Comparison of Receiver Function, Cross-Convolution and Transfer Function Methods

Bill Menke, April 11, 2020

Two components of ground motion, $A(t)$ and $B(t)$, are assumed to be related to the source-side wavelet $s(t)$ by receiver-side Earth response functions $a(t)$ and $b(t)$:

$$A(t) = a(t) * s(t) \quad \text{and} \quad B(t) = b(t) * s(t) \tag{1}$$

The functions $a(t)$ and $b(t)$ always exist in a well-posed wave propagation problem.

Earth structure inversion methods seek to define an error scalar $E(\mathbf{m})$ that is sensitive to Earth structure $\mathbf{m}$ but independent of (or at least insensitive to) source-side wavelet $s(t)$.

RF Method. The Receiver Function method (Langston and Burdick 1977) uses deconvolution, represented here as an inverse time series (e.g., $x^{-1}(t)$ with $x^{-1}(t) * x(t) = \delta(t)$) to manipulate Equation (1) and define a receiver function $R(t)$:

$$R(t) = A^{-1}(t) * B(t) = a^{-1}(t) * b(t) \tag{2}$$

The error E that compares observed and predicted receiver functions is:

$$E(\mathbf{m}) = \left\| R_{obs}(t) - R_{pre}(t, \mathbf{m}) \right\| = \left\| A_{obs}^{-1}(t) * B_{obs}(t) - A_{pre}^{-1}(t, \mathbf{m}) * B_{pre}(t, \mathbf{m}) \right\| \tag{3}$$

Note that E is independent of $s(t)$.

The CC Method. The Cross-Convolution Method (Menke and Levin, 2003) manipulates Equation (1) using convolutions to define the cross-convolution $C(t)$:

$$B(t) * [a(t) * s(t)] = A(t) * [b(t) * s(t)]$$

$$C(t, \mathbf{m}) = s(t) * c(t, \mathbf{m}) \quad \text{with} \quad c(t, \mathbf{m}) \equiv \left[B_{obs}(t) * a_{pre}(t, \mathbf{m}) - A_{obs}(t) * b_{pre}(t, \mathbf{m}) \right]$$

$$E(\mathbf{m}) = \|C(t)\| \;\; \leq \;\; \|s(t)\| \, \|c(t, \mathbf{m})\| \tag{4}$$

While $C(t, \mathbf{m})$ depends on the source-side wavelet $s(t)$, the dependence is only through a leading convolution. Thus, $C(t)$ can be understood to be a “filtered” version of $c(t, \mathbf{m})$ - a quantity that does not depend upon $s(t)$. The error E it is zero only when $c(t, \mathbf{m}) = 0$, and thus the solution $\{\mathbf{m} \mid E(\mathbf{m}) = 0\}$ does not depend upon $s(t)$.

C. Transfer Function Method. The transfer function method (Frederiksen 2020) uses the transfer function $T(t)$ that takes component $A(t)$ into component $B(t)$:

$$B(t) = T(t) * A(t) \tag{5}$$

Or equivalently, Earth response $a(t)$ into Earth response $b(t)$:

$$b(t) = T(t) * a(t) \tag{6}$$

The error is then defined as:

$$E(\mathbf{m}) = \left\| B_{obs}(t) - T_{pre}(t, \mathbf{m}) * A_{obs}(t) \right\| \tag{7}$$

Note that the error does not explicitly depend upon $s(t)$, although both $A_{obs}(t)$ and $B_{obs}(t)$ depend upon it implicitly.

Here are what I see as the main advantages and disadvantages of the three methods:

A. An advantage of the receiver function method is that E is independent of $s(t)$; that is, it will give the same result for two earthquakes with different $s(t)$s. The big disadvantage is that it involves two applications of deconvolution, one applied to the observations and one applied to predictions. While either can be unstable, the behavior of the data deconvolution tends to be worse. (Technically, neither $A^{-1}(t)$ or $a^{-1}(t)$ is guaranteed to exist, but usable approximations to them can usually be constructed). A more minor disadvantage is a lack of "symmetry"; in the presence of noise, a method based on the receiver function $A^{-1}(t) * B(t)$ will not necessarily yield the same result as one based on $B^{-1}(t) * A(t)$.

B. The big advantage of the cross-convolution method is that E is defined in way that avoids deconvolutions. Another advantage is that the cross-convolution is exactly symmetrical in $A(t)$ and $B(t)$ (or at least up to an overall sign). A disadvantage is that in the practical case where $\min E(\mathbf{m})$ is small but not exactly zero, then the estimated Earth model $\mathbf{m}^{est} = \operatorname{argmin} E(\mathbf{m})$ does depends upon $s(t)$, at least weakly. (Because of the structure of $C(t)$, this difference can be understood to be associated with choosing a frequency range over which $E(\mathbf{m})$ is measured). Another disadvantage is that $E(\mathbf{m})$ has a "spurious" zero in cases where $a_{pre}(t, \mathbf{m}) = b_{pre}(t, \mathbf{m}) = 0$. This case is relatively easy to detect, but can be annoying. (An example of this case is modeling a reflection; the special case "no reflection is predicted" gives $E(\mathbf{m}) = 0$, even when the data contain a reflection).

C. An advantage of the transfer function method is that it avoids a data deconvolution. A minor disadvantage is that, like the cross-convolution method, the minimum E still depends implicitly upon $s(t)$, but as in the cross-convolution case, it can be interpreted as a choice of bandwidth. A more serious disadvantage is that the transfer function implicitly contains a deconvolution, as can be seen by rewriting Equation (6) as:

$$T(t) = b(t) * a^{-1}(t) \tag{8}$$

Consequently, while $a(t)$ and $b(t)$ always exist in a well-posed wave propagation problem, the transfer function $T(t)$ does not always exist. (In the language of time series, it only exists when $\mathbf{a}$ is minimum-phase, with $\mathbf{a}$ a discrete version of $a(t)$). I see this as the major disadvantage of the method. A minor disadvantage is the lack of symmetry; in the presence of noise, the choice $b(t) = T(t) * a(t)$ will not necessarily lead to the same results as $a(t) = T(t) * b(t)$.

When using receiver functions or transfer function methods, I recommend that the deconvolutions be calculated using a technique that produces sensible results. What I mean is that, given a time series $f(t)$, one should construct its inverse $f^{-1}(t)$ so that $f^{-1}(t) * f(t)$ is a close a possible single-sized pulse (i.e., close to Gaussian). This can be accomplished by incorporation of prior information on smoothness into the deconvolution process (see Section 12.2 of Menke, 2018). Be aware that simple-minded techniques, such as spectral-domain water-level methods, can produce very oscillatory results.

References

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