The Derivative of Shear Velocity with Respect to Compressional Velocity at Constant Bulk Modulus and Density

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Compressional velocity V_P and shear velocity V_S are functions of bulk modulus K, shear modulus μ and density ρ:

$$V_P = \left(\frac{K + \frac{4}{3}\mu}{\rho}\right)^{½} \quad \text{and} \quad V_S = \left(\frac{\mu}{\rho}\right)^{½}$$

Now imagine that shear modulus, only, varies with a parameter θ, such as temperature. The derivatives are:

$$\frac{dV_P}{d\theta} = \left(\frac{1}{2}\right)\left(\frac{4}{3}\right)\left(\frac{1}{\rho}\right)V_P^{-1}\frac{d\mu}{d\theta} \quad \text{and} \quad \frac{dV_S}{d\theta} = \left(\frac{1}{2}\right)\left(\frac{1}{\rho}\right)V_S^{-1}\frac{d\mu}{d\theta}$$

The ratio is:

$$\frac{dV_S}{dV_P} = \frac{dV_S}{d\theta}\Big/\frac{dV_P}{d\theta} = \left(\frac{3}{4}\right)\left(\frac{V_P}{V_S}\right)$$

I checked this formula numerically. For a Poisson solid with $V_P/V_S = \sqrt{3}$,

$$\frac{dV_S}{dV_P} = \frac{3\sqrt{3}}{4} \approx 1.30$$

There is some experimental evidence that $dV_S/dV_P \approx 1.0$ in the Earth's mantle, a value about 30% smaller than this predicted value. The discrepancy implies that $dV_P/d\theta$ is larger than predicted. This behavior arises when $dK/d\theta \neq 0$, since then:

$$\frac{dV_P}{d\theta} = \left(\frac{1}{2}\right)\left(\frac{dK}{d\theta} + \frac{4}{3}\frac{d\mu}{d\theta}\right)\left(\frac{1}{\rho}\right)V_P^{-1} > \left(\frac{1}{2}\right)\left(\frac{4}{3}\frac{d\mu}{d\theta}\right)\left(\frac{1}{\rho}\right)V_P^{-1}$$

(presuming that $dK/d\theta$ and $d\mu/d\theta$ have the same sign).