Is it possible to estimate horizontal shaking from the observations of the aspect ratio of many un-toppled glacial boulders?

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Background

We model glacial boulders as simple right-cylinders with length, L, and width (diameter), W. The aspect ratio is defined as $a = L/W$. We consider only elongated boulder, for which $a \geq 1$. The aspect ratios of a large set of boulders are described by the probability density function, $p(a)$.

The critical angle, θ_c, is the tilt at which the bottom edge of the boulder is directly under its center of mass.

$$\theta_c = \tan^{-1}\left(\frac{W}{L}\right) = \tan^{-1}\left(\frac{1}{a}\right)$$

The boulder topples over when its tilt exceeds θ_c,.

We are concerned here with the statistics of boulders that are left un-toppled by natural processes. We divide the history of an ensemble of boulders into three stages: State 1, within the glacial ice; Stage 2, immediately after deposition onto the Earth's rocky surface; and Stage 3, immediately after having experienced shaking from an earthquake (Figure 1).

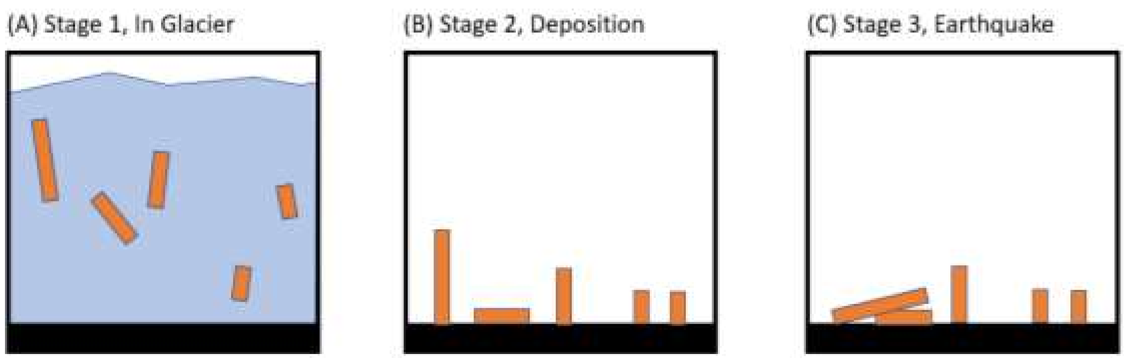

Figure 1. Phases of history of glacier boulders.

Distribution of aspect ratios of boulders within a glacier

Our idealization accounts for the abrasion of the boulders produced by the natural movement of the glacier by assuming that they are close to round ($a \approx 1$). Consequently, we choose a simple probability density function (p.d.f) of aspect ratios $1 \leq a < \infty$, that is peaked near $a = 1$, and that monotonically decreases with aspect ratio:

$$p(a) = c\exp\{-c(a-1)\}$$

Here, c is the shape parameter or decay constant. The percent of very high aspect boulders decreases with c.

When the boulders are within the glacier, they are being continually rotated by the force of the moving ice. Consequently, we assume they are uniformly distributed in angle, $0 \leq \theta \leq 90$ (measured in degrees from vertical), with p.d.f.

$$p(\theta) = \frac{1}{90}$$

The joint p.d.f., $p(a, \theta)$, is the probability function of the simultaneous occurrence of a specific aspect ratio of a boulder contained within the glacier, along with a specific orientation. As the boulder is carried along by the glacier, the boulder experiences abrasion from impacts with other boulders, will result in a change in aspect ratio. The flow of the ice will have also changed the orientation of the boulder. Though they are both results glacier movement, there is no causative relation between each outcome. Therefore, the aspect ratio and angle distribution are assumed to be independent of one another, so:

$$p(a, \theta) = p(a)p(\theta)$$

While we believe this form of $p(a, \theta)$ is a reasonable approximation, a better one could be determined by actual observations of boulders contained within glaciers.

