Moments of a Convolution

By Bill Menke, March 28, 2020

I presume that these formulas are "well known", but I couldn't track down a reference and so I worked them out for myself.

Let $f(t)$ and $g(t)$ be two functions. The first three moments of $f(t)$ are defined as:

$$M_0(f) = \int f(t)\, dt \quad \text{and} \quad M_1(f) = \int t\, f(t)\, dt \quad \text{and} \quad M_2(f) = \int t^2\, f(t)\, dt$$

And similarly for $g(t)$. The convolution of the two functions is:

$$(f * g)(\tau) = \int f(\tau - t)\, g(t)\, dt$$

The following relationships hold:

$$M_0(f * g) = M_0(f)\, M_0(g)$$

$$M_1(f * g) = M_0(f) M_1(g) + M_1(f) M_0(g)$$

$$M_2(f * g) = M_0(f)\, M_2(g) + M_2(f)\, M_0(g) + 2\, M_1(f)\, M_1(g)$$

Proof of 1: The zeroth moment of a convolution is:

$$M_0(f * g) = \int \left\{ \int f(\tau - t)\, g(t)\, dt \right\} d\tau = \int \left\{ g(t) \int f(\tau - t)\ d\tau \right\} dt$$

Introducing a change of variables:

$$x = \tau - t \quad \text{and} \quad \tau = t + x \quad \text{and} \quad dx = d\tau$$

leads to:

$$M_0(f * g) = \int \left\{ g(t) \int f(x)\ dx \right\} dt = M_0(f) \int g(t)\, dt = M_0(f)\, M_0(g)$$

Proof of 2: The first moment of a convolution is:

$$M_1(f * g) = \int \tau \left\{ \int f(\tau - t)\, g(t)\, dt \right\} d\tau = \int \left\{ g(t) \int \tau\, f(\tau - t)\ d\tau \right\} dt$$

Introducing a change of variables:

$$x = \tau - t \quad \text{and} \quad \tau = t + x \quad \text{and} \quad dx = d\tau$$

leads to:

$$M_1(f * g) = \int \left\{ g(t) \int (t + x)\, f(x)\ dx \right\} dt =$$

$$= \int g(t) \left\{ t \int\ f(x)\ dx + \int x\, f(x)\ dx \right\} dt =$$

$$= M_0(f) \int t\, g(t)\, dt +\ M_1(f) \int\ g(t)\, dt =$$

$$= M_0(f) M_1(g) + M_1(f) M_0(g)$$

Proof of 3: The second moment of a convolution is:

$$M_2(f * g) = \int \tau^2 \left\{ \int f(\tau - t)\, g(t)\, dt \right\} d\tau = \int g(t) \left\{ \int \tau^2 f(\tau - t)\ d\tau \right\} dt$$

Introducing a change of variables:

$$x = \tau - t \quad \text{and} \quad \tau = t + x \quad \text{and} \quad dx = d\tau$$

$$(t + x)^2 = t^2 + x^2 + 2xt$$

leads to:

$$\mathrm{var}(f * g) = \int g(t) \left\{ \int (t + x)^2 f(x)\ d\tau \right\} dt$$

$$= \int g(t) \left\{ \int (t^2 + x^2 + 2xt)\, f(x)\, dx \right\} dt$$

$$= \int g(t) \left\{ t^2 \int\ f(x)\, dx + \int x^2\, f(x)\, dx + 2t \int x\, f(x)\, dx \right\} dt =$$

$$= \int g(t) \{ t^2 M_0(f) + \mathrm{var}(f) + 2t\, M_1(f) \} dt =$$

$$= M_0(f)\, M_2(g) + M_2(f)\, M_0(g) + 2\, M_1(f)\, M_1(g)$$