Notes on Attenuation
Bill Menke, 7/10/2020

Functional form of attenuation. The amplitude $A(x)$ of a wave is observed to decline with the distance x of propagation.

Suppose that the wave's amplitude is A_0 at $x = 0$. An important issue is the form of the function $A(x)$.

If we assume that the attenuation is linear, then only amplitude ratios matter. That is, if a starting amplitude A_0 decreases to $A(x)$, then a starting amplitude of $2A_0$ decreases to $2A(x)$. This means that $A(x) = A_0 f(x)$ where $f(x)$ is independent of A_0.

In a homogenous medium, we might expect that the attenuation experienced between 0 and $x_2 = 2x_1$ is the same as if the wave had propagated twice through a distance x_1. If the amplitude decreases from A_0 to $A_0 f(x_1)$ in between $(0, x_1)$, then it should decrease from $A_0 f(x_1)$ to $A_0[f(x_1)]^2$ between (x_1, x_2),. Thus, $f(2x_1) = [f(x_1)]^2$. More generally, similar equation should hold for any $x_2 = rx_1$:

$$f(rx) = [f(x)]^r$$

This equation, together with the boundary condition $f(0) = 1$, can be viewed as an "functional equation" for $f(x)$. Its solution is $f(x) = \exp(-ax)$, where a is a decay constant, as can be verified by substitution:

$$\exp(-arx) = [\exp(-ax)]^r$$

A constructive derivation is given at the end of this note.

Frequency-dependence. Suppose that the wave is sinusoidal with angular frequency ω and that it propagates through a medium with velocity v. Waves of different frequencies may decay at different rates, or in other words, the decay constant $a(\omega)$ may be frequency-dependent.

A simple behavior that was observed in early experiments is that the decrease in amplitude depends only on the number n of wavelengths propagated; that is $f(x) = \exp(-(\text{constant})\, n)$. Historically, this constant was denoted as π/Q, where Q is "quality factor". Since wavelength λ, velocity v and angular frequency ω are related by $\omega\lambda/2\pi = v$, the number of wavelengths is $n = x/\lambda = \omega x/2\pi v$. So:

$$\exp\left(-\frac{\pi}{Q}n\right) = \exp\left(-\frac{\pi}{Q}\frac{\omega x}{2\pi v}\right) = \exp\left(-\frac{\omega x}{2Qv}\right) = \exp(-ax)$$

Hence, the decay rate is $a = \omega/(2Qv)$ and the attenuation is:

$$\frac{A}{A_0} = \exp\left(-\frac{\omega x}{2Qv}\right) = \exp(-ax) = \exp\left(-\frac{\omega t^*}{2}\right)$$

Here we have defined the "tee-star" $t^* \equiv x/(Qv)$.

In these historically-important formula, quality factor Q, tee-star t^* and velocity v are treated as frequency-independent constants. However, in the 1960's, a problem was discovered with this idea. It turns out that wave propagation through a medium that is constant-Q and constant-v is "acausal", meaning that an observer at x will detect a non-zero displacement *before* an earthquake at $x = 0$ occurs. This "unphysical" result is understood to mean that a frequency-independent Q is impossible. One of the interesting aspects of the (graduate-level) mathematics behind this result is the realization that not only must quality factor $Q(\omega)$ depend upon frequency, but so must velocity $v(\omega)$. A velocity that is frequency-dependent is said to be "dispersive". Furthermore, the analysis shows that either one of these functions is sufficient to exactly determine the other. The formula relating $Q(\omega)$ to $v(\omega)$ (and vice versa) are called Kramer-Kronig relations.

Azimi et al. (1968) propose a quasi-constant-Q model in which $Q(\omega)$ was approximately constant below a reference frequency ω_0 and increases as ω at high frequencies:

$$Q(\omega) = Q_0\left(1 + \frac{\omega}{\omega_0}\right)$$

The velocity $v(\omega)$ associated with the quasi-constant $Q(\omega)$ (above) was worked out by Azimi et al. (1968) using the Kramer-Kronig relations:

$$\frac{1}{v(f)} = \frac{1}{v_\infty} - \left(\frac{1}{Q_0 v_\infty}\right) \frac{\log\left(\frac{\omega}{\omega_0}\right)}{\pi\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)}$$

An analysis of the limit $\omega/\omega_0 \rightarrow 1$ is provided at the end of this note. Rock physics results have shown that the quasi-constant- Q model is not a good approximation for rocks, and that a better approximation is the power law formula:

$$Q(\omega) = Q_0\left(\frac{\omega}{\omega_0}\right)^\alpha$$

where ω_0 is a reference frequency, Q_0 is the quality factor at the reference frequency and the exponent $\alpha \approx 0.4$. This Q formula implies that quality factor increases with frequency; that is, at low frequencies the wave experiences high attenuation and at very high frequencies it experiences almost none.