Glacial deposition modifying distribution of aspect ratios

As the glacial ice melts, the boulder is deposited on the rocky surface of the Earth, assumed horizontal in our scenario. A boulder will only settle into an upright position when a downward pointing vector from its center of mass intersects its base. Otherwise, gravity will cause the boulder to topple.

The conditional probability of remaining upright after deposition for a specific aspect ratio and angle, $p(u = y|a, \theta)$, can be obtained by comparing the angle of deposition, θ, to the critical angle, θ_c, beyond which the boulder topples. We will denote the outcome of deposition as, u, which can take the values, y (meaning "yes", the boulder remains upright after deposition), and n (meaning "no", the boulder has toppled).

$$p(u = y|a, \theta) = \begin{cases} 1 & \text{if } \theta \leq \theta_c \\ 0 & \text{if } \theta > \theta_c \end{cases}$$

Bayes formula can be used to determine the posterior probability, $p(a, \theta|u = y)$, of aspect ratio and angle of deposition, given that the boulder remains upright after deposition (outcome $u = y$):

$$p(a, \theta|u = y) = \frac{p(u = y|a, \theta)p(a, \theta)}{\iint p(u = y|a, \theta)p(a, \theta)da d\theta}$$

Here, $\iint p(u = y|a, \theta)p(a, \theta)da d\theta$ represents the sum of all the combinations of (a, θ) that lead to the outcome u=y.

The posterior probability, $p(a|u = y)$, of aspect ratio alone is obtained by integrating over angle

$$p(a|u = y) = \int p(a, \theta|u = y)d\theta$$

In subsequent sections, we will abbreviate $p(a|u = y)$ as $p_d(a)$ ("d" for "deposition"). The p.d.f., $p_d(a)$, describes the distribution of aspect ratios of un-toppled boulders after the deposition process.

Earthquake shaking modifying distribution of aspect ratios

Consider an initially upright boulder subject to horizontal acceleration, h (measured as a fraction of gravity). Horizontal acceleration combines with gravity to make an effective tilt, $\theta_g = \tan^{-1}(h)$. A boulder will tip over if

$$\theta_g > \theta_c \quad \text{or} \quad \tan^{-1}(h) > \tan^{-1}(1/a)$$

Thus, for every horizontal acceleration, h, there is a critical aspect ratio, a_c

$$a_c = \frac{1}{h}$$

In this idealization, a boulder experiencing horizontal acceleration, h, will tip over if is aspect ratio, a_c, exceeds $1/h$. Here, outcome is denoted, v, and can take the values, y (meaning "yes", the boulder remains upright after acceleration), and n (meaning "no", the boulder has toppled). This untoppled outcome is described by the conditional probability of remaining upright after acceleration, $p(v = y|a; h)$. Here, the semicolon is used to set off deterministic variables from random ones. The simplest choice for the conditional p.d.f. is

$$p(v = y|a; h) = H(a_c - a) \ \text{ with } \ a_c = \frac{1}{h}$$

Here, $H(a_c - a)$ is the step function, whose value is one when $a < a_c$ and zero when $a > a_c$. In the real world, some boulders with $a < a_c$ topple, and some with $a > a_c$ remain upright. We account for this behavior by replacing the step function with the sigmoid function, f:

$$p(v = y|a; h, s) = f(a, h, s)$$

with

$$f(a, h, s) = \frac{1}{1 + \exp(z)} \quad \text{and} \quad z = 4s(a - a_c) \ \text{ and } \ a_c = \frac{1}{h}$$

Here, s is the slope of the sigmoid function at the inflection point, $a = a_c$ In the $s \to \infty$ limit, the sigmoid function tends to the step function. An exemplary sigmoid function is shown (Figure 1).

