.

We know 21 three-dimensional angular patterns associated with G_{kp}^X and 21 more associated with G_{kp}^V. If among these 42 patterns there are 21 of them that are linearly independent, then the 21 unknowns are uniquely determined. However, we expect that at most 19 patterns can be determined uniquely; the other two correspond to isotropic perturbations (that is, variations in Lame parameter λ and μ) that have $X = V = 0$.

We approach this problem numerically. We evaluate G_{ks}^{X} and G_{ks}^{V} on a $5° \times 5° \times 5°$ grid of $(\varphi_k, \lambda_k, \delta_k)$. We then use singular value decomposition to determine the number of independent patterns (which is equal to the number of non-zero singular values). We find:

Data	Number of Non-zero Singular values
X only	5
V only	14
X and V	19

A plot of the singular values is shown in Figure 1. When both X and V are used, all 19 coefficients can be recovered; the inversion is unique, up to isotropic perturbations (Figure 2). On the other hand, for X-only and V-only inversions, fewer than 19 coefficients can be recovered; these inversions are non-unique. The seven unresolved combinations of elastic parameters are shown for the V -only case (Figure 3).

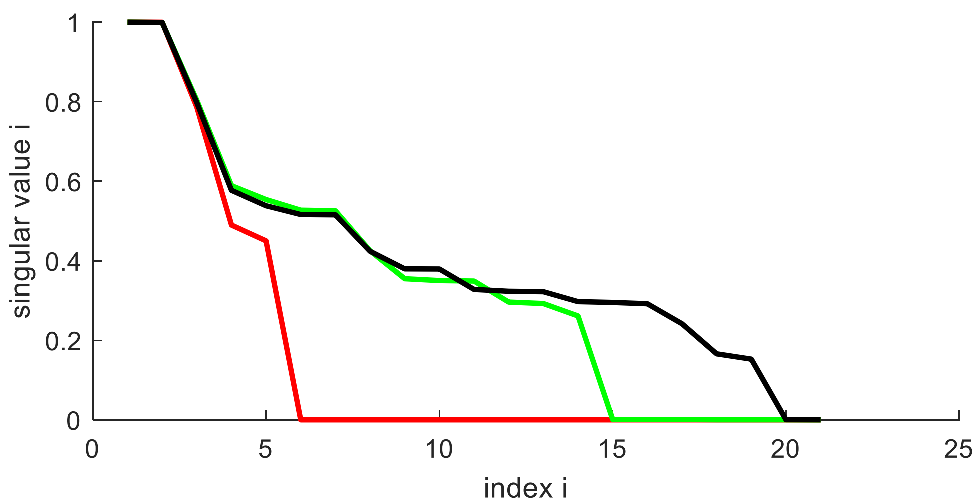

Figure 1. Singular values, normalized and sorted by size. (Red) X only, (green) V only, (black) both X and V. Two choices of isotropic medium, with Lame coefficients (λ, μ) of $(1.1,1)$ and $(2.1,1)$, respectively, gave this same result.

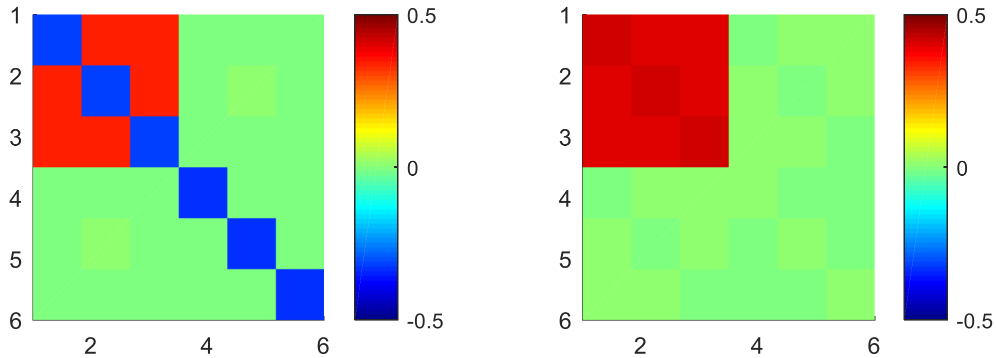

Figure 2. The two unresolved linear combinations of elastic parameters for a "both X and V" inversion, here shown as 6×6 Voigt matrices C_{ij}. They correspond to two different linear combinations of the Lame parameters λ and μ. Only the elements C_{14}, C_{15}, C_{16}, C_{24}, C_{25}, C_{26}, C_{34}, C_{35}, C_{36}, C_{45}, C_{46} and C_{56} are fully-resolved.

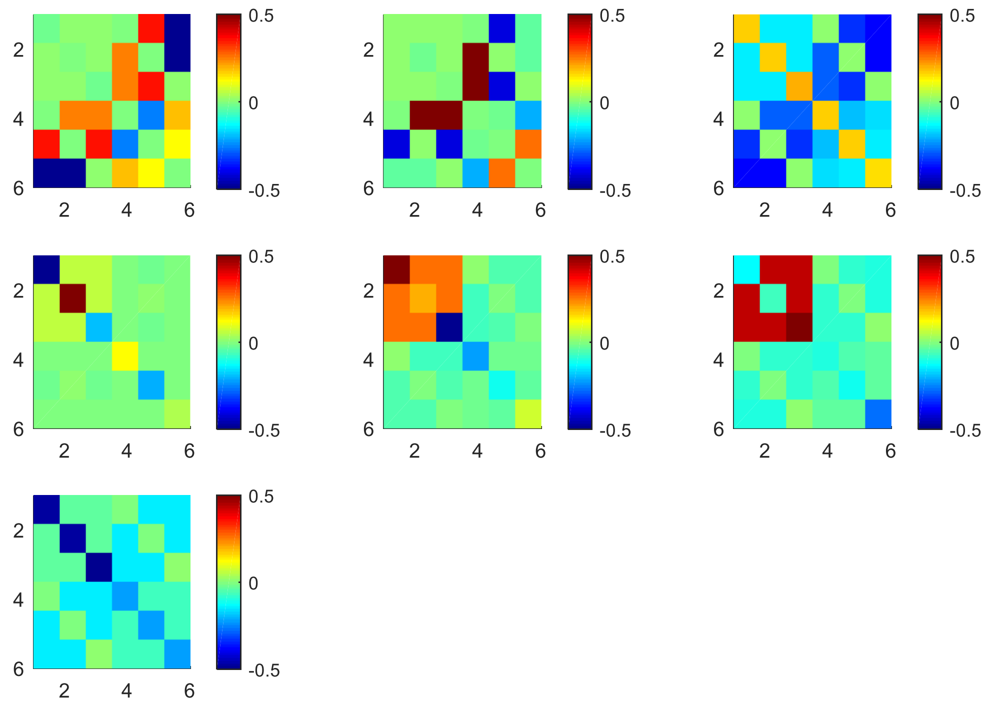

Figure 3. The seven unresolved linear combinations of elastic parameters for a V-only inversion, here shown as 6×6 Voigt matrices C_{ij}. Only the elements C_{14}, C_{25} and C_{36} are fully-resolved.