Time and Frequency Domain Inversions for a Bateman-type Attenuation Operator

William Menke, March 4, 2021

(Drawing upon work that Levi Borevitz did for his 2020 Summer Intern Project)

I consider a causal Bateman-style attenuation operator with quality factor of the form:

$$Q(f) = Q_0 \left(\frac{f}{f_0}\right)^{\alpha}$$

where Q_0 is the quality factor at reference frequency f_0 and $0 < \alpha < 1$ is an exponent.

The following text is from Levi Borevitz's Summer Intern Project (Borevitz and Menke, 2020):

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Starting with $u_0^{obs}(t)$ we take its Fourier transform

$$\hat{u}_0^{obs}(\omega) = \int_{-\infty}^{+\infty} u_0^{obs}(t) \;\; \exp\{-i\omega t\}\, dt$$

The unattenuated pulse $\hat{u}_0(\omega)$ is changed into the attenuated pulse $\hat{u}_1(\omega)$ through multiplication by the Bateman function $B(A, t_0, t_0^*, \alpha, \omega)$, which we abbreviate as $B(\omega)$:

$$\hat{u}_0^{pre}(\omega) = B(\omega)\, \hat{u}_0^{obs}(\omega)$$

$$\text{where} \quad B(\omega) = A \;\exp\{-a\} \exp\{-i\omega(\varphi + t_0)\} \quad \text{and}$$

$$a = ½\omega t^* \quad \text{and} \quad \varphi = bc\left(\frac{\omega}{\omega_0}\right)^{-\alpha} \quad \text{and} \quad b = ½t_0^* \quad \text{and} \quad c = \cot(½\pi\alpha)$$

The attenuated pulse $u_0^{pre}(t)$ is the inverse Fourier transform of $\hat{u}_0^{pre}(\omega)$:

$$u_0^{pre}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}_0^{pre}(\omega, A) \exp\{+i\omega t\}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(\omega, A)\, \hat{u}_0^{obs}(\omega) \exp\{+i\omega t\}\, d\omega$$

Differentiation with respect to parameters $A,\ t_0,\ t_0^*$ and α is performed inside the integral. For example,

$$\frac{\partial u_0^{pre}}{\partial A} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\partial B}{\partial A}\, \hat{u}_0(\omega) \exp\{+i\omega t\}\, d\omega$$

Here, the derivative $\partial u_1/\partial A$ is understood to be a function of time t and the derivative $\partial B/\partial A$ is understood to be a function of frequency ω. Derivatives of $B(\omega)$ with respect to $A,\ t_0,\ t_0^*$ and α are:

$$\frac{\partial B}{\partial A} = \exp\{-a\}\exp\{-iw(\varphi + t_0)\}$$

$$\frac{\partial B}{\partial\, t_0} = -A\, i\, \omega\ \exp\{-a\}\exp\{-iw(\varphi + t_0)\}$$

$$\frac{\partial B}{\partial t_0^*} = -A\left(\frac{\omega}{\omega_0}\right)^{-\alpha}\ \exp\{-a\}\exp\{-iw(\varphi + t_0)\}\,(\pi\, f + \frac{1}{2} i\, \omega\ \cot{(c)})$$

$$\frac{\partial B}{\partial \alpha} = A\ \exp\{-a\}\exp\{-iw(\varphi + t_0)\}(\frac{1}{4} i\, \omega\, \pi t_0^* f\left(\frac{\omega}{\omega_0}\right)^{-\alpha} \ln\left(\frac{\omega}{\omega_0}\right) + (\ \pi t_0^*\left(\frac{\omega}{\omega_0}\right)^{-\alpha} \csc^2\,(½\pi\alpha)$$
$$+ \mathrm{c}\ln\left(\frac{\omega}{\omega_0}\right))\ )$$

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I recoded Levi's MATLAB software into PYTHON. The function bateman() takes unattenuated and undelayed $u_0(t)$, together with the four parameters $A,\ t_0,\ t_0^*$ and α, and return an attenuated and delayed pulse $u(t)$ and the four derivatives $\partial u/\partial A$, $\partial u/\partial t_0$, $\partial u/\partial t_0^*$ and $\partial u/\partial\alpha$ (Figure 1).

