

The Bateman Attenuation Model
 Bill Menke, May 28, 2020

The dispersion relation is (Aki and Richards, 2009, page 169, eqn 13):

$$\frac{1}{c(\omega)} = \frac{1}{c_\infty} + \mathcal{H} \left[\frac{\alpha(\omega)}{\omega} \right]$$

These authors define the Hilbert transform \mathcal{H} as (Aki and Richards, 2009, page 169, eqn 13)

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\alpha(\zeta) d\zeta}{\zeta - \omega} = \mathcal{H}[\alpha(\omega)],$$

We relate the attenuation rate α to a quality factor Q by $\alpha(\omega) = \omega/(2Qc_\infty)$. The quality factor $Q(\omega)$ is an odd function of angular frequency ω (Aki and Richards, 2009, page 164). Consequently, attenuation rate $\alpha(\omega)$ is an even function of frequency.

Quality factor is $Q(\omega)$ is assumed to obey the empirical law $Q = Q_0 \operatorname{sgn}(\omega) (|\omega|/\omega_0)^\alpha$, where $Q_0 > 0$ and $\omega_0 > 0$ are constants and $0 < \alpha < 1$ (typically, $\alpha \approx 0.4$). Consequently:

$$\alpha(\omega) = \frac{\omega}{2Qc_\infty} = \frac{|\omega_0|^\alpha}{2Q_0c_\infty} |\omega|^{1-\alpha} \quad \text{and} \quad \frac{\alpha(\omega)}{\omega} = \left(\frac{\omega_0^\alpha}{2Q_0c_\infty} \right) \operatorname{sgn}(\omega) |\omega|^{-\alpha}$$

So:

$$\mathcal{H} \left[\frac{\alpha(\omega)}{\omega} \right] = \left(\frac{\omega_0^\alpha}{2Q_0c_\infty} \right) \mathcal{H} [\operatorname{sgn}(\omega) |\omega|^{-\alpha}]$$

Bateman (1954, page 249, eqn 30) gives the following Hilbert transform pair.:

	$f(x)$	$\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$
(30)	$\operatorname{sgn} x x ^{\nu-1} \quad 0 < \operatorname{Re} \nu < 1$	$\tan(\frac{1}{2}\nu\pi) y ^{\nu-1}$

I have checked the formula numerically. The discrete Hilbert transform matches the it, except for a sign and a slight d.c. shift. I attribute the sign to a sign flip in Matlab's definition of the Hilbert transform, and the d.c. shift to an issue with the zero-frequency component in the underlying discrete Fourier transform.

Note that the Bateman (1954) definition of the Hilbert transform is the same as the one by Aki and Richards (2009). Identifying $\nu - 1 = -\alpha$, we see that $\nu = 1 - \alpha$. Note that the condition is satisfied; that is, $0 < \nu < 1$ for $\alpha \approx 0.4$. We then have:

$$\begin{aligned} c^{-1}(\omega) - c_{\infty}^{-1}(\omega) &= \mathcal{H} \left[\frac{\alpha(\omega)}{\omega} \right] = \left(\frac{\omega_0^{\alpha}}{2Q_0 c_{\infty}} \right) \mathcal{H} [\text{sgn}(\omega) |\omega|^{-\alpha}] = \\ &= \left(\frac{1}{2Q_0 c_{\infty}} \right) \tan\{\frac{1}{2}\pi(1 - \alpha)\} \left| \frac{\omega}{\omega_0} \right|^{-\alpha} \end{aligned}$$