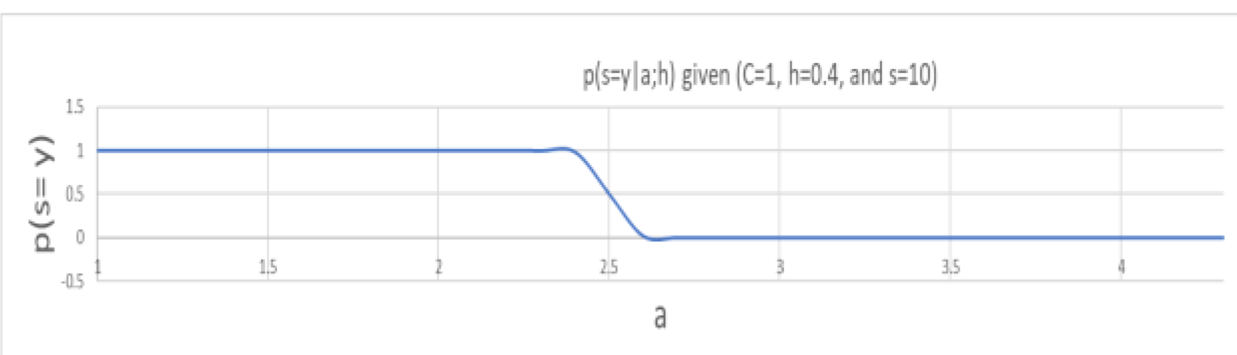

Figure 1. Exemplary sigmoid function with $a_c = 2.5$ and $s = 10$.

Bayes formula can be used to determine this the posterior probability, $p(a|v = y; h, s)$, of aspect ratio, given that the boulder remains upright after experiencing horizontal acceleration, h (outcome $v = y$):

$$p(a|v = y; h, s) = \frac{p(v = y|a; h, s)\, p_d(a)}{\int p(v = y|a; h, s) p(a) da}$$

Here, $p_d(a)$ is shorthand for $p(a|u = y)$, as determined in the previous section. We abbreviate $p(a|v = y; h, s)$ as $p_e(a; a_c, s)$ ("e" for "earthquake"), with $a_c = 1/h$.

