Amplitude of a Rayleigh Wave at Constant Energy Flux

Bill Menke, November 16, 2021

Scenario: Suppose a Rayleigh wave propagates from a region with one set of material properties into a region with another. If we ignore reflections and conversions at the boundary, we should expect the horizontal energy flux to be constant. Consequently, we should expect the displacement amplitude of the wave to change, because the depth distribution of the energy depends on the material properties of the region.

This scenario is motivated by ocean waves, which grow in amplitude as they approach a coastline, because the same energy is squeezed into a thinner and thinner water column.

Part 1: Energy Flux of an Elastic Wave

The energy flux, F_i, (rate of energy transport per unit area) of an elastic wave in the direction, i, is (Synge, 1956-1957):

$$F_i = -\tau_{ij}\dot{u}_j$$

The flux conserves the local energy density, E, in the sense that:

$$\dot{E} = -F_{i,i}$$

For a shear body wave with $\mathbf{u} = [0,0,A]^T \cos(\omega x/b - \omega t)$, where ω is angular frequency and b is shear velocity

$$\tau_{xz} = \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) = -\omega\rho b A \sin(\omega x/b - \omega t)$$

$$\dot{u}_z = \omega A \sin(\omega x/b - \omega t)$$

$$F_x = -\tau_{xz}\dot{u}_x = \omega^2 \rho b A^2 \sin^2(\omega x/b - \omega t)$$

so the time-averaged flux is $\langle F_x \rangle = -\tau_{xz}\dot{u}_x = ½\omega^2\rho b A^2$. If we imagine the shear wave propagating from one medium to another at constant flux (that is, no reflections, conversions or focusing), then the amplitude decreases as the shear velocity increases. That is.

$$½\omega^2\rho b A^2 = ½\omega^2\rho(b + \Delta b)(A + \Delta b)^2$$

$$`1 = \left(1 + \frac{\Delta b}{b}\right)\left(1 + \frac{\Delta A}{A}\right)^2 \approx \left(1 + \frac{\Delta b}{b}\right)\left(1 + \frac{2\Delta A}{A}\right)$$

$$\frac{\Delta A}{A} \approx -½\frac{\Delta b}{b}$$

The rest of this note is concerned with calculating the time-averaged flux for a Rayleigh surface wave (as contrasted to a shear body wave). A Rayleigh wave in a half-space is analyzed, because of its simplicity.

Part 2: Slowness of a Rayleigh wave in an elastic half-space

According to Aki and Richards (2009, Equation 5.56), the slowness, p, of the Rayleigh wave in a half-space is given by the zero of the Rayleigh function, $R(p)$:

$$R(p) = \left(\frac{1}{b^2} - 2p^2\right)^2 - 4p^2\left(p^2 - \frac{1}{b^2}\right)^{½}\left(p^2 - \frac{1}{a^2}\right)^{½}$$

Here a and b are the compressional and shear velocity of the half-space, respectively. The equation, $R(p) = 0$, can be solved numerically via Newton's method, using the analytic derivative:

$$\frac{dR}{dp} = -8p\left(\frac{1}{b^2} - 2p^2\right) - 8p\left(p^2 - \frac{1}{b^2}\right)^{½}\left(p^2 - \frac{1}{a^2}\right)^{½}$$

$$-4p^3\left(p^2 - \frac{1}{b^2}\right)^{-½}\left(p^2 - \frac{1}{a^2}\right)^{½} - 4p^3\left(p^2 - \frac{1}{b^2}\right)^{½}\left(p^2 - \frac{1}{a^2}\right)^{-½}$$

The phase velocity of the Rayleigh wave is $V_r = 1/p$.