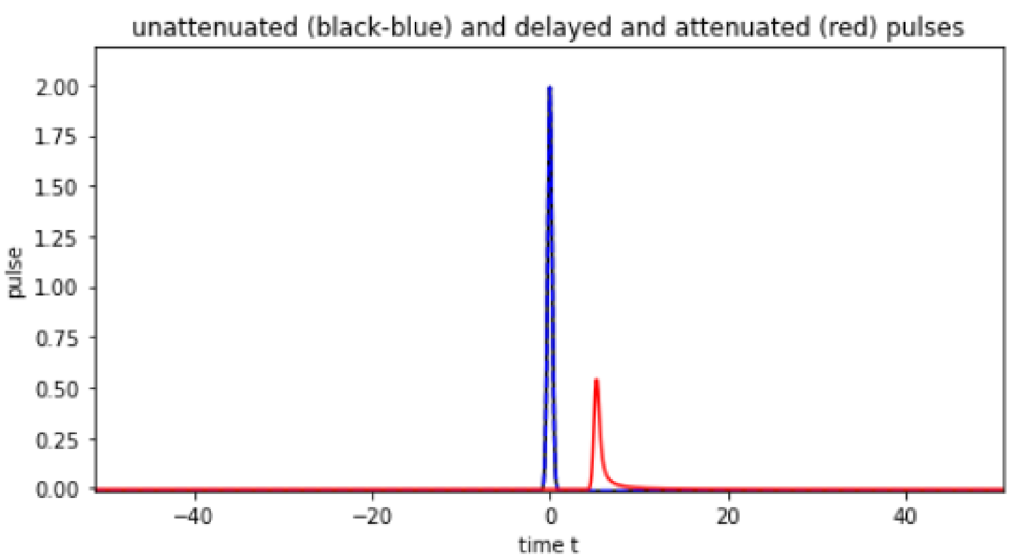
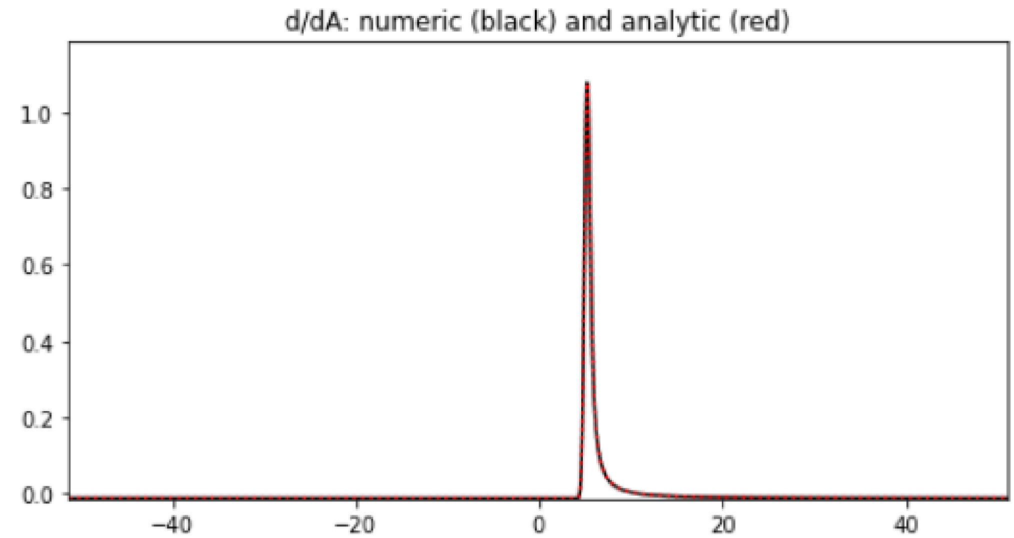


Figure 1A. Unattenuated and undelayed pulse $u_0(t)$ (blue) and attenuated and delayed pulse $u(t)$ (red). This case if for $A = ½,\ t_0 = 5,\ t_0^* = ½$ and $\alpha = 0.4$ using a Gaussian pulse of standard deviation $\sigma_t = 0.25$ and a $f_0 = ½(2\pi\sigma_t)^{-1}$ (one half the bandwidth of the pulse).

