The Linearized Resolution Matrix

Bill Menke, October 12, 2022

Consider a linear inverse problem, $\mathbf{Gm} = \mathbf{d}$, with $N = 2$ data, $\mathbf{d}$, and $M = 3$ model parameters, $\mathbf{m}$. Taken individually, each datum constrains the model parameters to lie on a plane (Figure 1A). Taken together, they constrain the model parameters to lie on a line, $\overline{AB}$. The solution is nonunique, because any point, $\mathbf{m}_A$ (yellow star in Figure 1B), on the line satisfies the data. Now, suppose we define two local coordinate systems centered on some point, $\mathbf{m}_0$ (green circle), that is near $\mathbf{m}_A$: the unprimed system $(\Delta m_1, \Delta m_2, \Delta m_3)$ with axes parallel to (m_1, m_2, m_3); and the primed system $(\Delta m'_1, \Delta m'_2, \Delta m'_3)$ (green arrows) with the $\Delta m'_3$ direction parallel to the line. The $(\Delta m'_1, \Delta m'_2)$ coordinates of $\mathbf{m}_A$ are unique, because the $\Delta m'_1$ and $\Delta m'_2$ directions are perpendicular to the line. The $\Delta m'_1$ direction represents a unique linear combination of $(\Delta m_1, \Delta m_2, \Delta m_3)$, and since we have some control of the orientation of the $\Delta m'_1$ direction (which only need be normal to $\overline{AB}$), we have some control over which linear combintion it represents. We can choose an orientation corrresponding to a localized average of $(\Delta m_1, \Delta m_2, \Delta m_3)$ centered – as best as is possible – on Δm_1, Δm_2 or Δm_3. These three linear combinations define the resolution matrx, $\mathbf{R}$.

In the linear case, the intersection of the two planes is a straight line, and the same results are obtained irrespective of the choice of $\mathbf{m}_0$, because if the $\Delta m'_3$ direction is parallel to line for one choice of $\mathbf{m}_0$, it is parallel to it for any choice.

Now, consider the analagous nonlinear case, $\mathbf{g}(\mathbf{m}) = \mathbf{d}$. Taken individually, each datum constrains the model parameters to lie on a curved surface (Figure 1C). Taken together, they constrain the model parameters to lie on a curve, $\overline{AB}$. The solution is nonunique, becase any point, $\mathbf{m}_A$, on the curve satisfies the data. Globally, the $\overline{AB}$ curve may have a complicated shape. For example, it may be closed or consist of several unconnected sections. However, as long as the curve is locally smooth, we can apply the same analysis procedure as in the linear case. For a point, $\mathbf{m}_A$, on the curve, and for a reference point, $\mathbf{m}_0$, near that point, we can define a local $(\Delta m_1, \Delta m_2, \Delta m_3)$ and $(\Delta m'_1, \Delta m'_2, \Delta m'_3)$ coordinate systems, with the $\Delta m'_3$ direction tangent to the curve. As before, we can choose an orientation for the $\Delta m'_1$ direction corrresponding to a localized average of $(\Delta m_1, \Delta m_2, \Delta m_3)$ centered – as best as is possible – on Δm_1, Δm_2 or Δm_3. These three linear combinations defne the resolution matrix. However, this matrix is only valid for the small region in which curve $\overline{AB}$ is approximately aligned with the $\Delta m'_3$ direction. For this reason, we denote the resolution matrix as, $\mathbf{R}_A$, meaning the local resolution near the point, $\mathbf{m}_A$. As in the linear case, the resolution matrix is independent of $\mathbf{m}_0$, as long as $\mathbf{m}_0$ is near $\mathbf{m}_A$. If we were to select a new solution, $\mathbf{m}_A$, that is far from the old one, the tangent of $\overline{AB}$ in its vicinity may be arbitrarily different than before. Consequently, the new resolution matrix will be different, too. Thus, in a nonlinear inverse problem, the resolution matrix, $\mathbf{R}_A$, is a function of the solution, $\mathbf{m}_A$.

Suppose that we solve the nonlinear inverse problem so that its solution is $\mathbf{m}_A = \mathbf{m}_0 + \Delta\mathbf{m}$. This solution is not unique, because it can be moved along the $\overline{AB}$ curve without changing the predicted data. In contrast, denoting the ith row of $\mathbf{R}_0$ as $\mathbf{r}_0^{(i)}$, the localized average, $\mathbf{a}^{(i)} \equiv \mathbf{r}_0^{(i)}\mathbf{m}_A = \mathbf{r}_0^{(i)}(\mathbf{m}_0 + \Delta\mathbf{m}_0) = \mathbf{r}_0^{(i)}\mathbf{m}_0 + \mathbf{r}_0^{(i)}\Delta\mathbf{m}_0$ is unique, in the sense that small perturbations of $\mathbf{m}_A$ along the $\overline{AB}$ curve leave it unchanged.

