Further Thoughts on Second Derivative in Amplitude Tomography

Bill Menke, June 30, 2021

Amplitude Tomography requires that the derivative d^2f/dx^2 where $f = \ln(s)$ be approximated as a linear operator acting on s. We use the approximation $\ln(1+t) \approx t$, valid when $|t| \ll 1$ to develop such a relationship. Suppose that $s = s_0 + \Delta s$, where s_0 is a constant, average value and Δs is spatially varying and small compared to s_0. Then:

$$f = \ln(s_0 + \Delta s) = \ln\left[s_0\left(1 + \frac{\Delta s}{s_0}\right)\right] = \ln(s_0) + \ln\left(1 + \frac{\Delta s}{s_0}\right) \approx \ln(s_0) + \frac{\Delta s}{s_0}$$

Her, we have uses the identity $\ln(ab) = \ln(a) + \ln(b)$. Because only Δs is spatially variable:

$$\frac{d^2f}{dx^2} = \frac{1}{s_0}\frac{d^2\Delta s}{dx^2} = \frac{1}{s_0}\frac{d^2s}{dx^2}$$

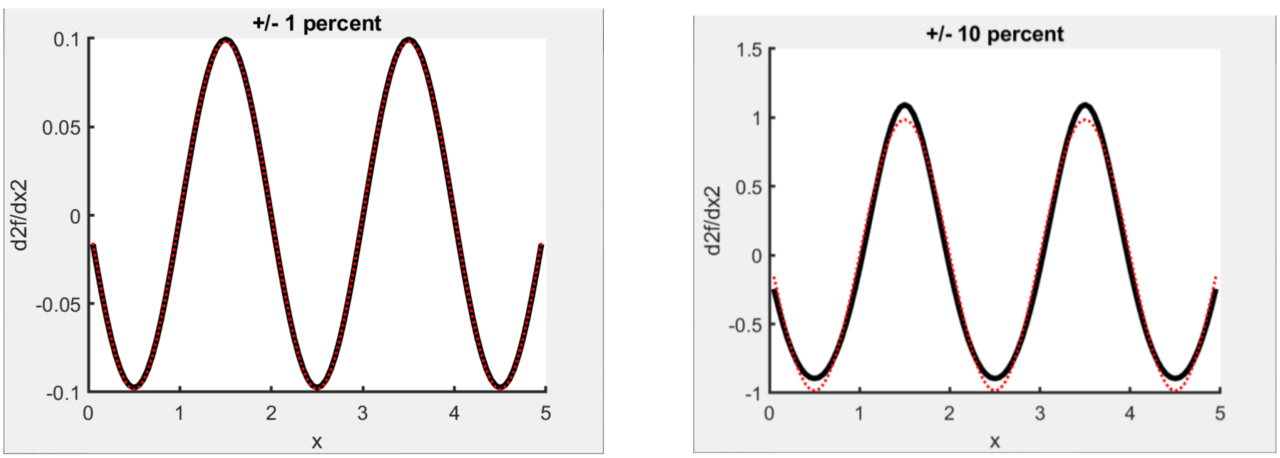

Figure: Comparison of exact derivative d^2f/dx^2 (black) with one based on the approximation (red), for signals with $\pm 1\%$ (top) and $\pm 10\%$ (bottom) fluctuation.

In practice, one computes the matrix of second derivatives (which is symmetric):

$$\mathbf{D} = \begin{bmatrix} \frac{\partial^2 s}{\partial x^2} & \frac{\partial^2 s}{\partial x \partial y} \\ \frac{\partial^2 s}{\partial x \partial y} & \frac{\partial^2 s}{\partial y^2} \end{bmatrix}$$

But one needs to know the derivative $d^2s/d\xi^2$, where ξ is the direction perpendicular to the ray, as shown in Figure 2.

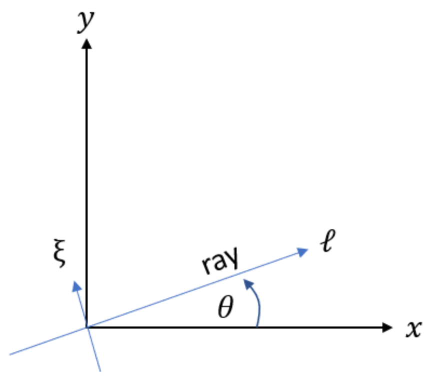

Fig. 2. Ray geometry. The angle θ of the ray is measured counter-clockwise with respect to the x-axis.

Note that when the angle $\theta = 0$, $d^2s/d\xi^{\,2} = d^2s/dy^{\,2}$; that is, D_{22}. The matrix $\mathbf{D}$ needs to be rotated to a new primed coordinate system:

$$\mathbf{D}' = \mathbf{RDR}^{\boldsymbol{T}} \quad \text{with} \quad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The second derivative in the ξ direction is D'_{22}, or:

$$\frac{d^2\Delta s}{d\xi^{\,2}} = \sin^2\theta \frac{d^2s}{dx^{\,2}} + 2\sin\theta\cos\theta \frac{d^2s}{dxdy} + \cos^2\theta \frac{d^2s}{dy^2}$$

I have checked this formula numerically on a test case $\Delta s = \cos(x)\cos(2y)$ and with $d^2\Delta s/d\xi^{\,2}$ calculated by the formula above and with finite differences.