Sensitivity Kernel for Ray Amplitude for Initially Straight Rays

Bill Menke, April 5, 2021

Ray equation for ray with position $\mathbf{x}(s)$, arc-length s, and slowness $u(\mathbf{x})$

$$\frac{d}{ds}u\frac{d\mathbf{x}}{ds} = \nabla u$$

$$\frac{d^2\mathbf{x}}{ds^2} + \frac{d\mathbf{x}}{ds}\left(\frac{d\mathbf{x}}{ds}\cdot u^{-1}\nabla u\right) = u^{-1}\nabla u$$

Define: $f \equiv \ln u$ so $\nabla f = u^{-1}\nabla u$ and the ray equation becomes:

$$\frac{d^2\mathbf{x}}{ds^2} + \frac{d\mathbf{x}}{ds}\left(\frac{d\mathbf{x}}{ds}\cdot \nabla f\right) = \nabla f$$

Case 1: Cartesian coordinates (x, y):

Spatial variation of f and its gradient, in Cartesian coordinated, with ε a small parameter:

$$f = f_0 + \varepsilon b_y(x)y + ½\varepsilon C_{yy}(x)y^2$$

$$\nabla f = \varepsilon \begin{bmatrix} db_x/dx + \left(dC_{yy}/dx\right)y \\ b_y + C_{yy}y \end{bmatrix}$$

Position vector for a ray that is initially parallel to the x-axis written inters of an unperturbed part and a perturbed part. Note that since $d\mathbf{x}/ds$ is a unit vector that initially is $[1 \quad 0]^T$, only its y-component can have a first order perturbation.

$$\mathbf{x} = \begin{bmatrix} x_0(s) \\ y_0 + \varepsilon y_1(s) \end{bmatrix}$$

$$\frac{d\mathbf{x}}{ds} = \begin{bmatrix} dx_0/ds \\ \varepsilon(dy_1/ds) \end{bmatrix}$$

$$\frac{d^2\mathbf{x}}{ds^2} = \begin{bmatrix} d^2x_0/ds^2 \\ \varepsilon(d^2y_1/ds^2) \end{bmatrix}$$

Second term on l.h.s. of ray equation:

$$\frac{d\mathbf{x}}{ds}\cdot\nabla f = \begin{bmatrix} dx_0/ds \\ \varepsilon(dy_1/ds) \end{bmatrix}\cdot\varepsilon\begin{bmatrix} db_x/dx + \left(dC_{yy}/dx\right)y_0 + \varepsilon\left(dC_{yy}/dx\right) \\ b_y + C_{yy}y_0 + \varepsilon C_{yy}y_1 \end{bmatrix}$$

$$= \varepsilon(dx_0/ds)\left(db_x/dx + \left(dC_{yy}/dx\right)y_0\right) + O(\varepsilon^2)$$

$$\frac{d\mathbf{x}}{ds}\left(\frac{d\mathbf{x}}{ds}\cdot\nabla f\right) = \begin{bmatrix} \varepsilon(dx_0/ds)^2 \\ O(\varepsilon^2) \end{bmatrix}\left(db_x/dx + \left(dC_{yy}/dx\right)y_0\right) + O(\varepsilon^2)$$

Ray equation written to first order in ε:

$$\begin{bmatrix} d^2x_0/ds^2 \\ \varepsilon(d^2y_1/ds^2) \end{bmatrix} + \begin{bmatrix} \varepsilon(dx_0/ds)^2 \\ O(\varepsilon^2) \end{bmatrix} \left(db_x/dx + \left(dC_{yy}/dx\right)y_0\right) = \varepsilon \begin{bmatrix} db_x/dx + \left(dC_{yy}/dx\right)y_0 \\ b_y + C_{yy}y_0 \end{bmatrix}$$

Zeroth order ray equation:

$$\varepsilon^0: \quad \frac{d^2x_0}{ds^2} = 0 \quad \text{or} \quad x = s \quad \text{and} \quad dx/ds = 1$$

Zeroth order ray equation for two rays, initially at y_0^A and y_0^B, respectively:

$$\varepsilon^1: \quad \frac{d^2y_1^A}{ds^2} = b_y + C_{yy}y_0^A$$

$$\varepsilon^1: \quad \frac{d^2y_1^B}{ds^2} = b_y + C_{yy}y_0^B$$

Equation for the difference, Δy and its solution:

$$\Delta y = y^B - y^A = (y_0^B - y_0^A) + \varepsilon(y_1^B - y_1^A) = \Delta y_0 + \varepsilon\Delta y_1$$

$$\varepsilon^1: \quad \frac{d^2\Delta y_1}{ds^2} = C_{yy}\Delta y_0$$

$$\frac{d\Delta y_1}{ds} = \Delta y_0 \int_0^s C_{yy}(s')\, ds'$$

$$\Delta y_1 = \Delta y_0 \int\limits_0^s \left[\int_0^{s'} C_{yy}(s")\, ds"\right] ds'$$