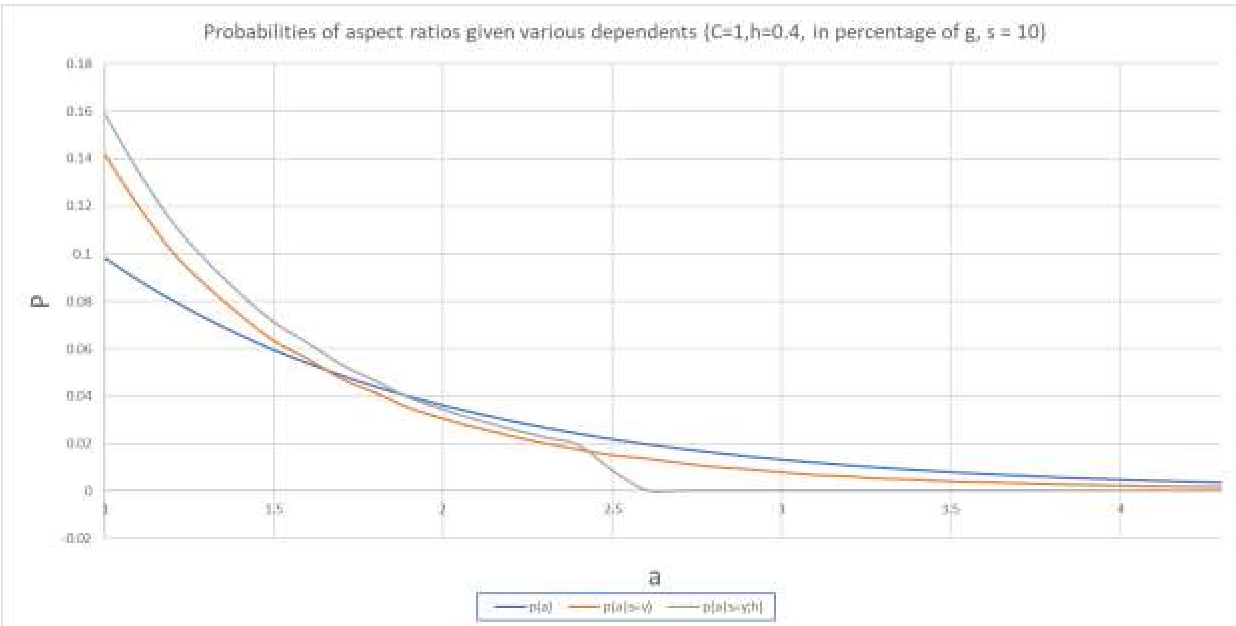

Figure 2. P.d.f's of aspect ratio, $p(a)$ (blue) $p_d(a)$ (orange), $p_e(a; a_c, s)$ (grey). a_c = 2.5 and $s = 10$

Maximum likelihood estimation of horizontal acceleration

We address the question of whether $p_e(a; a_c, s)$ helps us understand observations of aspect ratio.

The observed data are a list of aspect ratios, a_i with $i = 1, \cdots, N$, of untoppled boulders that survived an unknown horizontal acceleration, h^{true}. The maximum of these observed aspect ratios is a_{max}.

Our first consideration is that a_c cannot be much smaller than a_{max}, because the probability of an observation much larger than a_c is very small; that is, $p_e(a; a_c, s)$ falls off very quickly beyond a_c, especially for higher slopes, s. Our second consideration is that, for large slopes, a_c cannot be larger than a_{max} as $p_e(a; a_c, s)$ transitions very quickly from an extremely high probability of survival to a very low (but still non-zero) probability, as a increases beyond a_c. These considerations suggest that the observations contain information about (a_c, s). They can help us narrow the range of possible values.

A numerical estimate of a_c can be obtained using the maximum likelihood method. Given trial values, a_c and s, the probability of each boulder is

$$p_i(a_c, s) = p_e(a_i; a_c, s)$$

The principle of maximum likelihood states that the best estimates of a_c and s are the ones that maximize the probability that all observation were observed (for fixed values of the N of observations). For trial values (s, a_c), the probability is

$$p(s, a_c) = p_1(a_c, s) \times p_2(a_c, s) \times \cdots \times p_N(a_c, s)$$

and its logarithm, called the likelihood function is

$$L(s, a_c) = \log p(s, a_c) = \log p_1(a_c, s) + \log p_2(a_c, s) + \cdots + \log p_N(a_c, s)$$

The best estimate (a_c^{est}, s^{est}) is the one that maximizes $L(s, a_c)$.

Note that the maximization cannot simply assign high probability to every $p_i(a_c, s)$. The area under $p_e(a_i; a_c, s)$ is fixed at 1, so high probabilities for some aspect ratios must be offset by low values at others. This trading off of probabilities requires $p(s, a_c)$ to have a maximum at some specific value of (a_c, s). For instance, although setting $a_c \gg a_{max}$ "widens" and "flattens" $p_e(a; a_c, s)$ in the range of the data, it does not lead to a maximal $p(s, a_c)$, because wide p.d.f.'s are necessarily low.

In practice, we use a grid search to find (a_c^{est}, s^{est}). Many different trial values of s and a_c are tried and the one that maximizes the likelihood of the slope and critical aspect ratio, $L(s, a_c)$ is identified. It gives the best estimate of a_c^{est} and s^{est}. the estimated horizontal acceleration is $h^{est} = 1/a_c^{est}$.

Numerical experiments (Figure 3) demonstrate the viability of the technique. Under favorable circumstance, (a_c^{est}, s^{est}) are close to their true values (a_c^{true}, s^{true}). The quality of the estimate is highly sensitive to which aspect ratios were observed.

