SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

CONTINUUM MECHANICS

BY

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2.48. Sketch the Mohr's circles and determine the maximum shear stress for each of the following stress states:

$$(a)\quad \sigma_{ij} = \begin{pmatrix} \tau & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (b)\quad \sigma_{ij} = \begin{pmatrix} \tau & 0 & 0 \\ 0 & -\tau & 0 \\ 0 & 0 & -2\tau \end{pmatrix}$$

Ans. (a) $\sigma_S = \tau$, (b) $\sigma_S = 3\tau/2$

2.49. Use the result given in Problem 1.58, page 39, together with the stress transformation law (*2.27*), page 50, to show that $\epsilon_{ijk}\epsilon_{pqm}\sigma_{ip}\sigma_{jq}\sigma_{km}$ is an invariant.

2.50. In a continuum, the stress field is given by the tensor

$$\sigma_{ij} = \begin{pmatrix} x_1^2 x_2 & (1-x_2^2)x_1 & 0 \\ (1-x_2^2)x_1 & (x_2^3 - 3x_2)/3 & 0 \\ 0 & 0 & 2x_3^2 \end{pmatrix}$$

Determine (a) the body force distribution if the equilibrium equations are to be satisfied throughout the field, (b) the principal stress values at the point $P(a, 0, 2\sqrt{a})$, (c) the maximum shear stress at P, (d) the principal deviator stresses at P.

Ans. (a) $b_3 = -4x_3$, (b) $a, -a, 8a$, (c) $\pm 4.5a$, (d) $-11a/3, -5a/3, 16a/3$

Chapter 3

Deformation and Strain

3.1 PARTICLES AND POINTS

In the kinematics of continua, the meaning of the word "point" must be clearly understood since it may be construed to refer either to a "point" in space, or to a "point" of a continuum. To avoid misunderstanding, the term "point" will be used exclusively to designate a location in fixed space. The word "particle" will denote a small volumetric element, or "material point", of a continuum. In brief, a *point* is a place in space, a *particle* is a small part of a material continuum.

3.2 CONTINUUM CONFIGURATION. DEFORMATION AND FLOW CONCEPTS

At any instant of time t, a continuum having a volume V and bounding surface S will occupy a certain region R of physical space. The identification of the particles of the continuum with the points of the space it occupies at time t by reference to a suitable set of coordinate axes is said to specify the *configuration* of the continuum at that instant.

The term *deformation* refers to a change in the shape of the continuum between some initial (undeformed) configuration and a subsequent (deformed) configuration. The emphasis in deformation studies is on the initial and final configurations. No attention is given to intermediate configurations or to the particular sequence of configurations by which the deformation occurs. By contrast, the word *flow* is used to designate the continuing state of motion of a continuum. Indeed, a configuration history is inherent in flow investigations for which the specification of a time-dependent velocity field is given.

3.3 POSITION VECTOR. DISPLACEMENT VECTOR

In Fig. 3-1 the undeformed configuration of a material continuum at time $t = 0$ is shown together with the deformed configuration of the same continuum at a later time $t = t$. For the present development it is useful to refer the initial and final configurations to separate coordinate axes as in the figure.

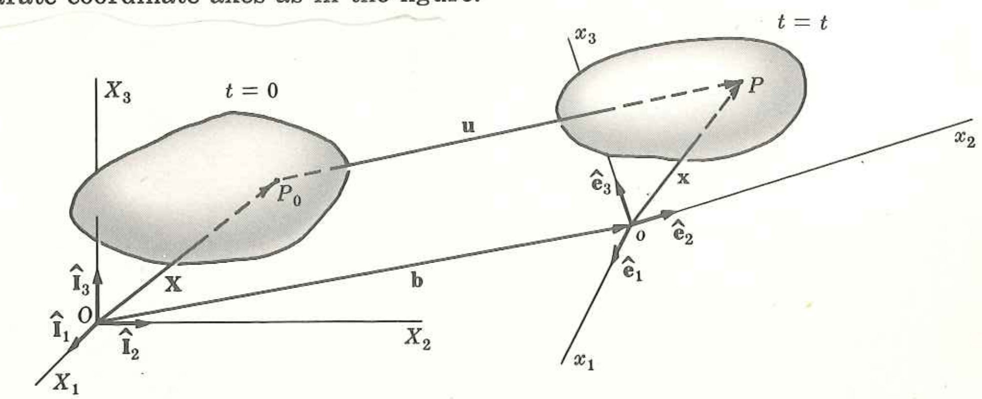

Fig. 3-1

Accordingly, in the initial configuration a representative particle of the continuum occupies a point P_0 in space and has the *position vector*

