This is a quick modification to the previous note. Here I consider forward scattering, only.

Summary: the $P \rightarrow P$ forward scattering is sufficient to solve for $\delta \alpha / \alpha$, $\delta \beta / \beta$ and $\delta \rho / \rho$ and the $P \rightarrow S$ forward scattering is sufficient to solve for $\delta \beta / \beta$ and $\delta \rho / \rho$. When both are measured, one can also solve for $r = \beta / \alpha$.

Rondenay (2009, DOI 10.1007/s10712-009-9071-5, Equations 14-15) gives the scattering functions for P wave interacting with a point heterogeneity as $f = Hh$:

$$
\begin{bmatrix}
    f_1(\theta) \\
    f_2(\theta)
\end{bmatrix} = 
\begin{bmatrix}
    \left(2 \frac{\delta \alpha}{\alpha} - 2r^2 \frac{\delta \beta}{\beta} + (1 - r^2) \frac{\delta \rho}{\rho}\right) & \left(2r^2 \frac{\delta \beta}{\beta} + r^2 \frac{\delta \rho}{\rho}\right) & 0 \\
    0 & 0 & \left(2r \frac{\delta \beta}{\beta} + r \frac{\delta \rho}{\rho}\right)
\end{bmatrix}
\begin{bmatrix}
    \frac{1}{\cos(\theta)} \\
    \frac{\sin(\theta)}{\cos(\theta)} \\
    \frac{\cos(2\theta)}{\sin(2\theta)} \\
    \frac{\sin(2\theta)}{\sin(2\theta)}
\end{bmatrix}
\begin{bmatrix}
    h_1 \\
    h_2 \\
    h_4
\end{bmatrix}
$$

with $r = \beta / \alpha$. For forward-scattered waves $-\pi/2 < \theta < +\pi/2$. The above angular functions $h$ are not orthogonal on this interval, but the functions $g_n(\theta)$, $n = 1 \ldots 5$ are:

$$
\begin{bmatrix}
    g_1(\theta) \\
    g_2(\theta) \\
    g_3(\theta) \\
    g_4(\theta) \\
    g_5(\theta)
\end{bmatrix} = 
\begin{bmatrix}
    \frac{1}{\cos(\theta) - c_1 - c_4 \cos(2\theta)} \\
    \frac{\sin(\theta) - c_3 \sin(2\theta)}{\cos(2\theta)} \\
    \frac{\cos(2\theta)}{\sin(2\theta)} \\
    \frac{\sin(2\theta)}{\sin(2\theta)}
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_4 \\
    c_3
\end{bmatrix} = 
\begin{bmatrix}
    \frac{2}{-\pi} \\
    \frac{-4}{3\pi} \\
    \frac{8}{3\pi}
\end{bmatrix}
$$

By orthogonal, I mean $\int_0^{2\pi} g_n(\theta) g_m(\theta) d\theta = N_n^2 \delta_{nm}$ with $N_n$ a normalization constant. I have verified the orthogonality numerically. We find the elements of $G$ by writing $f = Gg$ and equating coefficients in $Hh = Gg$ :

first row:

$$
f_1 = H_{11} h_1 + H_{12} h_2 + H_{14} h_4 = G_{11} g_1 + G_{12} g_2 + G_{14} g_4 =
\begin{align}
    &= G_{11} h_1 + G_{12} (h_2 - c_1 h_1 - c_4 h_4) + G_{14} h_4 \\
    &= G_{12} h_2 + (G_{11} - G_{12} c_1) h_1 + (G_{14} - G_{12} c_4)
\end{align}
$$

$G_{12}$ = $H_{12}$ and $H_{11}$ = $G_{11}$ - $G_{12} c_1$ and $H_{14}$ = $G_{14}$ - $G_{12} c_4$

second row:
\[ f_2 = H_{23} h_3 + H_{25} h_5 = G_{23} g_3 + G_{25} g_5 = G_{23} (h_3 - c_3 h_5) + G_{25} h_5 = G_{23} h_3 + (G_{25} - G_{23} c_3) h_5 \]

\[ G_{23} = H_{23} \quad \text{and} \quad H_{25} = G_{25} - G_{23} c_3 \quad \text{and} \quad G_{25} = H_{25} + H_{23} c_3 \]