Part 3: Displacement of a Rayleigh Wave

According to Aki and Richards (2009, Equations 5.52 and 5.53), the displacement of the Rayleigh wave is the sum of contributions from evanescent compressional and shear waves:

$$P\begin{bmatrix} ap \\ i\sqrt{a^2p^2 - 1}\end{bmatrix}\exp\left(-\omega\sqrt{p^2 - \frac{1}{b^2}}z\right)\exp(i\omega px) + S\begin{bmatrix} i\sqrt{b^2p^2 - 1} \\ -bp\end{bmatrix}\exp\left(-\omega\sqrt{p^2 - \frac{1}{b^2}}z\right)\exp(i\omega px)$$

Here, ω is angular frequency and P and S are the amplitudes of the compressional and shear evanescent waves, respectively. We now eliminate S from this equation. According to Aki and Richards (2009, Equation 5.55 and 5.68), the conditions that the shear traction and normal tractions be zero on the free surface at $z = 0$ imply:

$$-2ipab\sqrt{p^2 - \frac{1}{a^2}} = (1 - 2b^2p^2)S \quad \text{and} \quad (1 - 2b^2p^2)P = 2i\left(\frac{b^3p}{a}\right)\sqrt{p^2 - \frac{1}{a^2}}S$$

Algebraic manipulation of the second equation yields:

$$(1 - 2b^2p^2)P - 2i\left(\frac{b^3p}{a}\right)\sqrt{p^2 - \frac{1}{b^2}}S = 0$$

$$½i\frac{(1 - 2b^2p^2)}{\sqrt{p^2 - \frac{1}{b^2}}}P = -S\left(\frac{b^3p}{a}\right) = -bp\left(\frac{b^2}{a}\right)S$$

$$-bpS = ½i\left(\frac{a}{b^2}\right)\frac{(1 - 2b^2p^2)}{\sqrt{p^2 - \frac{1}{b^2}}}P$$

Another algebraic manipulation of the second equation yields:

$$(1-2b^2p^2)P = 2\left(\frac{b^2p}{a}\right) ib\sqrt{p^2-\frac{1}{b^2}}S$$

$$i\sqrt{b^2p^2-1}S = ½\left(\frac{a}{b^2p}\right)(1-2b^2p^2)P$$

Hence, S in the displacement equation can be eliminated in favor of P:

$$\begin{bmatrix} U_1 \\ U_3 \end{bmatrix} \exp(i\omega px) = P\left\{\begin{bmatrix} u_x^P \\ u_z^P \end{bmatrix} \exp\left(-\omega\sqrt{p^2-\frac{1}{b^2}z}\right) + \begin{bmatrix} u_x^S \\ u_z^S \end{bmatrix} \exp\left(-\omega\sqrt{p^2-\frac{1}{b^2}z}\right)\right\} \exp(i\omega px)$$

with

$$u_x^P = apP$$

$$u_x^S = ½\left(\frac{a}{p}\right)(1/b^2-2p^2)$$

$$u_z^P = i\sqrt{a^2p^2-1}$$

$$u_z^S = ½ia\frac{(1/b^2-2p^2)}{\sqrt{p^2-\frac{1}{b^2}}}P$$

I have coded these expressions and verified that the stresses:

$$\tau_{xz} = \mu\left(\frac{dU_1}{dz}+\frac{dU_3}{dx}\right) \quad \text{and} \quad \tau_{xz} = (\lambda+2\mu)\frac{dU_1}{dx}+\lambda\frac{dU_3}{dz}$$

are zero on the free surface (which verifies that they are correct).

Part 4: Horizontal Energy Flux of a Rayleigh Wave

According to menke_research_note155 (which is based on Synge 1956-7), for a real displacement, the positive and negative frequency components of an elastic wave are complex conjugate pairs:

$$\begin{aligned} u_i &= U_i \exp(-i\omega t) + \bar{U}_i \exp(+i\omega t) \\ &= 2U_i^R \cos(\omega t) + 2U_i^I \sin(\omega t) \end{aligned}$$