In Figure 3, the higher likelihoods, $L(s, a_c)$, occur within a smaller region of the a_c axis than the s axis, so the a_c^{est} has a higher degree of precision than s^{est}. In this case the comparatively narrow range of which a high likelihood of a_c occurs also results in a_c^{est} possessing a higher accuracy then s^{est}. For the specific $L(s, a_c)$ function in Figure 3, a group of calculated a_c^{est} (say from a group of repeated experiments) will be closer to each other and the a_c^{true} then a group of s^{est} would be to each other and s^{true}. Note that the likelihood of a point on the a_c axis does not exhibit a high variation along the s axis. The maximum likelihood of $L(s, a_c)$ is close to parallel with the s axis, indicating that a_c^{est} is not much affected by inaccuracies in s^{est}.

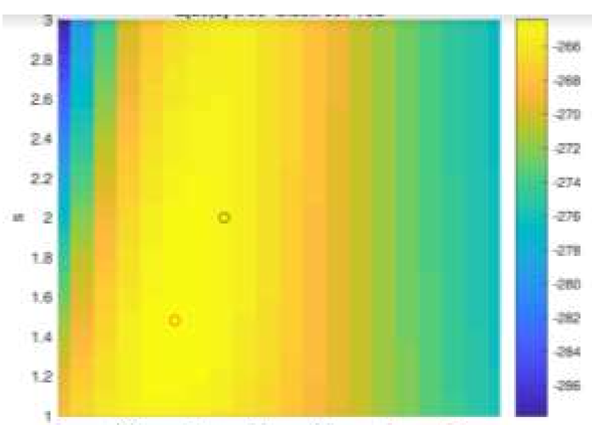

Figure 3. Grid search, showing likelihood function, $L(s, a_c)$ (colors), (a_c^{true}, s^{true}) (black circle) and (a_c^{est}, s^{est}) (red circle).

The accuracy of s^{est} is high when some of the observed aspect ratios, a_i are made just beyond a_c^{true}. This is because the slope at the inflection point, a_c, of $p_e(a_i; a_c, s)$ is directly related to the slope, s, of the sigmoid function, $p(v = y|a; h, s)$, at the inflection point of the sigmoid. Therefore, observations collected just preceding and beyond a_c^{true}, which are approximately on the slope at the inflection point of $p_e(a_i; a_c, s)$ will have a probability that is relatively highly sensitive to the value of s. A much smaller range of s values will thereby produce a high $p(s, a_c)$ for sequences that contain observations just beyond a_c compared to sequences that do not contain observations in that range.

Conclusions

We have demonstrated *in principle* that it is possible to estimate the horizontal shaking from the observations of the aspect ratio of many boulders.

Our procedure demonstrates the feasibility of developing a formula for the probability of aspect ratio, $p_e(a; h, s)$, given that the boulder remains upright after experiencing horizontal acceleration, h. This p.d.f. depends upon knowledge of the probability of aspect ratios given that they survive untoppled after deposition, $p_d(a)$, and the probability of a boulder surviving horizontal shaking, given an aspect ratio, $p(v = y|a; h, s)$.

We provide a simple model for $p_d(a)$ that is based on ideas about how glacial boulders are modified by glacial ice. However, an empirical p.d.f. based on observations of boulders within the glacier could be substituted, were it available. We also provide a simple model for $p(v = y|a; h, s)$, based on the notion that a boulder will topple if the horizontal acceleration of the earthquake results in an effective tilt that exceeds a critical value that depends on aspect ratio.

Given a set of observed aspect rations of untoppled boulders, a standard maximum likelihood procedure can be used to determine the parameters, s and h. A numerical test in which a grid search was used to find the maximum likelihood point produced estimates of these parameters that were acceptably close to their true values.

In or model, aspect ratio is a measure of ‘fragility’, in the sense that boulders with high aspect rations are most likely to topple, and hence are the most fragile. However, our general procedure could be extended to other measures of fragility.