$$\mathbf{X} = X_1\hat{\mathbf{I}}_1 + X_2\hat{\mathbf{I}}_2 + X_3\hat{\mathbf{I}}_3 = X_K\hat{\mathbf{I}}_K \tag{3.1}$$

with respect to the rectangular Cartesian axes $OX_1X_2X_3$. Upper-case letters are used as indices in (*3.1*) and will appear as such in several equations that follow, but their use as summation indices is restricted to this section. In the remainder of the book upper-case subscripts or superscripts serve as labels only. Their use here is to emphasize the connection of certain expressions with the coordinates $(X_1X_2X_3)$, which are called the *material coordinates*. In the deformed configuration the particle originally at P_0 is located at the point P and has the position vector

$$\mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 = x_i\hat{\mathbf{e}}_i \tag{3.2}$$

when referred to the rectangular Cartesian axes $ox_1x_2x_3$. Here lower-case letters are used as subscripts to identify with the coordinates $(x_1x_2x_3)$ which give the current position of the particle and are frequently called the *spatial coordinates*.

The relative orientation of the material axes $OX_1X_2X_3$ and the spatial axes $ox_1x_2x_3$ is specified through direction cosines α_{kK} and α_{Kk}, which are defined by the dot products of unit vectors as

$$\hat{\mathbf{e}}_k \cdot \hat{\mathbf{I}}_K = \hat{\mathbf{I}}_K \cdot \hat{\mathbf{e}}_k = \alpha_{kK} = \alpha_{Kk} \tag{3.3}$$

No summation is implied by the indices in these expressions since k and K are distinct indices. Inasmuch as Kronecker deltas are designated by the equations $\hat{\mathbf{I}}_K \cdot \hat{\mathbf{I}}_P = \delta_{KP}$ and $\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_p = \delta_{kp}$, the *orthogonality conditions* between spatial and material axes take the form

$$\alpha_{Kk}\alpha_{Kp} = \alpha_{kK}\alpha_{pK} = \delta_{kp}; \quad \alpha_{Kp}\alpha_{Mp} = \alpha_{pK}\alpha_{pM} = \delta_{KM} \tag{3.4}$$

In Fig. 3-1 the vector $\mathbf{u}$ joining the points P_0 and P (the initial and final positions, respectively, of the particle), is known as the *displacement vector*. This vector may be expressed as

$$\mathbf{u} = u_k\hat{\mathbf{e}}_k \tag{3.5}$$

or alternatively as

$$\mathbf{U} = U_K\hat{\mathbf{I}}_K \tag{3.6}$$

in which the components U_K and u_k are interrelated through the direction cosines α_{kK}. From (*1.89*) the unit vector $\hat{\mathbf{e}}_k$ is expressed in terms of the material base vectors $\hat{\mathbf{I}}_K$ as

$$\hat{\mathbf{e}}_k = \alpha_{kK}\hat{\mathbf{I}}_K \tag{3.7}$$

Therefore substituting (*3.7*) into (*3.5*),

$$\mathbf{u} = u_k(\alpha_{kK}\hat{\mathbf{I}}_K) = U_K\hat{\mathbf{I}}_K = \mathbf{U} \tag{3.8}$$

from which

$$U_K = \alpha_{kK}u_k \tag{3.9}$$

Since the direction cosines α_{kK} are constants, the components of the displacement vector are observed from (*3.9*) to obey the law of transformation of first-order Cartesian tensors, as they should.

The vector $\mathbf{b}$ in Fig. 3-1 serves to locate the origin o with respect to O. From the geometry of the figure,

$$\mathbf{u} = \mathbf{b} + \mathbf{x} - \mathbf{X} \tag{3.10}$$

Very often in continuum mechanics it is possible to consider the coordinate systems $OX_1X_2X_3$ and $ox_1x_2x_3$ superimposed, with $\mathbf{b} \equiv 0$, so that (*3.10*) becomes

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \tag{3.11}$$

In Cartesian component form this equation is given by the general expression

$$u_k = x_k - \alpha_{kK}X_K \tag{3.12}$$

However, for superimposed axes the unit triads of base vectors for the two systems are identical, which results in the direction cosine symbols α_{kK} becoming Kronecker deltas. Accordingly, (*3.12*) reduces to

$$u_k = x_k - X_k \tag{3.13}$$

in which only lower-case subscripts appear. In the remainder of this book, unless specifically stated otherwise, the material and spatial axes are assumed *superimposed* and hence only lower-case indices will be used.