As an example, consider the underdetermined linear problem:

$$d_1 = m_1 + m_2$$

$$d_2 = m_2 + m_3$$

By inspection, the linear combination, $m_1 - m + m_3$, is orthogonal to both equations and is unresolved. The averages, **r**=[½, ½,0], **r**=[½, ¼, ½] and **r**=[0, ½, ½] =, are unique, because they are linear combinations of the two data equations. Furthermore, they are somewhat localized around m_1, m_2 and m_3, respectively. It can easily be verified that $\mathbf{r}^T[1,-1,1]^T = 0$ in all three cases. Thus, a reasonable resolution relationship is:

$$\mathbf{m}^{pre} = \mathbf{R}\mathbf{m} \text{ with } \mathbf{R} = \begin{bmatrix} ½ & ½ & 0 \\ ¼ & ½ & ¼ \\ 0 & ½ & ½ \end{bmatrix}$$

Now consider the following nonlinear equations, which correspond to cylindrical surfaces:

$$d_1 = [m_1^2 + m_2^2]^{½}$$

$$d_2 = [m_2^2 + m_3^2]^{½}$$

In the vicinity of $\mathbf{m}_0 = [1,1,1]^T$, the first equation is approximately:

$$d_1 = [(1+\Delta m_1)^2 + (1+\Delta m_2)^2]^{½} \approx [2 + 2\Delta m_1 + 2\Delta m_2]^{½} \approx$$

$$[2]^{½}[1 + \Delta m_1 + \Delta m_2]^{½} \approx [2]^{½}[1 + ½\Delta m_1 + ½\Delta m_2]$$

or

$$d'_1 \equiv [2]^{½}d_1 - 2 = \Delta m_1 + \Delta m_2$$

Similarly, the second equation is approximately:

$$d'_2 \equiv [2]^{½}d_2 - 2 = \Delta m_2 + \Delta m_3$$

So, near of $\mathbf{m}_0 = [1,1,1]^T$, the nonlinear equations have the same form as the linear ones, and the approximate resolution relationship is:

$$\Delta\mathbf{m}^{pre} = \mathbf{R}_{\boldsymbol{A}}\Delta\mathbf{m} \quad \text{with} \quad \mathbf{R}_{\boldsymbol{A}} = \begin{bmatrix} ½ & ½ & 0 \\ ¼ & ½ & ¼ \\ 0 & ½ & ½ \end{bmatrix}$$

It also follows that:

$$\mathbf{m}^{pre} = \mathbf{R}_{\boldsymbol{A}}\mathbf{m} = \mathbf{R}_{\boldsymbol{A}}(\mathbf{m}_0 + \Delta\mathbf{m}) = \mathbf{R}_{\boldsymbol{A}}\mathbf{m}_0 + \mathbf{R}_{\boldsymbol{A}}\Delta\mathbf{m}$$

is unique, because $\mathbf{R}_{\boldsymbol{A}}\Delta\mathbf{m}$ is unique and $\mathbf{R}_{\boldsymbol{A}}\mathbf{m}_0$ is constant.

The standard Newton method approach for solving a nonlinear problem employs the Taylor series approximation:

$$\Delta\mathbf{d}^{(i)} \equiv \mathbf{d} - \mathbf{g}(\mathbf{m}^{(i)}) \approx \nabla_m \mathbf{g}|_{\mathbf{m}^{(i)}}\left(\mathbf{m} - \mathbf{m}^{(i)}\right) \equiv \mathbf{G}^{(i)}\Delta\mathbf{m}^{(i)}$$

which is solved iteratively as:

$$\mathbf{m}^{(0)} = \mathbf{m}_0$$

$$\mathbf{m}^{(i+1)} = \mathbf{m}^{(i)} + \mathbf{G}_{(i)}^{-g}\left[\mathbf{d} - \mathbf{g}(\mathbf{m}^{(i)})\right]$$

where $\mathbf{G}_{(i)}^{-g}$ is a generalized inverse based on the linearized data kernel, $\mathbf{G}^{(i)}$. The linearized equations, $\Delta\mathbf{d}^{(i)} = \mathbf{G}^{(i)}\Delta\mathbf{m}^{(i)}$, define N planar surfaces that are tangent to the N curved surfaces, $\mathbf{d} = \mathbf{g}(\mathbf{m}^{(i)})$ the point, $\mathbf{m}^{(i)}$. The intersection of these planes approximates the tangent to the $\overline{AB}$ curve, as it was defined above. Hence, the local resolution matrix, $\mathbf{R}_{(i)} = \mathbf{G}_{(i)}^{-g}\mathbf{G}^{(i)}$, is the same as discussed above (with reference point, $\mathbf{m}^{(i)}$). Furthermore, insofar as the sequence of approximate solutions, $\mathbf{m}^{(0)}$, $\mathbf{m}^{(1)}$, $\mathbf{m}^{(2)}$, $\cdots$ are all near one another, one would expect $\mathbf{R}_{(i)}$ to be approximately independent of i.