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A: maximum fractional deviation 9.041567e-09
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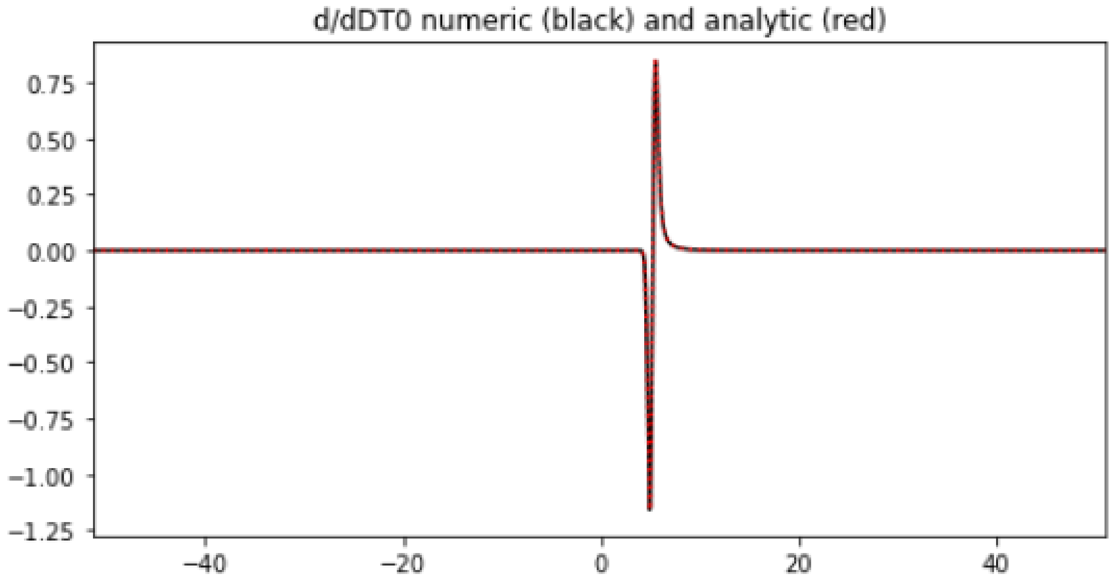
Figure 1B. The derivative $\partial u/\partial A$, computed using the analytic formula in bateman() (black) and via the finite difference method (red).

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DT0: maximum fractional deviation 3.235705e-08
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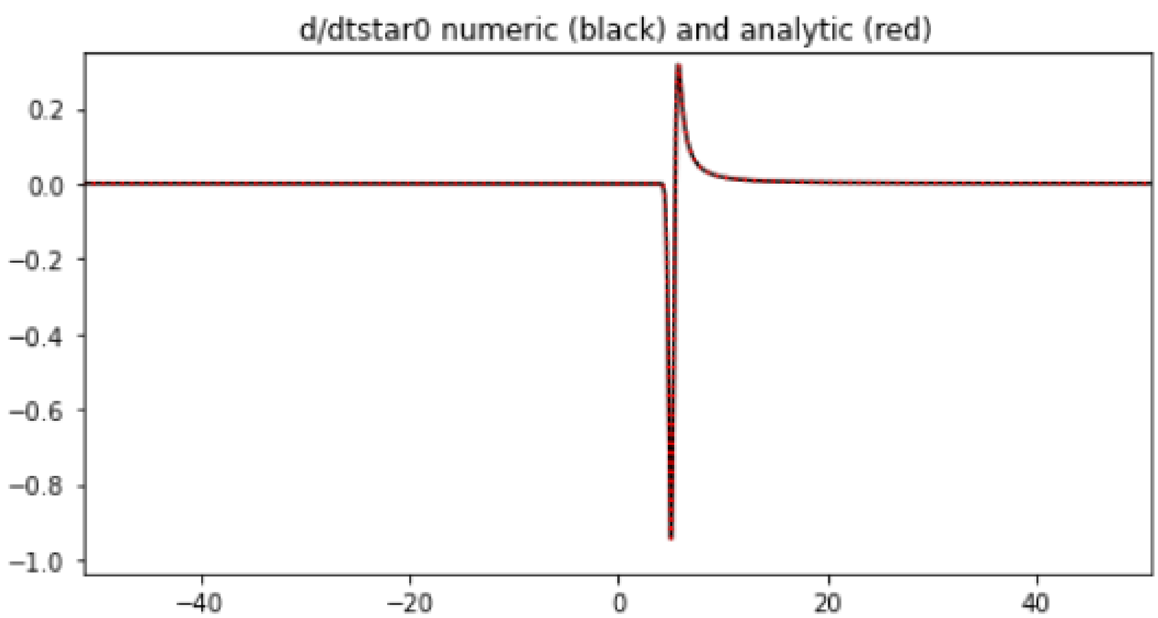
Figure 1C. The derivative $\partial u/\partial t_0$, computed using the analytic formula in bateman() (black) and via the finite difference method (red).