Then, the time-averaged energy flux, $\langle F_1 \rangle$, of an elastic wave in the x-direction is:

$$\frac{\langle F_1 \rangle}{-2\omega} = (\lambda+2\mu)\left(U_{1,1}^R U_1^I - U_{1,1}^I U_1^R\right) +$$

$$+\mu\left(U_{3,3}^R U_1^I + U_{1,3}^R U_3^I + U_{3,1}^R U_3^I - U_{3,3}^I U_1^R - U_{1,3}^I U_3^R - U_{3,1}^I U_3^R\right)$$

Here, the superscripts R and I refer to real and imaginary parts. Now suppose:

$$U_i = q_i(z) \exp(i\omega px)$$

then

$U_1 = q_1 \exp(i\omega p x)$ and $U_3 = q_3 \exp(i\omega p x)$

$U_{1,1} = i\omega p\, p_1 \exp(i\omega p x)$ and $U_{3,1} = i\omega p\, p_3 \exp(i\omega p x)$

$U_{1,3} = p_{1,3} \exp(i\omega p x)$ and $U_{3,3} = p_{3,3} \exp(i\omega p x)$

I checked the formulas numerically against a more general form of the energy flux equation.

The vertically integrated horizontal flux, $\langle F_i \rangle_T$, can be calculated as:

$$\langle F_1 \rangle_T = \int_0^\infty \langle F_1 \rangle \, dz$$

The quantity

$$A_r = \frac{-2U_3^I(x=0, z=0)}{\sqrt{\langle F_1 \rangle_T}}$$

is a measure of the change in vertical amplitude at constant energy flux, as parameters such as a and b are varied. (We have omitted U_3^R, since it is zero).

The ratio $R = A_r/A_r^{(0)}$, where $A_r^{(0)}$ is for a reference condition, quantifies the degree to which the change in a material property changes the amplitude of the Rayleigh wave, at constant energy flux.

Part 5: Experiment: varying compressional velocity at constant shear velocity

In this case, there is a weak increase in Rayleigh wave phase velocity, V_r, with increasing compressional velocity and the ratio, $R = A_r/A_r^{(0)}$, decreases with compressional velocity (same as the body wave case).