We now write the non-zero components of \( \mathbf{G} \) as:

\[ G_{11} = H_{11} + H_{12} c_1 = 2 \left( \frac{\delta \alpha}{\alpha} \right) - 2 r^2 \left( \frac{\delta \beta}{\beta} \right) + (1 - r^2) \left( \frac{\delta \rho}{\rho} \right) + \left( \frac{2}{\pi} \right) \left( \frac{\delta \rho}{\rho} \right) = 2 \left( \frac{\delta \alpha}{\alpha} \right) - 2 r^2 \left( \frac{\delta \beta}{\beta} \right) + (1 + \left( \frac{2}{\pi} \right) - r^2) \left( \frac{\delta \rho}{\rho} \right) \]

\[ G_{12} = H_{12} = \left( \frac{\delta \rho}{\rho} \right) \]

\[ G_{14} = H_{14} + H_{12} c_4 = 2 r^2 \left( \frac{\delta \beta}{\beta} \right) + r^2 \left( \frac{\delta \rho}{\rho} \right) + \left( \frac{4}{3 \pi} \right) \left( \frac{\delta \rho}{\rho} \right) = 2 r^2 \left( \frac{\delta \beta}{\beta} \right) + \left( r^2 + \left( \frac{4}{3 \pi} \right) \right) \left( \frac{\delta \rho}{\rho} \right) \]

\[ G_{23} = H_{23} = \left( \frac{\delta \rho}{\rho} \right) \]

\[ G_{25} = H_{25} + H_{23} c_3 = 2 r \left( \frac{\delta \beta}{\beta} \right) + r \left( \frac{\delta \rho}{\rho} \right) + \left( \frac{8}{3 \pi} \right) \left( \frac{\delta \rho}{\rho} \right) = 2 r \left( \frac{\delta \beta}{\beta} \right) + \left( r + \left( \frac{8}{3 \pi} \right) \right) \left( \frac{\delta \rho}{\rho} \right) \]

Applying the integral to the equation \( \mathbf{f} = \mathbf{Gg} \) yields:

\[
\begin{bmatrix}
\int d_{1n}^1 \\
\int d_{2n}^1 
\end{bmatrix} = N_n^{-2} \int_0^{2\pi} \int_0^{\pi} f_1(\theta) g_n(\theta) \, d\theta
\]

\[
N_n^{-2} \sum_{m=1}^{5} G_{1m}^m \int_0^{2\pi} g_n(\theta) g_m(\theta) \, d\theta = \sum_{m=1}^{5} G_{1m}^m \delta_{nm} = \left[ G_{1n}^1 \right]
\]

The non-zero elements provide five equations linking \( \frac{\delta \alpha}{\alpha} \), \( \frac{\delta \beta}{\beta} \) and \( \frac{\delta \rho}{\rho} \) to the \( ds \):

\[
2 \left( \frac{\delta \alpha}{\alpha} \right) - 2 r^2 \left( \frac{\delta \beta}{\beta} \right) + (1 + \left( \frac{2}{\pi} \right) - r^2) \left( \frac{\delta \rho}{\rho} \right) = d_{11}
\]

\[
\left( \frac{\delta \rho}{\rho} \right) = d_{12}
\]
\[ 2r^2 \left( \frac{\delta \beta}{\beta} \right) + \left( r^2 + \left( \frac{4}{3\pi} \right) \right) \left( \frac{\delta \rho}{\rho} \right) = d_{14} \]
\[ \left( \frac{\delta \rho}{\rho} \right) = d_{23} \]
\[ 2r \left( \frac{\delta \beta}{\beta} \right) + \left( r + \left( \frac{8}{3\pi} \right) \right) \left( \frac{\delta \rho}{\rho} \right) = d_{25} \]