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tstar: maximum fractional deviation 1.906717e-08
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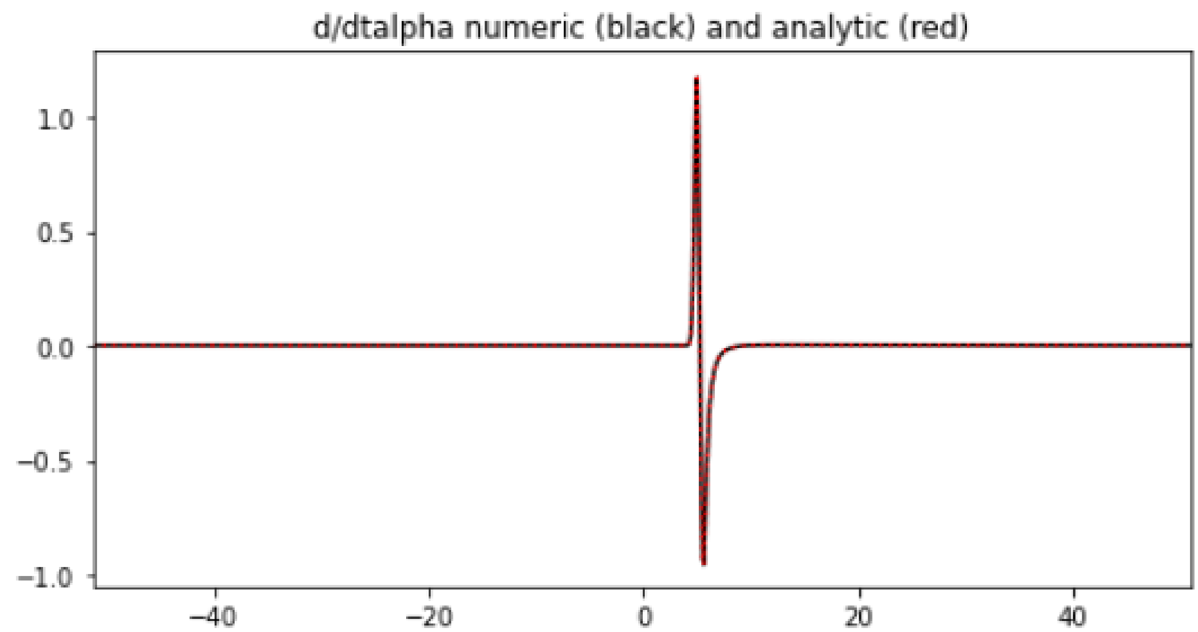
Figure 1D. The derivative $\partial u/\partial t_0^*$, computed using the analytic formula in bateman() (black) and via the finite difference method (red).

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alpha: maximum fractional deviation 4.735560e-08
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Figure 1E. The derivative $\partial u/\partial \alpha$, computed using the analytic formula in bateman() (black) and via the finite difference method (red).

In all cases, the analytic calculation of the derivative matches the result of a finite difference calculation very closely. Note that the shape of the derivatives $\partial u/\partial t_0$ and $-\partial u/\partial \alpha$ are similar to one another, implying that t_0 and α will trade off in an inversion.

I also coded frequency-domain versions of the derivatives in the function fbateman(): $\partial s/\partial A$, $\partial s/\partial t_0^*$ and $\partial s/\partial \alpha$ where $s(f) = |\hat{u}(f)|$ is the amplitude spectral density (a.s.d.) (Figure 2).