Solution for constant C_{yy}:

$$\Delta y_1 = ½\Delta y_0 C_{yy} s^2$$

An approximation that removes one integral. Using integration by parts we can write:

$$\int_0^{s'} u\, dv" = uv - \int_0^{s'} v\, du \quad \text{with} \quad u = C_{yy} \;\text{ and }\; dv = 1\, ds$$

$$\int_0^{s'} C_{yy}\, 1\, ds" = s'^{C_{yy}(s')} - \int_0^{s'} s\frac{dC_{yy}}{ds}\, ds" \approx s'C_{yy}(s') \quad \text{when} \quad \frac{dC_{yy}}{ds} \text{ small}$$

$$\Delta y_1 \approx \Delta y_0 \int\limits_0^s s'C_{yy}(s')\, ds'$$

An approximate solution (to first order in ε) for Δy involving an exponential:

$$\varepsilon\frac{d\Delta y_1}{ds} = \frac{d\Delta y}{ds} = \varepsilon\Delta y_0 \int_0^s C_{yy}(s')\, ds' \approx \varepsilon\Delta y \int_0^s C_{yy}(s')\, ds'$$

$$\frac{d\Delta y}{ds} \approx \varepsilon \Delta y \int_0^s C_{yy}(s')\, ds'$$

$$\frac{d\Delta y}{\Delta y} \approx \varepsilon \int_0^{s'} C_{yy}(s")\, ds"\, ds'$$

$$\ln \Delta y = \varepsilon \int_0^s \int_0^{s'} C_{yy}(s")\, ds"\, ds'$$

$$\Delta y = \Delta y_0 \exp\left\{\varepsilon \int_0^s \int_0^{s'} C_{yy}(s")\, ds"\, ds'\right\} \approx \Delta y_0 \exp\left\{\varepsilon \int_0^s s' C_{yy}(s')\, ds'\right\}$$

Proof that the approximate solution matches the exact solution to first order in ε:

$$\Delta y \approx \Delta y_0 \left(1 + \varepsilon \int_0^s \int_0^{s'} C_{yy}(s")\, ds"\, ds'\right)$$

$$\Delta y - \Delta y_0 = \varepsilon \Delta y_1 = \varepsilon \Delta y_0 \int_0^s \int_0^{s'} C_{yy}(s")\, ds"\, ds'$$

$$\Delta y_1 = \Delta y_0 \int_0^s \int_0^{s'} C_{yy}(s")\, ds"\, ds'$$

Given a point heterogeneity $C_{yy} = C\delta(s - s_0)$, then for $s > s_0$:

$$\Delta y_1 = C\Delta y_0 \int\limits_0^s \left[\int_0^{s'} \delta(s" - s_0)\ \ ds"\right] ds' = C\Delta y_0 \int\limits_0^s H(s' - s_0)\, ds' = C\Delta y_0 (s - s_0)$$

Thus, Δy_1 grows linearly with distance from the heterogeneity. This effect corresponds to ray divergence and convergence.

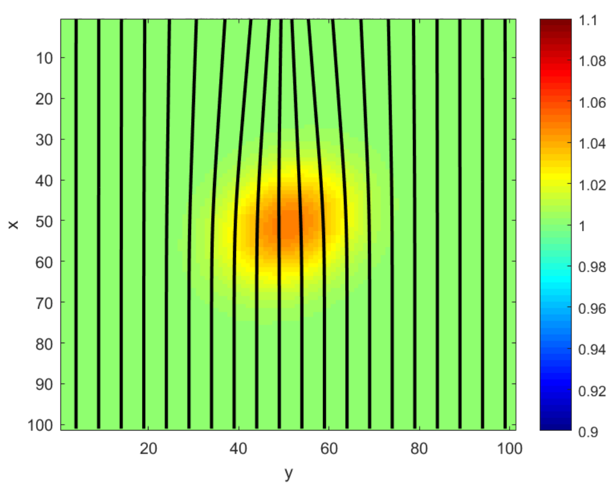

Figure 1. Exact ray tracing through a medium with a Gaussian slowness anomaly.