Let’s assume that \( r \) is known. Then, the top three equations give:
\[ \left( \frac{\delta \rho}{\rho} \right) = d_{12} \]
\[ \left( \frac{\delta \beta}{\beta} \right) = \frac{d_{14} - \left( r^2 + \left( \frac{4}{3\pi} \right) \right) d_{12}}{2r^2} \]
\[ \left( \frac{\delta \alpha}{\alpha} \right) = \frac{d_{11} + 2r^2 \left( \frac{\delta \beta}{\beta} \right) - \left( 1 + \frac{2}{\pi} \right) - r^2 \right)}{2} d_{12} \]

And from the bottom two equations:
\[ \left( \frac{\delta \rho}{\rho} \right) = d_{23} \]
\[ \left( \frac{\delta \beta}{\beta} \right) = \frac{d_{25} - \left( r + \left( \frac{8}{3\pi} \right) \right) d_{23}}{2r} \]

I have checked these five equations numerically.

So, the \( P \rightarrow P \) interaction (equations 1-3) are sufficient to solve for \( \delta \alpha / \alpha, \delta \beta / \beta \) and \( \delta \rho / \rho \) and that the \( P \rightarrow S \) interaction (equations 4-5) are sufficient to solve for \( \delta \beta / \beta \) and \( \delta \rho / \rho \). In order to solve for \( r \), one must eliminate it from equations 3 and 5, and then solve the resulting equation for \( \delta \beta / \beta \):

\[ r^2 = \frac{d_{14} - \left( \frac{4}{3\pi} \right) d_{12}}{2 \left( \frac{\delta \beta}{\beta} \right) + d_{12}} \]
\[ r = \frac{d_{25} - \left( \frac{8}{3\pi} \right) d_{12}}{2 \left( \frac{\delta \beta}{\beta} \right) + d_{12}} \]
Eliminating $r$ yields a quadratic equation for $\delta \beta / \beta$:

$$\frac{d_{14} - \left( \frac{4}{3\pi} \right) d_{12}}{2 \left( \frac{\delta \beta}{\beta} \right) + d_{12}} = \frac{\left( d_{25} - \left( \frac{8}{3\pi} \right) d_{12} \right)^2}{\left( 2 \left( \frac{\delta \beta}{\beta} \right) + d_{12} \right)^2}$$

$$(d_{14} - \left( \frac{4}{3\pi} \right) d_{12}) \left( 2 \left( \frac{\delta \beta}{\beta} \right) + d_{12} \right)^2 = \left( 2 \left( \frac{\delta \beta}{\beta} \right) + d_{12} \right) \left( d_{25} - \left( \frac{8}{3\pi} \right) d_{12} \right)^2$$

$$\left( \frac{\delta \beta}{\beta} \right)^2 + c \left( \frac{\delta \beta}{\beta} \right) + (d_{12})^2 - cd_{12} = 0$$

so $A = 4$ and $B = (4d_{12} - 2c)$ and $C = ((d_{12})^2 - cd_{12})$

The discriminant, $D^2 = B^2 - 4AC$ is

$$D^2 = 16(d_{12})^2 + 4c^2 - 16cd_{12} - 16(d_{12})^2 + 16cd_{12} = 4c^2 \text{ and } D = 2c$$

and the solution is

$$\left( \frac{\delta \beta}{\beta} \right) = \frac{-4d_{12} + 2c \pm 2c}{8}$$

The $-d_{12}/2$ solution is unphysical, because it leads to a singular value of $r$ when inserted into the original equations. Hence:

$$\left( \frac{\delta \beta}{\beta} \right) = \frac{c - d_{12}}{2}$$

The quantity $\delta \alpha / \alpha$ can then be determined using the first equation. Thus, when both P and S wave forward scattering is measured, the data can also determine $r$.

I’ve not checked the last half of this note.