3.4 LAGRANGIAN AND EULERIAN DESCRIPTIONS

When a continuum undergoes deformation (or flow), the particles of the continuum move along various paths in space. This motion may be expressed by equations of the form

$$x_i = x_i(X_1, X_2, X_3, t) = x_i(\mathbf{X}, t) \quad \text{or} \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{3.14}$$

which give the present location x_i of the particle that occupied the point $(X_1X_2X_3)$ at time $t = 0$. Also, (*3.14*) may be interpreted as a mapping of the initial configuration into the current configuration. It is assumed that such a mapping is one-to-one and continuous, with continuous partial derivatives to whatever order is required. The description of motion or deformation expressed by (*3.14*) is known as the *Lagrangian* formulation.

If, on the other hand, the motion or deformation is given through equations of the form

$$X_i = X_i(x_1, x_2, x_3, t) = X_i(\mathbf{x}, t) \quad \text{or} \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{3.15}$$

in which the independent variables are the coordinates x_i and t, the description is known as the *Eulerian* formulation. This description may be viewed as one which provides a tracing to its original position of the particle that now occupies the location (x_1, x_2, x_3). If (*3.15*) is a continuous one-to-one mapping with continuous partial derivatives, as was also assumed for (*3.14*), the two mappings are the unique inverses of one another. A necessary and sufficient condition for the inverse functions to exist is that the Jacobian

$$J = \left|\frac{\partial x_i}{\partial X_j}\right| \tag{3.16}$$

should not vanish.

As a simple example, the Lagrangian description given by the equations

$$\begin{aligned} x_1 &= X_1 + X_2(e^t - 1) \\ x_2 &= X_1(e^{-t} - 1) + X_2 \\ x_3 &= X_3 \end{aligned} \tag{3.17}$$

has the inverse Eulerian formulation,

$$\begin{aligned} X_1 &= \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}} \\ X_2 &= \frac{x_1(e^{-t} - 1) - x_2}{1 - e^t - e^{-t}} \\ X_3 &= x_3 \end{aligned} \tag{3.18}$$

3.5 DEFORMATION GRADIENTS. DISPLACEMENT GRADIENTS

Partial differentiation of (3.14) with respect to X_j produces the tensor $\partial x_i/\partial X_j$ which is called the *material deformation gradient*. In symbolic notation, $\partial x_i/\partial X_j$ is represented by the dyadic

$$\mathbf{F} = \mathbf{x}\nabla_{\mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial X_1}\hat{\mathbf{e}}_1 + \frac{\partial \mathbf{x}}{\partial X_2}\hat{\mathbf{e}}_2 + \frac{\partial \mathbf{x}}{\partial X_3}\hat{\mathbf{e}}_3 \tag{3.19}$$

in which the differential operator $\nabla_{\mathbf{X}} = \frac{\partial}{\partial X_i}\hat{\mathbf{e}}_i$ is applied from the right (as shown explicitly in the equation). The matrix form of $\mathbf{F}$ serves to further clarify this property of the operator $\nabla_{\mathbf{X}}$ when it appears as the consequent of a dyad. Thus

$$\mathcal{F} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} \partial x_1/\partial X_1 & \partial x_1/\partial X_2 & \partial x_1/\partial X_3 \\ \partial x_2/\partial X_1 & \partial x_2/\partial X_2 & \partial x_2/\partial X_3 \\ \partial x_3/\partial X_1 & \partial x_3/\partial X_2 & \partial x_3/\partial X_3 \end{bmatrix} = [\partial x_i/\partial X_j] \tag{3.20}$$

Partial differentiation of (3.15) with respect to x_j produces the tensor $\partial X_i/\partial x_j$ which is called the *spatial deformation gradient*. This tensor is represented by the dyadic

$$\mathbf{H} = \mathbf{X}\nabla_{\mathbf{x}} \equiv \frac{\partial \mathbf{X}}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial \mathbf{X}}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial \mathbf{X}}{\partial x_3}\hat{\mathbf{e}}_3 \tag{3.21}$$

having a matrix form

$$\mathcal{H} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} \partial X_1/\partial x_1 & \partial X_1/\partial x_2 & \partial X_1/\partial x_3 \\ \partial X_2/\partial x_1 & \partial X_2/\partial x_2 & \partial X_2/\partial x_3 \\ \partial X_3/\partial x_1 & \partial X_3/\partial x_2 & \partial X_3/\partial x_3 \end{bmatrix} = [\partial X_i/\partial x_j] \tag{3.22}$$

The material and spatial deformation tensors are interrelated through the well-known chain rule for partial differentiation,

$$\frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial X_k} = \delta_{ik} \tag{3.23}$$