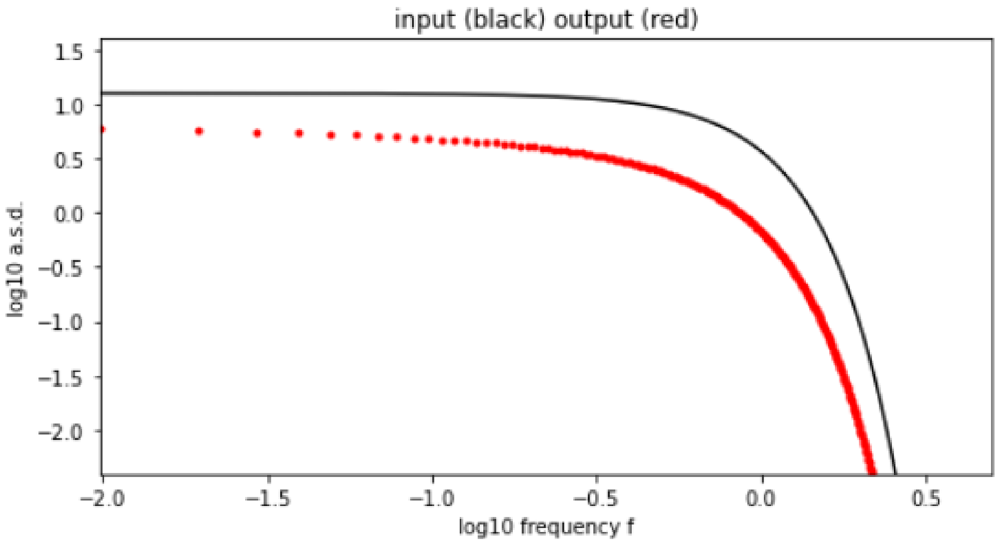

Figure 2A. Amplitude spectral density (a.s.d) of unattenuated and undelayed pulse $u_0(t)$ (black) and attenuated and delayed pulse $u(t)$ (red). This case if for $A = ½$, $t_0 = 5$, $t_0^* = ½$ and $\alpha = 0.4$ using a Gaussian pulse of standard deviation $\sigma_t = 0.25$ and a $f_0 = ½(2\pi\sigma_t)^{-1}$ (one half the bandwidth of the pulse).

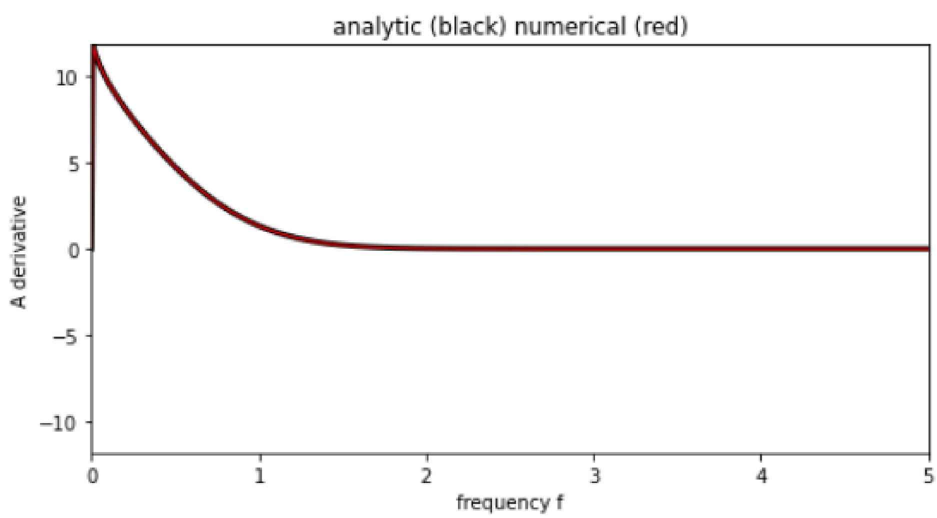

Figure 2B. The derivative $\partial s/\partial A$, computed using the analytic formula in fbateman() (black) and via the finite difference method (red).

