Further Thoughts on Second Derivative in Amplitude Tomography

Bill Menke, June 30, 2021

Amplitude Tomography requires that the derivative \( \frac{d^2 f}{dx^2} \) where \( f = \ln(s) \) be approximated as a linear operator acting on \( s \). We use the approximation \( \ln(1 + t) \approx t \), valid when \( |t| \ll 1 \) to develop such a relationship. Suppose that \( s = s_0 + \Delta s \), where \( s_0 \) is a constant, average value and \( \Delta s \) is spatially varying and small compared to \( s_0 \). Then:

\[
f = \ln(s_0 + \Delta s) = \ln \left[ s_0 \left( 1 + \frac{\Delta s}{s_0} \right) \right] = \ln(s_0) + \ln \left( 1 + \frac{\Delta s}{s_0} \right) \approx \ln(s_0) + \frac{\Delta s}{s_0}
\]

Here, we have used the identity \( \ln(ab) = \ln(a) + \ln(b) \). Because only \( \Delta s \) is spatially variable:

\[
\frac{d^2 f}{dx^2} = \frac{1}{s_0} \frac{d^2 s}{dx^2} = \frac{1}{s_0} \frac{d^2 s}{d\xi^2}
\]

Figure: Comparison of exact derivative \( \frac{d^2 f}{dx^2} \) (black) with one based on the approximation (red), for signals with \( \pm 1\% \) (top) and \( \pm 10\% \) (bottom) fluctuation.

In practice, one computes the matrix of second derivatives (which is symmetric):

\[
D = \begin{bmatrix}
\frac{\partial^2 s}{\partial x^2} & \frac{\partial^2 s}{\partial x \partial y} \\
\frac{\partial^2 s}{\partial x \partial y} & \frac{\partial^2 s}{\partial y^2}
\end{bmatrix}
\]

But one needs to know the derivative \( \frac{d^2 s}{d\xi^2} \), where \( \xi \) is the direction perpendicular to the ray, as shown in Figure 2.
Fig. 2. Ray geometry. The angle $\theta$ of the ray is measured counter-clockwise with respect to the $x$-axis.

Note that when the angle $\theta = 0$, $d^2s/d\xi^2 = d^2s/dy^2$; that is, $D_{22}$. The matrix $D$ needs to be rotated to a new primed coordinate system:

$$D' = RDR^T \quad \text{with} \quad R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

The second derivative in the $\xi$ direction is $D'_{22}$, or:

$$\frac{d^2\Delta s}{d\xi^2} = \sin^2 \theta \frac{d^2s}{dx^2} + 2 \sin \theta \cos \theta \frac{d^2s}{dx dy} + \cos^2 \theta \frac{d^2s}{dy^2}$$

I have checked this formula numerically on a test case $\Delta s = \cos(x) \cos(2y)$ and with $d^2\Delta s/d\xi^2$ calculated by the formula above and with finite differences.