Partial differentiation of the displacement vector u_i with respect to the coordinates produces either the *material displacement gradient* $\partial u_i/\partial X_j$, or the *spatial displacement gradient* $\partial u_i/\partial x_j$. From (3.13), which expresses u_i as a difference of coordinates, these tensors are given in terms of the deformation gradients as the material gradient

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \quad \text{or} \quad \mathbf{J} \equiv \mathbf{u}\nabla_{\mathbf{X}} = \mathbf{F} - \mathbf{I} \tag{3.24}$$

and the spatial gradient

$$\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} \quad \text{or} \quad \mathbf{K} \equiv \mathbf{u}\nabla_{\mathbf{x}} = \mathbf{I} - \mathbf{H} \tag{3.25}$$

In the usual manner, the matrix forms of $\mathbf{J}$ and $\mathbf{K}$ are respectively

$$\mathcal{J} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} \partial u_1/\partial X_1 & \partial u_1/\partial X_2 & \partial u_1/\partial X_3 \\ \partial u_2/\partial X_1 & \partial u_2/\partial X_2 & \partial u_2/\partial X_3 \\ \partial u_3/\partial X_1 & \partial u_3/\partial X_2 & \partial u_3/\partial X_3 \end{bmatrix} = [\partial u_i/\partial X_j] \tag{3.26}$$

and

$$\mathcal{K} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{bmatrix} = [\partial u_i/\partial x_j] \tag{3.27}$$

3.6 DEFORMATION TENSORS. FINITE STRAIN TENSORS

In Fig. 3-2 the initial (undeformed) and final (deformed) configurations of a continuum are referred to the superposed rectangular Cartesian coordinate axes $OX_1X_2X_3$ and $ox_1x_2x_3$. The neighboring particles which occupy points P_0 and Q_0 before deformation, move to points P and Q respectively in the deformed configuration.

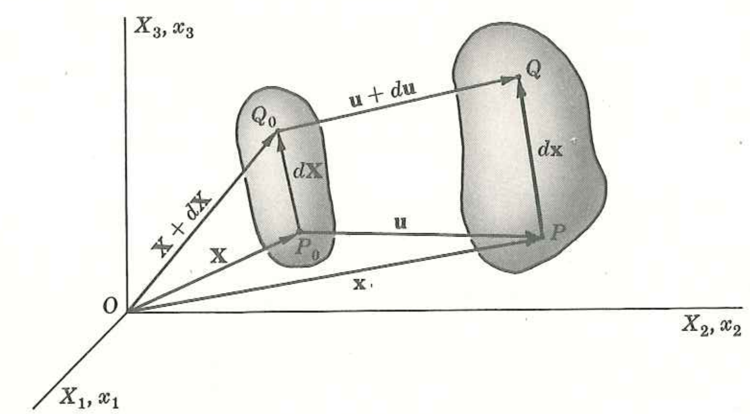

Fig. 3-2

The square of the differential element of length between P_0 and Q_0 is

$$(dX)^2 = d\mathbf{X}\cdot d\mathbf{X} = dX_i\,dX_i = \delta_{ij}\,dX_i\,dX_j \tag{3.28}$$

From (3.15), the distance differential dX_i is seen to be

$$dX_i = \frac{\partial X_i}{\partial x_j}dx_j \quad \text{or} \quad d\mathbf{X} = \mathbf{H}\cdot d\mathbf{x} \tag{3.29}$$

so that the squared length $(dX)^2$ in (3.28) may be written

$$(dX)^2 = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}dx_i\,dx_j = C_{ij}\,dx_i\,dx_j \quad \text{or} \quad (dX)^2 = d\mathbf{x}\cdot\mathbf{C}\cdot d\mathbf{x} \tag{3.30}$$

in which the second-order tensor

$$C_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} \quad \text{or} \quad \mathbf{C} = \mathbf{H}_c\cdot\mathbf{H} \tag{3.31}$$

is known as *Cauchy's deformation tensor*.

In the deformed configuration, the square of the differential element of length between P and Q is

$$(dx)^2 = d\mathbf{x}\cdot d\mathbf{x} = dx_i\,dx_i = \delta_{ij}\,dx_i\,dx_j \tag{3.32}$$

From (3.14) the distance differential here is

$$dx_i = \frac{\partial x_i}{\partial X_j}dX_j \quad \text{or} \quad d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} \tag{3.33}$$

so that the squared length $(dx)^2$ in (3.32) may be written

$$(dx)^2 = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}dX_i\,dX_j = G_{ij}\,dX_i\,dX_j \quad \text{or} \quad (dx)^2 = d\mathbf{X}\cdot\mathbf{G}\cdot d\mathbf{X} \tag{3.34}$$

in which the second-order tensor

$$G_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} \quad \text{or} \quad \mathbf{G} = \mathbf{F}_c\cdot\mathbf{F} \tag{3.35}$$

is known as *Green's deformation tensor*.