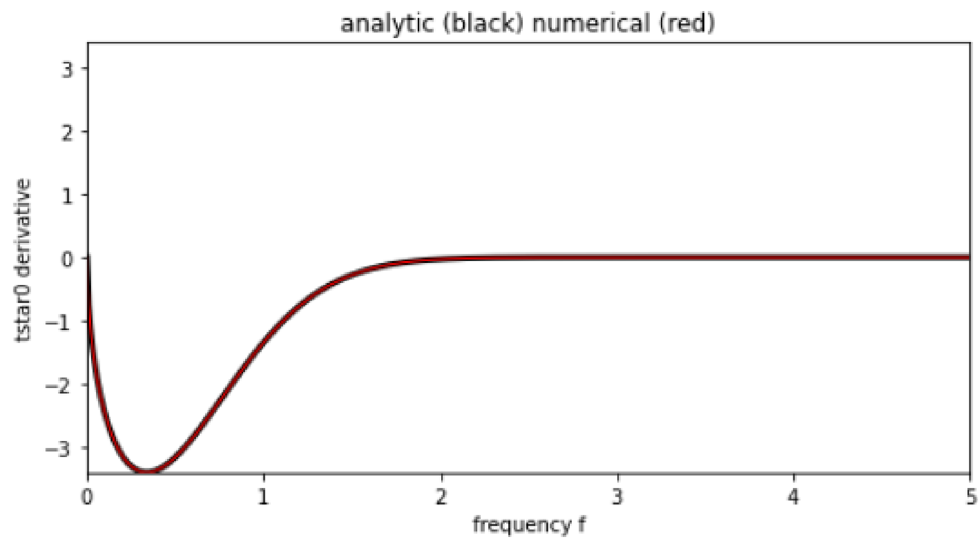

Figure 2C. The derivative $\partial s/\partial t_0^*$, computed using the analytic formula in fbateman() (black) and via the finite difference method (red).

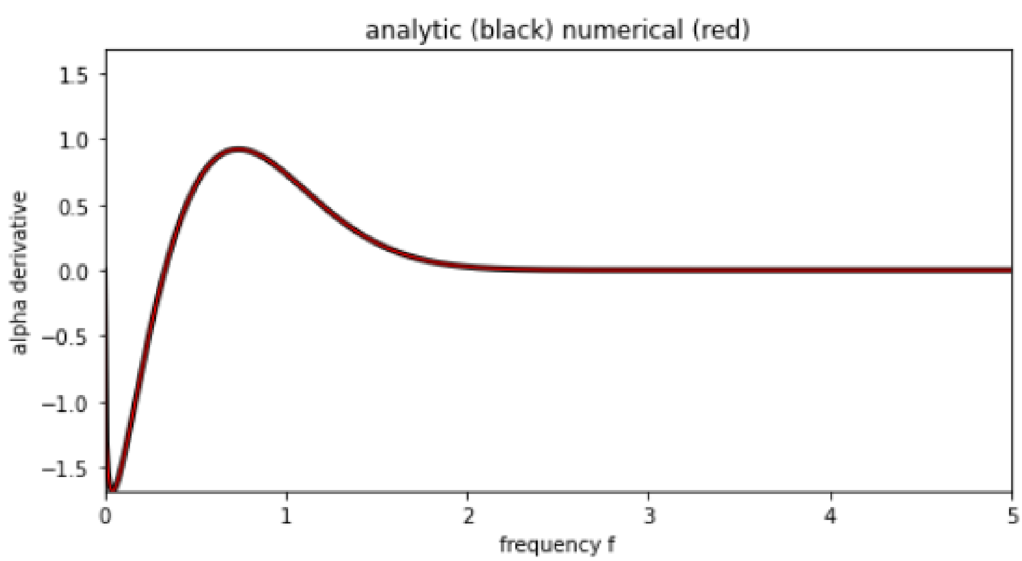

Figure 2D. The derivative $\partial s/\partial \alpha$, computed using the analytic formula in fbateman() (black) and via the finite difference method (red).

In all cases, the analytic calculation of the derivative matches the result of a finite difference calculation very closely. The shapes are sufficiently dissimilar that an inversion should work reasonably well.

Finally, I coded a test inversion that consisted of these steps:

(1) Estimate lag t_0 by cross-correlating pulse $u(t)$ and $u_0(t)$.

(2) Estimate A by regressing $u(t)$ against $u_0(t - t_0)$.

(3) Estimate t_0^* by regressing $\ln s(f)$ against $\ln s_0(f)$ in the frequency band (f_2, f_2) at fixed $\alpha =$ ½.

(4) Refine estimates of A, t_0^* and α using fbateman() and Newton's method.

(5) Refine estimates of A, t_0 and t_0^* using bateman() and Newton's method

(6) Refine estimates of A, t_0, t_0^* and α using bateman() and Newton's method and prior information that t_0 shouldn't change much.

The inversion produces accurate results (Table 1 and Figure 3).

Table 1. Error improvement for each step in the inversion process.