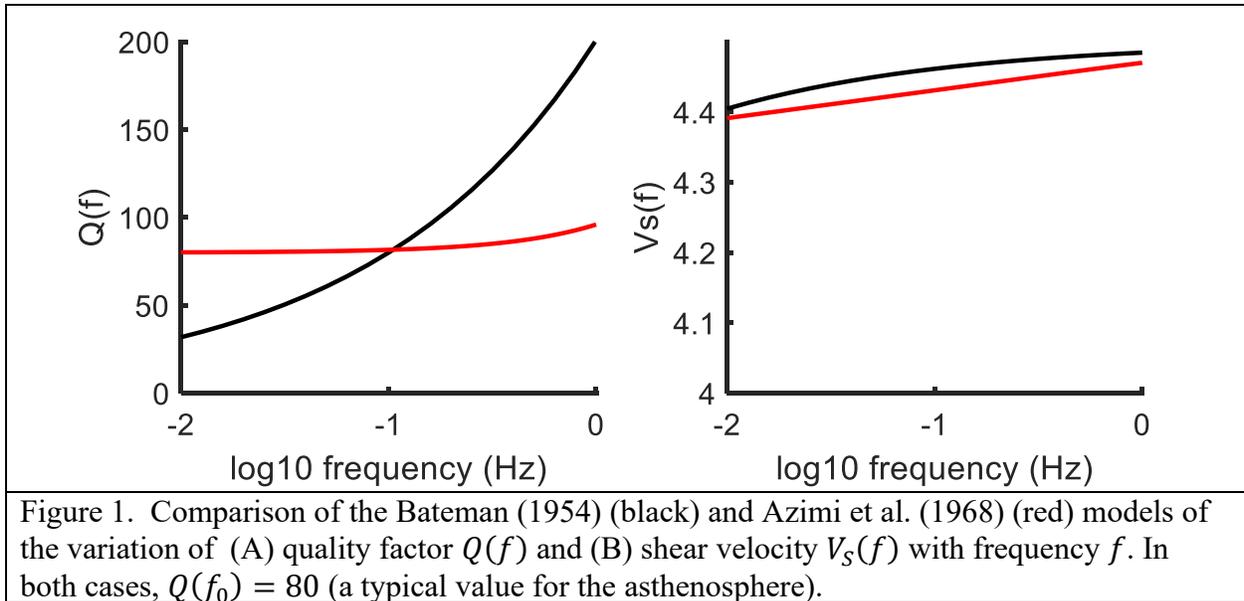
Note that $\tan\{\frac{1}{2}\pi(1 - \alpha)\} = \tan\{\frac{1}{2}\pi - \frac{1}{2}\pi\alpha\} = \cot\{\frac{1}{2}\pi\alpha\}$, and also that $\omega/\omega_0 = f/f_0$, where f is frequency, so the formula reduces to:

$$\Delta s(f) \equiv c^{-1}(f) - c_{\infty}^{-1}(f) = \left(\frac{1}{2Q_0 c_{\infty}} \right) \cot\{\frac{1}{2}\pi\alpha\} \left| \frac{f}{f_0} \right|^{-\alpha}$$

Here Δs is the slowness change. When $\alpha = 0.4$, $\cot\{\frac{1}{2}\pi\alpha\} \approx 1.38$.

In the example below, we simulate shear wave propagation through typical asthenosphere with $c_{\infty} = 4.5$ km/s, $Q_0 = 80$, $f_0 = 0.1$ Hz, $\alpha = 0.4$ and compare the results with the quasi-constant Q Azimi et al. (1968) model. Based on this simulation, we make the following observations:

1. The velocity dispersion of the two models is very similar, with velocity increasing with frequency at a rate of about 0.05 km/s per \log_{10} Hz (Figures 1). The Azimi et al. (1968) dispersion is more linear than the Bateman (1954) model.
2. The Bateman (1954) attenuation operator is narrower than the Azimi et al. (1968) attenuation operator, because the latter has a lower quality factor at the higher frequencies (Figure 2).