The velocity $v(\omega)$ associated with the power-law $Q(\omega)$ (above) has been worked out by Bateman (1954) using the Kramer-Kronig relations:

$$\frac{1}{v(f)} = \frac{1}{v_\infty} + \left(\frac{1}{2Q_0 v_\infty}\right) \cot\{½\pi\alpha\} \left|\frac{f}{f_0}\right|^{-\alpha}$$

Here v_∞ is the velocity at infinite frequency.

Asthenospheric S waves have $Q_0 \approx 80$ at a reference frequency of $f_0 = \omega_0/2\pi = 0.1$ Hz. The $Q(\omega)$ and $v(\omega)$ for the quasi-constant-Q and power-law models is shown in Figure 1.

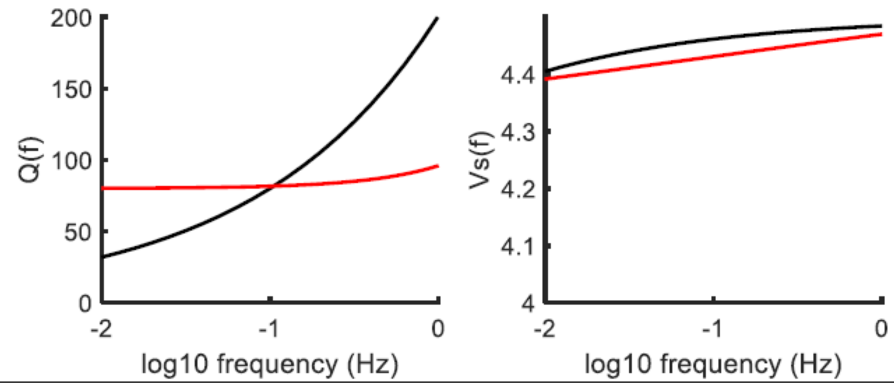

Figure 1. Comparison of the Bateman (1954) (black) and Azimi et al. (1968) (red) models of the variation of (A) quality factor $Q(f)$ and (B) shear velocity $V_S(f)$ with frequency f. In both cases, $Q(f_0) = 80$ (a typical value for the asthenosphere).

Note 1: Constructive solution to $f(rx) = [f(x)]^r$ with boundary condition $f(0) = 1$. We take the logarithm of both sides of the equation: $\log f(rx) = r \log f(x)$ and expand $\log f(x)$ in a Taylor series:

$$\log f(x) = c_0 + c_1 x + ½ c_2 x^2 + \cdots$$

We then insert the series into the equation:

$$c_0 + c_1 r x + ½ c_2 r^2 x^2 + \cdots = r c_0 + r c_1 x + ½ c_2 r x^2 + \cdots$$

and match terms of the same power of x:

$$\begin{bmatrix} c_0 \\ c_1 r x \\ ½ c_2 r^2 x^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} r c_0 \\ r c_1 x \\ ½ c_2 r x^2 \\ \vdots \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \text{(arbitrary)} \\ 0 \\ \vdots \end{bmatrix}$$

Consequently, $\log f(x) = c_1 x$ and $f(x) = \exp(c_1 x)$. The notion that $f(x)$ declines with x requires c_1 to be negative, that is $c_1 = -a$, where $a > 0$. If other solutions exist, they must either not have a logarithm or not have a Taylor series. One solution without a logarithm is $f(x) = 0$, but it does not satisfy the boundary condition. Any solutions without a Taylor series must be singular at the origin, and could not satisfy the boundary condition, either. Consequently, the decaying exponential is the only "physical" solution.

Note 2. The velocity in the quasi-constant-Q model contains the factor:

$$g\left(\frac{\omega}{\omega_0}\right) = \frac{\log\left(\frac{\omega}{\omega_0}\right)}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)} \quad \text{or} \quad g(y) = \frac{\log(y^{½})}{(1-y)} = \frac{-½ \log(y)}{(y-1)} \quad \text{with} \quad y = \left(\frac{\omega}{\omega_0}\right)^2$$

Both the numerator and denominator tend to zero as $y \to 1$, so analysis is needed assure that their ratio is finite. Using the fact that $\lim_{y\to 1} (\log y)/(y-1) = 1$, we find that $\lim_{y\to 1} g(y) = -½$. Consequently, the velocity formula is well-behaved at $\omega = \omega_0$.