Effective Pressure Diffusivity of a Long, Thin Pipe

Bill Menke, May 28, 2022, after a conversation with Jasper Barr

Derivation:

$$\text{compressibility: } \beta = -\frac{1}{V}\frac{\partial V}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

with volume, V, density, ρ, and pressure, p.

$$\text{mass convervation: } \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v)$$

with fluid velocity, v.

$$\text{Darcy's Law: } v = -k\nabla p$$

with hydraulic conductivity, k.

$$\text{incompressible approximation: } \nabla \cdot (\rho v) = \rho \nabla \cdot v$$

$$\text{homeogeneity approximation: } \nabla \cdot (k\nabla p) = k\nabla \cdot (\nabla p) = k\nabla^2 p$$

Putting it all together into the pressure diffusion equation:

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} = \beta\frac{\partial p}{\partial t} = \nabla \cdot (k\nabla p) = k\nabla^2 p$$

$$\frac{\partial p}{\partial t} = \frac{k}{\beta}\nabla^2 p = \kappa \nabla^2 p$$

with diffusivity, $\kappa = k/\beta$. Poiseuille flow in a long, thin pipe:

$$-\nabla \mathrm{p} = -\frac{\Delta \mathrm{p}}{L} = \frac{8\mu Q}{\pi R^4} = \frac{8\mu \bar{v}}{R^2}$$

$$\bar{v} = -\frac{R^2}{8\mu}\nabla \mathrm{p}$$

with dynamic viscosity, μ, pipe radius, R, and volumetric flow rate, Q (i.e. m^3/s of flow in the pipe). We can write $Q = \pi R^2 \bar{v}$, where $\bar{v}$ is the average fluid velocity. Equating the v in Darcy's law with $\bar{v}$, we have:

$$\kappa = \frac{k}{\beta} = \frac{R^2}{8\mu\beta}$$

Thus, diffusivity increases with pipe radius and decreases with dynamic viscosity and compressibility.

The one dimensional Green function for a source at $z = 0$, $t = 0$ is:

$$g(t) = \frac{E_0}{\sqrt{2\pi}}\frac{1}{\sqrt{2\kappa t}}\exp\left\{-½\frac{z^2}{2\kappa t}\right\}$$

Note that $g(t)$ is a Gaussian of area, E_0, and variance, $2\kappa t$, so that

$$\int_{-\infty}^{\infty} g(t)\, dz = E_0 \text{ at all times } t \geq 0$$

Pressure is a measure of energy density, so its integral over all z is the total energy, E_0, in the system, which by conservation of energy, is constant with time.

Reference

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