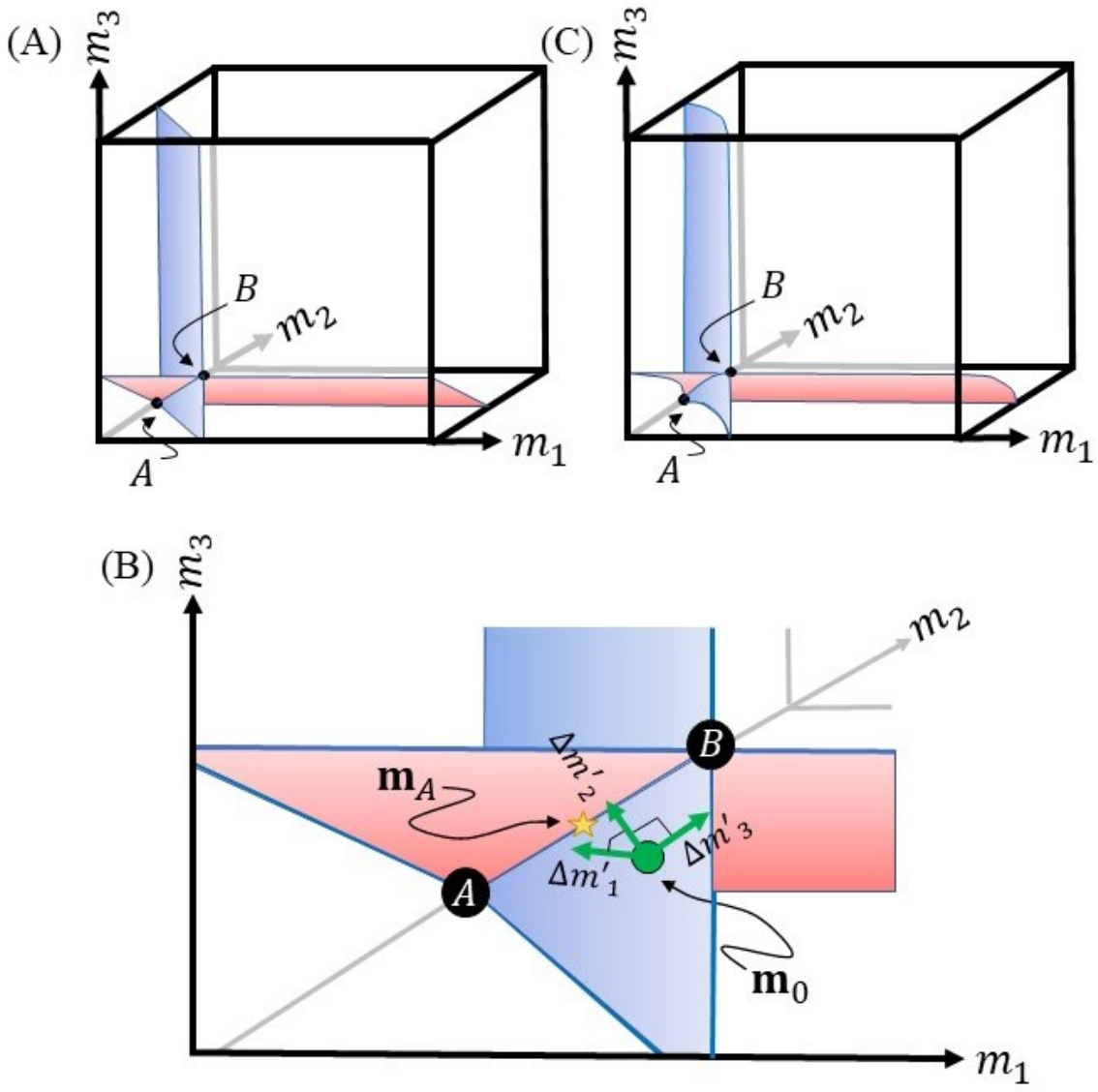

Figure 1. Space of model parameters. (A) Linear inverse problem. (B) Enlargement of region of solution in the linear inverse problem. (C) Nonlinear inverse problem. See text for more discussion.