	A	DT0	tstar0	alpha	Error
true	0.5000	5.0000	0.5000	0.4000	0.0000000
lagged	1.0000	5.3000	0.0000	0.5000	8.5722248
rgress	0.3073	5.3000	0.0000	0.5000	0.1401392
logasd	0.3073	5.3000	0.6075	0.5000	0.6139334
asd	0.5000	5.3000	0.5000	0.4000	0.6246414
pass1	0.5000	5.0000	0.5000	0.4000	0.0000000
pass2	0.5000	5.0000	0.5000	0.4000	0.0000000

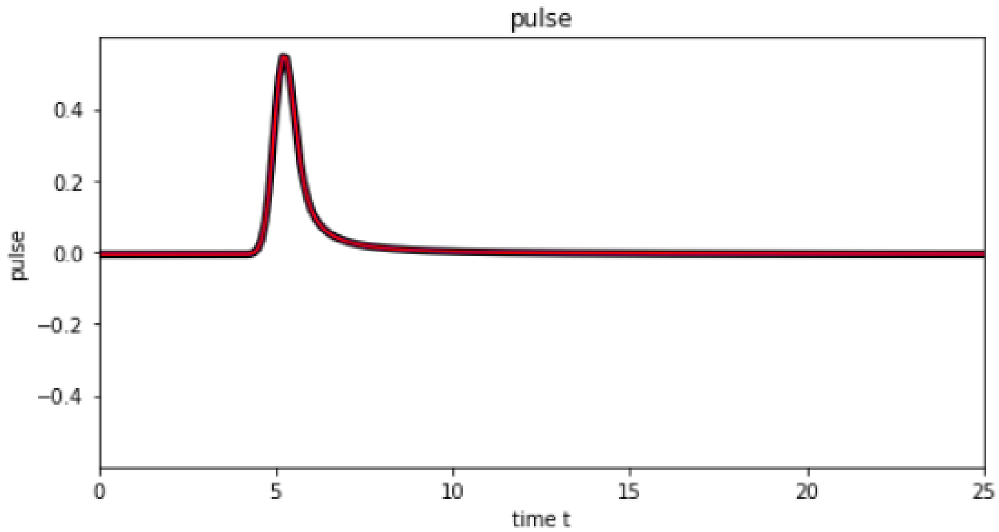

Fig. 3A. Results of inversion, true pulse $u(t)$ (black) and estimated pulse $u(t)$ (red).

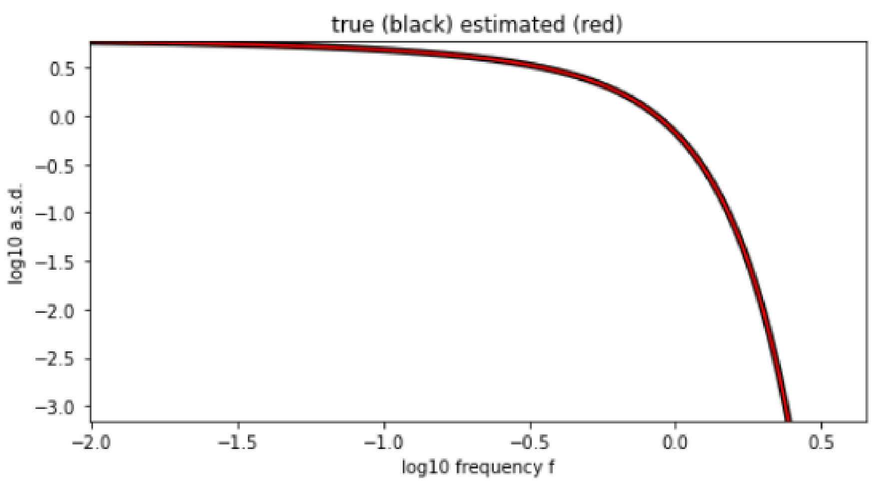

Fig. 3B. Results of inversion, amplitude spectral densities of true pulse $u(t)$ (black) and estimated pulse $u(t)$ (red).

Borevitz, Levi and Menke, William (2020), Estimating Amplitude, Delay, Attenuation and its Frequency Dependence of Seismic Waves Simultaneously with Applications to Alaska, 2020 Summer Intern Project Report, Lamont-Doherty Earth Observatory of Columbia University (Palisades NY), 20pp.