Is Cross-correlating Displacement or Velocity Better in Ambient Noise Studies?

Bill Menke, February 26, 2020 (after conversation with Josh Russell)

Suppose we correlate two signals $a(t)$ and $b(t)$ to get $c(t) = a(t) \star b(t)$ where $\star$ signifies cross-correlation. In the frequency domain, we have $c(\omega) = a(\omega)\bar{b}(\omega)$ where $c(\omega)$ is the cross spectrum and where $\bar{}$ signifies complex conjugation.

Now suppose that we form two other signals, $a'(t) = T(t) \star a(t)$ and $b'(t) = T(t) \star b(t)$ where $T(t)$ is some operator. For example, the choice $T(t) = -\partial\delta/\partial t$, where $\delta(t)$ is the Dirac impulse function, implements differentiation. The cross correlation is $c'(t) = a'(t) \star b'(t)$. Following the general approach of Menke and Menke (2016, their equation 9.24), we rewrite the equation using convolutions, noting that $a(t) \star b(t) = a(t) * b(-t)$, where $*$ signifies convolution:

$$c'(t) = a'(t) \star b'(t) = \big(T(t) * a(t)\big) * \big(T(-t) * b(-t)\big) =$$

$$= \big(T(t) * T(-t)\big) * \big(a(t) * b(-t)\big) = \big(T(t) \star T(t)\big) * \big(a(t) \star b(t)\big) =$$

$$= X(t) * \big(a(t) \star b(t)\big) \quad \text{with} \quad X(t) \equiv \big(T(t) \star T(t)\big)$$

Here, $X(t)$ is the autocorrelation of $T(t)$, a function which is symmetric about $t = 0$. Consequently, $c'(t)$ is related to $c(t)$ by a symmetric filter that does not introduce any phase shifts.

In the frequency domain:

$$c'(\omega) = \big(T(\omega)\bar{T}(\omega)\big) * \big(a(\omega)\,\bar{b}(\omega)\,\big) = |T(\omega)|^2 c(\omega) = X(\omega)c(\omega)$$

Note that $X(\omega)$ is a real-valued function. Consequently, $c'(\omega)$ is related to $c(\omega)$ by modulation by an envelope function $X(\omega)$.

Thus, any zero-crossing in $c(\omega)$ will also be zero-crossings of $c'(\omega)$, and as long as $T(\omega)$ has no zero-crossings, $c'(\omega)$ will have exactly the same number of zero-crossings as $c(\omega)$. The only effect of $T(\omega)$ is to modulate the overall shape of the cross-spectrum $c'(\omega)$.

So, whether one cross-correlates displacement $u(t)$ or velocity $v(t)$ or raw seismogram output[1] $s(t)$ is unimportant, except for aesthetics in plotting, since the choice effects that relative amplitude of different frequency bands and can be used to suppress parts of the cross-spectrum that is dominate by noise, or to emphasize frequency bands that are important to the analysis.

[1]Presuming the response of the two seismometers are the same.

Reference:

Menke, W. and J. Menke, Environmental Data Analysis with MATLAB, Second Edition (textbook), Academic Press (Elsevier), 342pp, 2016.