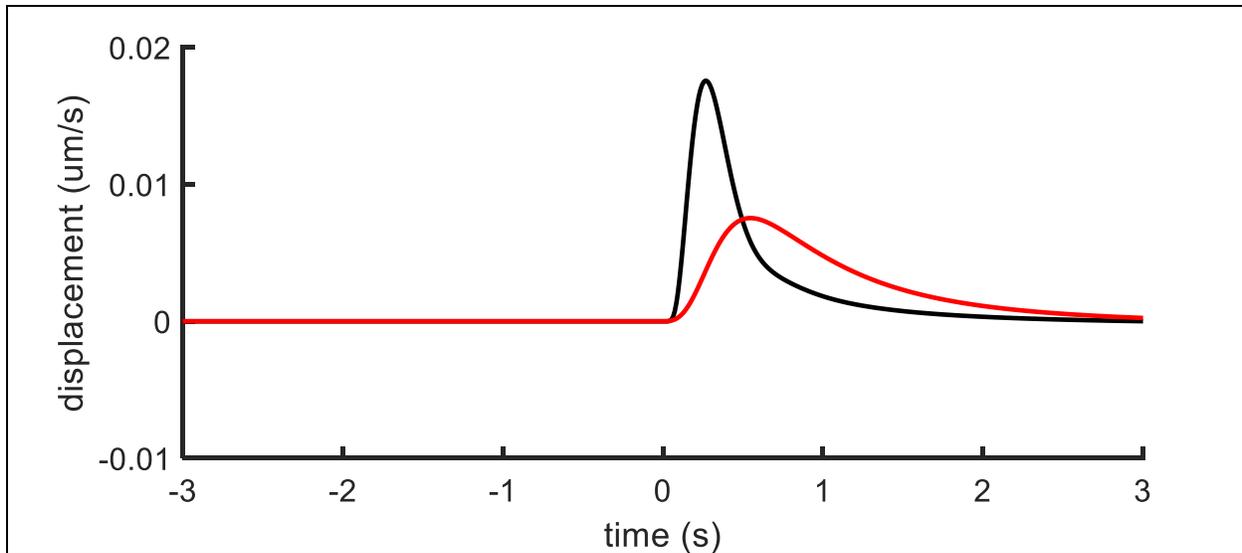


Figure 2. Comparison of the Bateman (1954) (black) and Azimi et al. (1968) (red) attenuation operators, for propagation through 300 km of attenuating material. The Bateman (1954) model has $c_{\infty} = 4.5$ km/s, $Q_0 = 80$, $f_0 = 0.1$ Hz, $\alpha = 0.4$. The Azimi et al. (1968) model has $c_{\infty} = 4.5$ km/s, $Q_0 = 80$, $f_0 = 0.1$ Hz and $f_{corner} = 5$ Hz. Both operators have been bandpass filtered between 0.01 and 1.0 Hz, to simulate a seismometer response.

References

Aki, K. & Richards, P.G. (2009). Quantitative Seismology (2 ed.), University Science Books (Sausalito, California), 700pp. ISBN 978-1-891389-64-4.

Azimi, S.A, Kalinin, A.V., Kalinin V.V and Pivovarov, B.L.(1968). Impulse and transient characteristics of media with linear and quadratic absorption laws. Izvestiya - Physics of the Solid Earth 2. p.88-93.

Bateman, H., Table of Integral transforms, Project NR 043-045, Erdélyi, A. (editor), McGraw Hill Book Company, Inc., New York, 1954, 450pp.

Exemplary MATLAB code

```
function [ t, pulse0, pulse, f, Qw, cw ] = bateman( N, Dt, x, cinf,
Q0, f0, alpha )
```

```
% Q model  $Q(f) = Q0 * (f/f0)^{\alpha}$ 
% input parameters:
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% N number of samples in pulse (e.g. 1024);
% Dt sampling interbal (e.g. 0.1)
% x propagation distance in km (e.g. 100)
% cinf base velocity in km/s (e.g. 4.5);
% Q0 refrence quality factor (e.g. 80)
% f0 reference frequency in hz (e.g. 0.1)
% alpha frequency dependence of quality factor (e.g. 0.4)
```

```

% returned values
% t time array
% pulse0 input pulse, a unit spike at time N/2
% pulse attenuated pulse
% f frequencies in Hz
% Qw frequency dependent quality factors
% cw frequency dependent phase velocities

% time series
t = Dt*[0:N-1]';
pulse0 = zeros(N,1);
pulse0(N/2)=1;

% standard fft setup
fny = 1/(2*Dt);
N2 = N/2+1;
df = fny / (N/2);
f = df*[0:N2-1]';
w = 2*pi*f;
w0 = 2*pi*f0;

% Using Bateman's (1954) Hilbert transform pair
%  $f(w) = \text{sgn}(w) \text{abs}(w)^{(\text{nu}-1)}$  when  $0 < \text{nu} < 1$ 
%  $H(f) = \tan(\text{nu} \pi / 2) \text{abs}(w)^{(\text{nu}-1)}$ 

Qw = Q0 * ((w/w0).^alpha);
a = w./(2*Qw*cinf);
a(1)=0;
b = 1/(2*Q0*cinf);
c = cot(pi*alpha/2);
ds = b*c*((w/w0).^(-alpha));
ds(1)=0;
sw = (1/cinf) + ds;
cw = 1./sw;
cw(1)=0;

dt = fft(pulse0);
dp = dt(1:N2);
dp = dp .* exp(-a*x) .* exp(-complex(0,1)*w.*ds.*x);
dtnew = [dp(1:N2);conj(dp(N2-1:-1:2))]; % fold out negative
frequencies
pulse = real(ifft(dtnew));

end

```