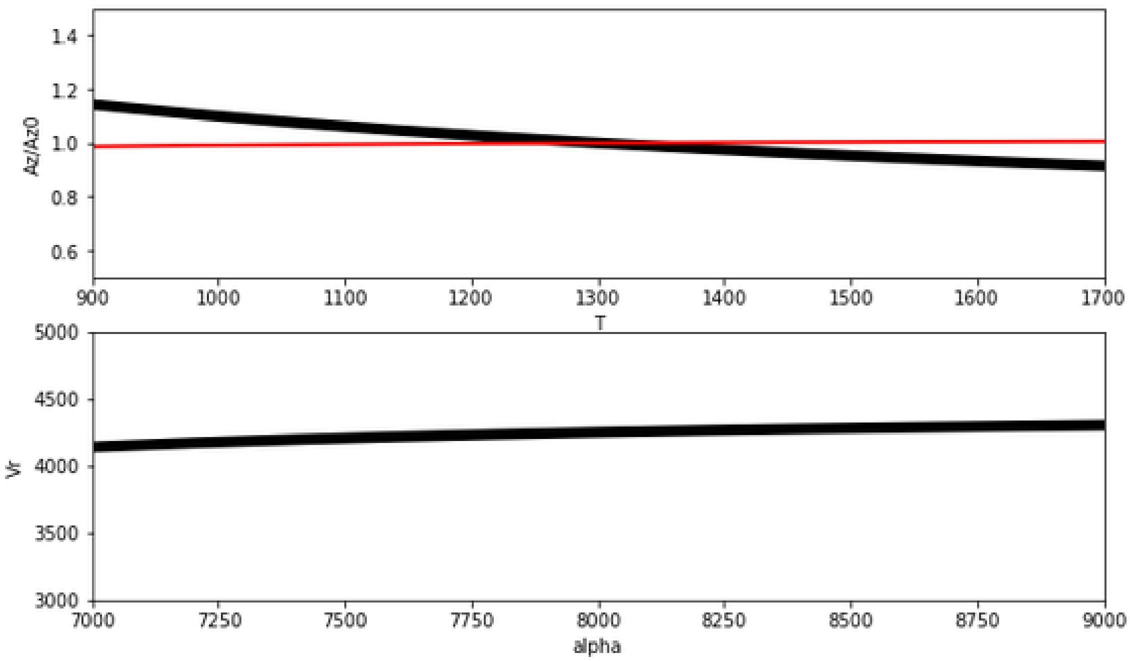


Fig. 1. Experiment with variable a and constant $b = 8000/\sqrt{3}$ (with $\omega = 1$). (Top) Ratio $R = A_r/A_r^{(0)}$ (black) and the function $1 + ½\Delta V_r/V_r$ (red). (Bottom) Rayleigh wave phase velocity, V_r.

The pattern of variation does not match the $R = 1 + ½\Delta V_r/V_r$ pattern put forward by Dalton and Ekstrom (2006).

Part 6: Experiment: varying shear velocity at constant compressional velocity.

In this case, there is a strong increase in Rayleigh wave phase velocity, V_r, with increasing shear velocity, and the ratio, $R = A_r/A_r^{(0)}$, weakly decreases with shear velocity.

