In summary, the functions \( f_1(\theta) \) and \( f_2(\theta) \), observed on the interval \( 0 \leq \theta < 2\pi \), are sufficient to uniquely characterize the point heterogeneity, by determining \( \delta \rho / \rho, \delta \beta / \beta, \delta \alpha / \alpha \), and as a bonus, give the velocity ratio \( \beta / \alpha \) of the background medium.

Rondenay (2009, DOI 10.1007/s10712-009-9071-5, Equations 14-15) gives the scattering functions for P wave interacting with a point heterogeneity as:

\[
\begin{align*}
\mathbf{f}^{P \rightarrow P}(x, \theta) &= \rho \left[ 2 \frac{\delta \alpha}{\alpha} + \frac{\delta \beta}{\beta} \left( 2 \left( \frac{\beta}{\alpha} \right)^2 (\cos 2\theta) - 1 \right) \right] \\
&\quad + \frac{\delta \rho}{\rho} \left( 1 + \cos \theta + \left( \frac{\beta}{\alpha} \right)^2 (\cos 2\theta - 1) \right), \\
\mathbf{f}^{P \rightarrow S}(x, \theta) &= \rho \left( \frac{\delta \beta}{\beta} \left( 2 \frac{\beta}{\alpha} \sin 2\theta \right) + \frac{\delta \rho}{\rho} \left( \sin \theta + \frac{\beta}{\alpha} \sin 2\theta \right) \right),
\end{align*}
\]  

(14)

These functions are plotted in Figure 1. The equations can be rearranged:

\[
\begin{bmatrix}
\mathbf{f}_1(\theta) \\
\mathbf{f}_2(\theta)
\end{bmatrix} = \begin{bmatrix}
\rho & 2 & 0 \\
0 & 2r \sin(2\theta) & 1 \cos(\theta) + r \sin(2\theta) \\
2 & 0 & \{1 - r^2\} (\cos(2\theta) - 1)
\end{bmatrix} \begin{bmatrix}
\delta \alpha / \alpha \\
\delta \beta / \beta \\
\delta \rho / \rho
\end{bmatrix}
\]

(15)

with \( r = \beta / \alpha \). Reorganizing this equation with trigonometric functions as right-hand vector:

\[
\begin{bmatrix}
\mathbf{f}_1(\theta) \\
\mathbf{f}_2(\theta)
\end{bmatrix} = \\
\begin{bmatrix}
\left( 2 \frac{\delta \alpha}{\alpha} - 2r^2 \frac{\delta \beta}{\beta} + (1 - r^2) \frac{\delta \rho}{\rho} \right) & \left( \frac{\delta \rho}{\rho} \right) & 0 \\
0 & \left( 2r^2 \frac{\delta \beta}{\beta} + r^2 \frac{\delta \rho}{\rho} \right) & 0 \\
2 & 0 & \left( 2r \frac{\delta \beta}{\beta} + r \frac{\delta \rho}{\rho} \right)
\end{bmatrix} \begin{bmatrix}
1 \cos(\theta) \\
\sin(\theta) \\
\cos(2\theta) \\
\sin(2\theta)
\end{bmatrix}
\]

\[= \mathbf{Mg}\]

Here the angular functions \( g_n(\theta), n = 1 \ldots 5 \) are given by:

\[
\begin{bmatrix}
g_1(\theta) \\
g_2(\theta) \\
g_3(\theta) \\
g_4(\theta) \\
g_5(\theta)
\end{bmatrix} = \\
\begin{bmatrix}
1 \cos(\theta) \\
\cos(\theta) \\
\sin(\theta) \\
\cos(2\theta) \\
\sin(2\theta)
\end{bmatrix}
\]

It can be shown that the angular functions are orthogonal:
\[ \int_0^{2\pi} g_n(\theta) g_m(\theta) \, d\theta = N_n^2 \delta_{nm} \]

Here \( N_n^2 \) is a normalization factor. Applying the integral to the equation yields:

\[
\begin{bmatrix} d_{1n} \\ d_{2n} \end{bmatrix} \equiv N_n^{-2} \left[ \int_0^{2\pi} f_1(\theta) g_n(\theta) \, d\theta \right] = N_n^{-2} \sum_{m=1}^{5} \left[ M_{1m} \right] \int_0^{2\pi} g_n(\theta) g_m(\theta) \, d\theta = \sum_{m=1}^{5} \left[ M_{1m} M_{2m} \right] \delta_{nm} = \left[ M_{2n} \right]
\]

The non-zero elements provide five equations linking \( \frac{\delta \alpha}{\alpha}, \frac{\delta \beta}{\beta} \) and \( \frac{\delta \rho}{\rho} \) to the \( d \)s:

\[
\frac{d_{11}}{2} \frac{\delta \alpha}{\alpha} - 2r^2 \frac{\delta \beta}{\beta} + (1 - r^2) \frac{\delta \rho}{\rho} = d_{12}
\]

\[
\frac{d_{12}}{\rho} = d_{12}
\]

\[
\frac{2r^2 \delta \beta}{\beta} + r^2 \frac{\delta \rho}{\rho} = d_{14}
\]

\[
\frac{d_{14}}{\rho} = d_{23}
\]

\[
2r \frac{\delta \beta}{\beta} + r \frac{\delta \rho}{\rho} = d_{25}
\]