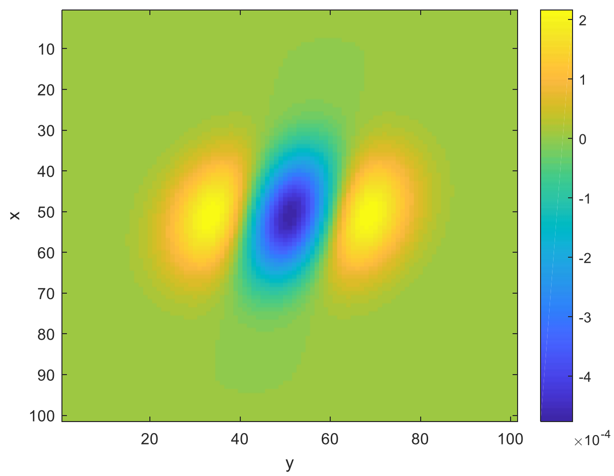

Figure 2. C_{yy} for the medium with a Gaussian slowness anomaly.

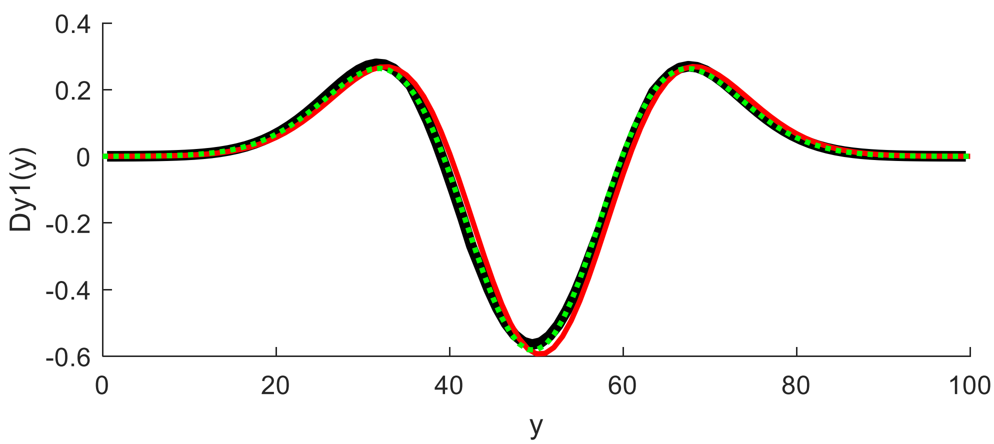

Figure 3. Exact Δy_1 (black), the first order approximation (red) and the single-integral approximation (green) at the ray endpoints. The correspondence is deceptively good. Placing the anomaly closer to or further from to the ray endpoints leads to misprediction of the amplitude of variation by a factor of two or more.

Case 2: Polar coordinates (r, θ):

Spatial variation of f and its gradient, in polar coordinates, with ε a small parameter:

$$f = f_0 + \varepsilon b_\theta(r)\theta + ½\varepsilon C_{\theta\theta}(r)\theta^2$$

$$\nabla f = \begin{bmatrix} \partial f / \partial r \\ r^{-1} \partial f / \partial \theta \end{bmatrix} = \varepsilon \begin{bmatrix} db_x/dr + (dC_{\theta\theta}/dr)\theta \\ r^{-1} b_y + r^{-1} C_{\theta\theta}\theta \end{bmatrix}$$

Position vector for a ray that is initially parallel to the r-axis written inters of an unperturbed part and a perturbed part. Note that since $d\mathbf{x}/ds$ is a unit vector that initially is $[1 \quad 0]^T$, only its θ-component can have a first order perturbation.

$$\begin{bmatrix} \mathrm{r} \\ \theta \end{bmatrix} = \begin{bmatrix} r_0(s) \\ \theta_0 + \varepsilon\theta_1(s) \end{bmatrix} \quad \text{and} \quad d\mathbf{x} = \begin{bmatrix} dr \\ rd\theta \end{bmatrix}$$

$$\frac{d\mathbf{x}}{ds} = \begin{bmatrix} dr_0/ds \\ \varepsilon r_0(d\theta_1/ds) \end{bmatrix}$$

$$\frac{d^2\mathbf{x}}{ds^2} = \begin{bmatrix} d^2r_0/ds^2 \\ \varepsilon r_0(d^2y_1/ds^2) + \varepsilon(dr_0/ds)(d\theta_1/ds) \end{bmatrix}$$