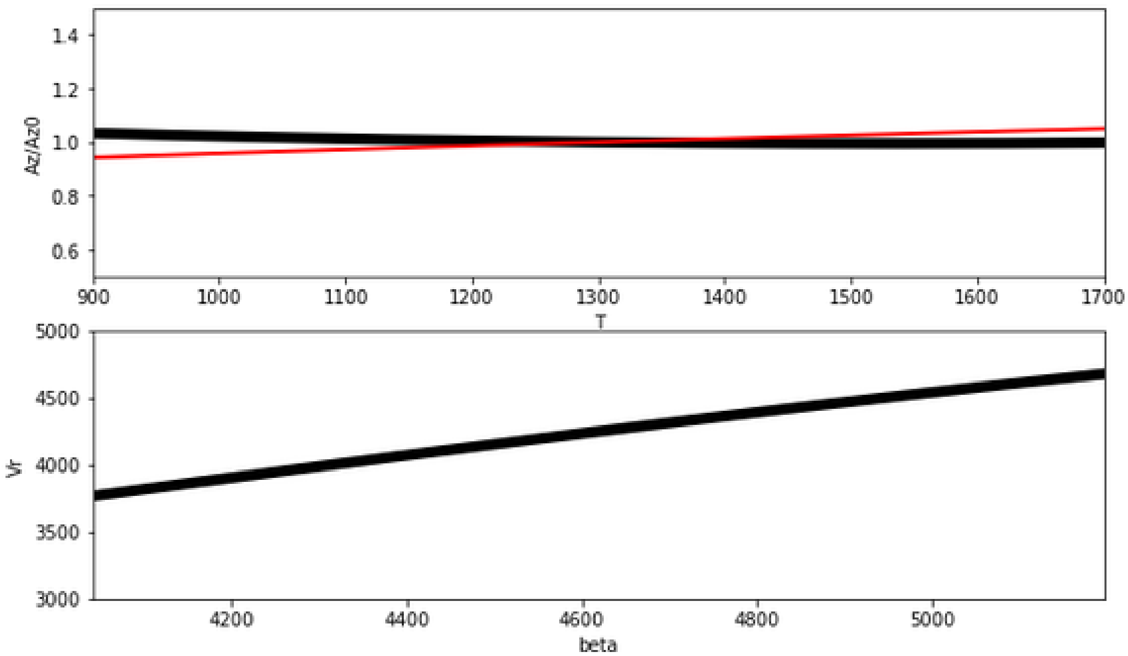
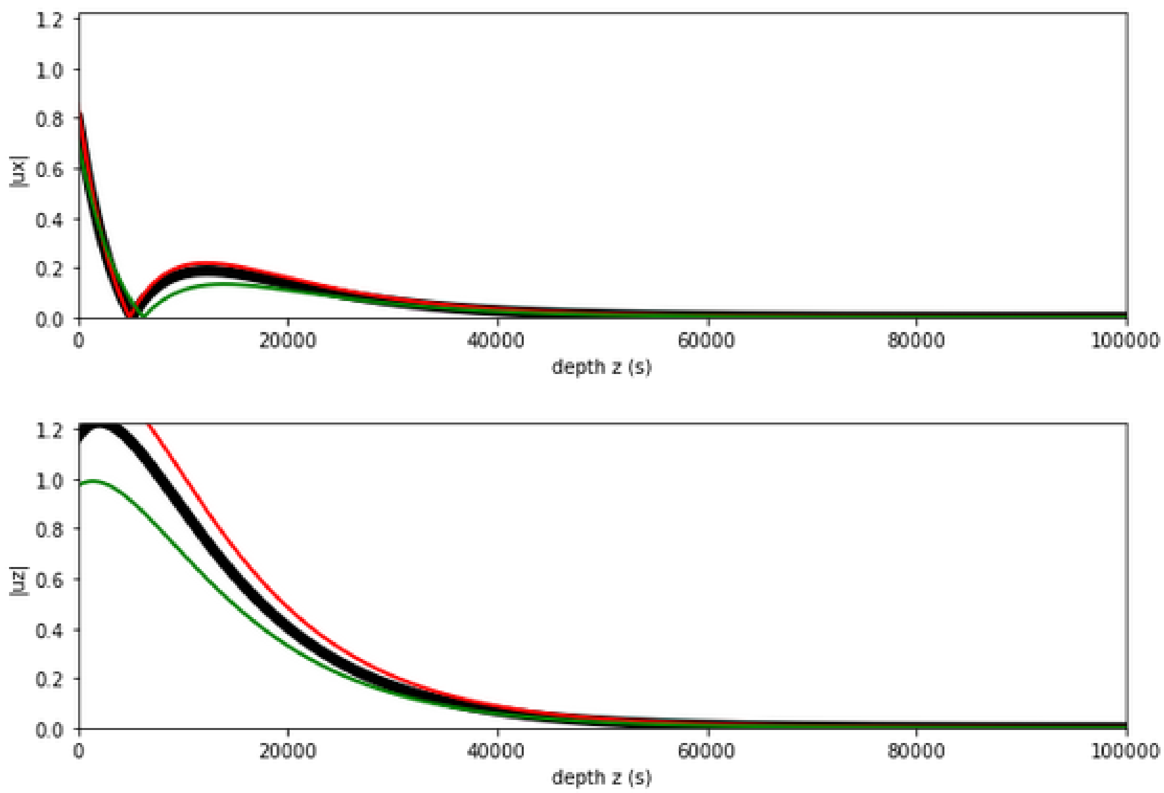


Fig. 2. Experiment with variable b and constant $a = 8000$ (with $\omega = 1$). (Top) Ratio $R = A_r/A_r^{(0)}$ (black) and the function $1 + ½\Delta V_r/V_r$ (red). (Bottom) Rayleigh wave phase velocity, V_r.

I was puzzled about why the change in R is different for compressional waves and shear waves, so I calculated the vertical wavefunctions:

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velocities 4674.17 4276.66 4621.39
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Fig. 3. Vertical wave functions for a base model (black $a = 8000$, $r = \sqrt{3}$, $\omega = 1$) and models in which only the compressional velocity is perturbed (to $a = 8000$+500, red) and only the shear velocity is perturbed (to $b = 8000/5 + 500$, green). (Top) Horizontal component. (Bottom) Vertical component.

The calculation shows that the major changes are in the vertical component, and that the sign of the changes differes between the two cases.