The five equations are solved as follows: From equations 2 and 4:

\[
\frac{d_{12}}{\rho} = d_{12} = d_{23}
\]

If \( r \) is presumed known, then from equations 3 and 5:

\[
\frac{d_{14}}{\beta} = \frac{d_{14} - r^2 d_{12}}{2r^2} = \frac{d_{25} - rd_{12}}{2r}
\]

Alternatively, when \( r \) is presumed unknown, from equations 3 and 5

\[
r^2 \left( 2 \frac{\delta \beta}{\beta} + d_{12} \right) = d_{14} \quad \text{so} \quad r^2 = \frac{d_{14}}{2 \frac{\delta \beta}{\beta} + d_{12}}
\]

\[
r \left( 2 \frac{\delta \beta}{\beta} + d_{12} \right) = d_{25} \quad \text{so} \quad r^2 = \frac{(d_{25})^2}{2 \left( 2 \frac{\delta \beta}{\beta} + d_{12} \right)^2}
\]
\[
\frac{d_{14}}{(2 \frac{\delta \beta}{\beta} + d_{12})} = \frac{(d_{25})^2}{(2 \frac{\delta \beta}{\beta} + d_{12})^2}
\]
\[
d_{14} \left(2 \frac{\delta \beta}{\beta} + d_{12}\right)^2 = (d_{25})^2 \left(2 \frac{\delta \beta}{\beta} + d_{12}\right)
\]
\[
4d_{14} \left(\frac{\delta \beta}{\beta}\right)^2 + (4d_{12}d_{14} - 2(d_{25})^2) \left(\frac{\delta \beta}{\beta}\right) + (d_{14}(d_{12})^2 - (d_{25})^2d_{12}) = 0
\]

\[
A \equiv 4d_{14} \quad \text{and} \quad B \equiv 4d_{12}d_{14} - 2(d_{25})^2 \quad \text{and} \quad C \equiv d_{14}(d_{12})^2 - (d_{25})^2d_{12}
\]
\[
B^2 - 4A = 16(d_{12})^2(d_{14})^2 + 4(d_{25})^4 - 16d_{12}d_{14}(d_{25})^2 - 16(d_{12})^2(d_{14})^2 + 16d_{12}d_{14}(d_{25})^2 = 4(d_{25})^4
\]
\[
\sqrt{B^2 - 4A} = 2(d_{25})^2
\]
\[
\left(\frac{\delta \beta}{\beta}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-4d_{12}d_{14} + 2(d_{25})^2 \pm 2(d_{25})^2}{8d_{14}}
\]

Note that for the $-$ of the $\pm$, \( \left(\frac{\delta \beta}{\beta}\right) = -\frac{1}{2}d_{12} \). We will show that this root is not allowed, since it leads to a singular \( r \). Consequently, the correct root is the $+$ case:
\[
\left(\frac{\delta \beta}{\beta}\right) = \frac{(d_{25})^2 - d_{12}d_{14}}{2d_{14}}
\]

Then from equation 5:
\[
r = \frac{d_{25}}{(2 \frac{\delta \beta}{\beta} + d_{12})}
\]

As anticipated, \( r \) is singular when \( \frac{\delta \beta}{\beta} = -\frac{1}{2}d_{12} \). Finally, from equation 1:
\[
\frac{\delta \alpha}{\alpha} = \frac{1}{2}d_{11} + r^2 \frac{\delta \beta}{\beta} - \frac{1}{2}(1 - r^2) \frac{\delta \rho}{\rho}
\]

Finally, note that the \( P \to P \) interaction (equations 1-3) are sufficient to solve for \( \delta \alpha/\alpha, \delta \beta/\beta \) and \( \delta \rho/\rho \) but not \( r \), and that the \( P \to S \) interaction (equations 4-5) are sufficient to solve for \( \delta \beta/\beta \) and \( \delta \rho/\rho \) but not \( \delta \alpha/\alpha \) or \( r \).

In summary, the functions \( f_1(\theta) \) and \( f_2(\theta) \), observed on the interval \( 0 \leq \theta < 2\pi \), are sufficient to uniquely characterize the point heterogeneity, by determining \( \delta \rho/\rho, \delta \beta/\beta, \delta \alpha/\alpha \), and as a bonus, give the velocity ratio \( \beta/\alpha \) of the background medium.

Note: I have checked the algebra numerically:
Output of a MATLAB script that checks the algebra:

```
>> Eqn14
error in eqn 1:  0.0000
error in eqn 2:  0.0000
error in eqn 3:  0.0000
error in eqn 4:  0.0000
error in eqn 5:  -0.0000
dror true 0.1100 est1 0.1100 est2 0.1100
dbob true 1.0700 est1 1.0700 est2 1.0700 (with correct r)
two versions of B^2-4C: 11.3906 11.3906
r denominator 2.2500 0.0000
dbob true 1.0700 est 1.0700 (with variable r)
r true 0.5774 est 0.5774
daoa true 0.5100 est 0.5100
Total L1 error 0.0000
```
Fig. 1. (Top row) The function $f_1(\theta)$ when (left) only $\delta \alpha / \alpha \neq 0$, (middle) only $\delta \beta / \beta \neq 0$; and (right) only $\delta \rho / \rho \neq 0$. (Bottom row) Same as top row, except for $f_2(\theta)$. Incident P wave propagates from left to right. Positive $f$'s are shown in red, negative in blue.