Second term on l.h.s. of ray equation:

$$\frac{d\mathbf{x}}{ds} \cdot \nabla f = \begin{bmatrix} dr_0/ds \\ \varepsilon r_0(d\theta_1/ds) \end{bmatrix} \cdot \varepsilon \begin{bmatrix} db_x/dr + (dC_{\theta\theta}/dr)\theta_0 + \varepsilon(dC_{\theta\theta}/dr)\theta_1 \\ r^{-1}b_y + r^{-1}C_{\theta\theta}\theta_0 + \varepsilon r^{-1}C_{\theta\theta}\theta_1 \end{bmatrix}$$

$$= \varepsilon(dr_0/ds)(db_x/dr + (dC_{\theta\theta}/dr)\theta_0) + O(\varepsilon^2)$$

$$\frac{d\mathbf{x}}{ds}\left(\frac{d\mathbf{x}}{ds} \cdot \nabla f\right) = \begin{bmatrix} \varepsilon(dr_0/ds)^2 \\ O(\varepsilon^2) \end{bmatrix} (db_x/dr + (dC_{\theta\theta}/dr)\theta_0) + O(\varepsilon^2)$$

Ray equation written to first order in ε:

$$\begin{bmatrix} d^2r_0/ds^2 \\ \varepsilon r_0(d^2y_1/ds^2) + \varepsilon(dr_0/ds)(d\theta_1/ds) \end{bmatrix} + \begin{bmatrix} \varepsilon(dr_0/ds)^2 \\ O(\varepsilon^2) \end{bmatrix} (db_x/dr + (dC_{\theta\theta}/dr)\theta_0)$$

$$= \varepsilon \begin{bmatrix} db_x/dr + (dC_{\theta\theta}/dr)\theta \\ r^{-1}b_y + r^{-1}C_{\theta\theta}\theta \end{bmatrix}$$

Zeroth order ray equation:

$$\varepsilon^0: \quad \frac{d^2r_0}{ds^2} = 0 \quad \text{or} \quad r_0 = s \quad \text{and} \quad dr_0/ds = 1$$

Zeroth order ray equation for two rays, initially at θ_0^A and θ_0^B, respectively:

$$\varepsilon^1: \; s\frac{d^2\theta_1^A}{ds^2} + \frac{d\theta_1^A}{ds} = b_y + s^{-1}C_{\theta\theta}y_0^A$$

$$\varepsilon^1: \; s\,\frac{d^2\theta_1^B}{ds^2} + \frac{d\theta_1^B}{ds} = b_y + s^{-1}C_{\theta\theta}y_0^B$$

Equation for the difference, Δy and its solution:

$$\Delta\theta = \theta^B - \theta^A = (\theta_0^B - \theta_0^A) + \varepsilon(\theta_1^B - \theta_1^A) = \Delta\theta_0 + \varepsilon\Delta\theta_1$$

$$\varepsilon^1: \; s\frac{d^2\Delta\theta_1}{ds^2} + \frac{d\Delta\theta_1}{ds} = s^{-1}C_{\theta\theta}\Delta\theta_0$$

$$s^2 \frac{d^2 \Delta\theta_1}{ds^2} + s \frac{d\Delta\theta_1}{ds} = C_{\theta\theta} \Delta\theta_0$$

$$s \frac{d}{ds} s \frac{d\Delta\theta_1}{ds} = C_{\theta\theta} \Delta\theta_0$$

$$\Delta\theta_1 = \Delta\theta_0 \int_0^s (s')^{-1} \int_0^{s'} (s")^{-1} C_{\theta\theta} ds"\, ds'$$

Defining $\Delta y_0 \equiv r_0 \Delta\theta_0$ and $\Delta y_1 \equiv r_0 \Delta\theta_1$ and $C_{yy} = r_0^2 C_{\theta\theta}$, we find that:

$$\Delta y_1 = \Delta y_0 \int_0^s (s')^{-1} \int_0^{s'} s" C_{yy} ds" ds'$$

For a constant C_{yy}:

$$\Delta y_1 = \Delta y_0\, C_{yy} \int_0^s (s')^{-1} ½ (s')^2\, ds' = \Delta y_0\, C_{yy} \int_0^s ½ s'\, ds' = ¼ \Delta y_0\, C_{yy} s^2$$

which differs from the Cartesian result by a factor of two.

My thinking on this issue was inspired by Equation 3 of Dalton and Ekstrom (2006), Global models of surface wave attenuation, JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 111, B05317, doi:10.1029/2005JB003997. This paper cites Woodhouse, J. H., and Y. K. Wong (1986), Amplitude, phase and path anomalies of mantle waves, Geophys. J. R. Astron. Soc., 87, 753–773, but I have not read that paper (yet).