Part 6: Experiment: compressional velocity and shear velocity vary with temperature

Here, we use linear approximations relating compressional and shear velocity to temperature (at 200 km depth), from menke_research_note207:

$$V_P(200) = 8.301 - (5.838 \times 10^{-4})(\theta - 1300)$$

$$V_S(200) = 4.518 - (5.596 \times 10^{-4})(\theta - 1300)$$

Here, θ is temperature. Because the compressional and shear velocity changes have the same sign their effect on the amplitude tends to cancel.

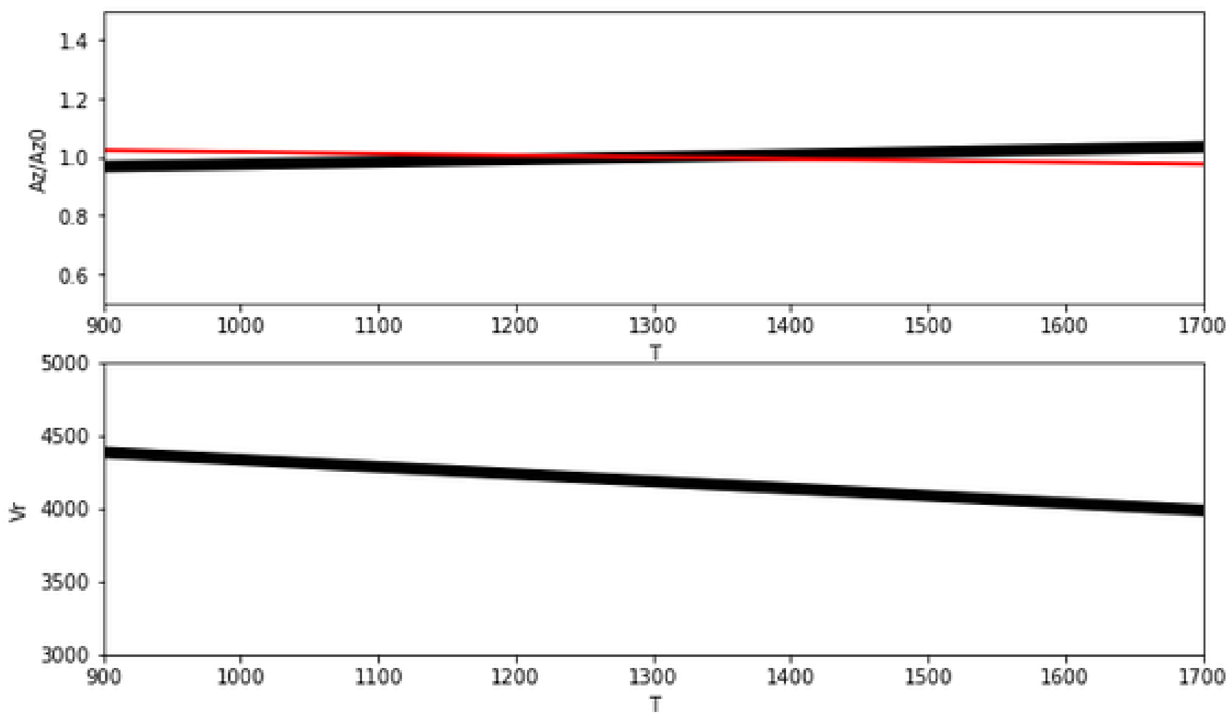

Fig. 4. Experiment with a and b varying with temperature, T. (Top) Ratio $R = A_r/A_r^{(0)}$ (black) and the function $1 + ½\Delta V_r/V_r$ (red). (Bottom) Rayleigh wave phase velocity, V_r.

References.

Aki, K. and P.G. Richards, Quantitative Seismology, Second Edition, University Science Books, Mill Valley, California (USA), 743 pp, 2009.

Dalton, C. and G. Ekstrom, Global models of surface wave attenuation, JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 111, B05317, doi:10.1029/2005JB003997, 2006.

Synge, J.L., Flux of energy for elastic waves in anisotropic media, Proceedings of the Royal Irish Academy: Section A, Mathematical and Physical Sciences, Volume 58, 1956-1957, 12-21.