The difference $(dx)^2 - (dX)^2$ for two neighboring particles of a continuum is used as the *measure of deformation* that occurs in the neighborhood of the particles between the initial and final configurations. If this difference is identically zero for all neighboring particles of a continuum, a *rigid displacement* is said to occur. Using (*3.34*) and (*3.28*), this difference may be expressed in the form

$$(dx)^2 - (dX)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i\, dX_j = 2L_{ij}\, dX_i\, dX_j$$

or

$$(dx)^2 - (dX)^2 = d\mathbf{X}\cdot(\mathbf{F}_c\cdot\mathbf{F} - \mathbf{I})\cdot d\mathbf{X} = d\mathbf{X}\cdot 2\mathbf{L}_G\cdot d\mathbf{X} \qquad (3.36)$$

in which the second-order tensor

$$L_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) \quad \text{or} \quad \mathbf{L}_G = \tfrac{1}{2}(\mathbf{F}_c\cdot\mathbf{F} - \mathbf{I}) \qquad (3.37)$$

is called the *Lagrangian* (or Green's) *finite strain tensor.*

Using (*3.32*) and (*3.30*), the same difference may be expressed in the form

$$(dx)^2 - (dX)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i\, dx_j = 2E_{ij}\, dx_i\, dx_j$$

or

$$(dx)^2 - (dX)^2 = d\mathbf{x}\cdot(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H})\cdot d\mathbf{x} = d\mathbf{x}\cdot 2\mathbf{E}_A\cdot d\mathbf{x} \qquad (3.38)$$

in which the second-order tensor

$$E_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}_A = \tfrac{1}{2}(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H}) \qquad (3.39)$$

is called the *Eulerian* (or Almansi's) *finite strain tensor.*

An especially useful form of the Lagrangian and Eulerian finite strain tensors is that in which these tensors appear as functions of the displacement gradients. Thus if $\partial x_i/\partial X_j$ from (*3.24*) is substituted into (*3.37*), the result after some simple algebraic manipulations is the Lagrangian finite strain tensor in the form

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \quad \text{or} \quad \mathbf{L}_G = \tfrac{1}{2}(\mathbf{J} + \mathbf{J}_c + \mathbf{J}_c\cdot\mathbf{J}) \qquad (3.40)$$

In the same manner, if $\partial X_i/\partial x_j$ from (*3.25*) is substituted into (*3.39*), the result is the Eulerian finite strain tensor in the form

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}_A = \tfrac{1}{2}(\mathbf{K} + \mathbf{K}_c - \mathbf{K}_c\cdot\mathbf{K}) \qquad (3.41)$$

The matrix representations of (*3.40*) and (*3.41*) may be written directly from (*3.26*) and (*3.27*) respectively.

3.7 SMALL DEFORMATION THEORY. INFINITESIMAL STRAIN TENSORS

The so-called *small deformation theory* of continuum mechanics has as its basic condition the requirement that the displacement gradients be small compared to unity. The fundamental measure of deformation is the difference $(dx)^2 - (dX)^2$, which may be expressed in terms of the displacement gradients by inserting (*3.40*) and (*3.41*) into (*3.36*) and (*3.38*) respectively. If the displacement gradients are small, the finite strain tensors in (*3.36*) and (*3.38*) reduce to infinitesimal strain tensors, and the resulting equations represent small deformations.

In (*3.40*), if the displacement gradient components $\partial u_i/\partial X_j$ are each small compared to unity, the product terms are negligible and may be dropped. The resulting tensor is the *Lagrangian infinitesimal strain tensor*, which is denoted by

$$l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{L} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{J} + \mathbf{J}_c) \qquad (3.42)$$

Likewise for $\partial u_i/\partial x_j \ll 1$ in (*3.41*), the product terms may be dropped to yield the *Eulerian infinitesimal strain tensor*, which is denoted by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{K} + \mathbf{K}_c) \qquad (3.43)$$

If both the displacement gradients and the displacements themselves are small, there is very little difference in the material and spatial coordinates of a continuum particle. Accordingly the material gradient components $\partial u_i/\partial X_j$ and spatial gradient components $\partial u_i/\partial x_j$ are very nearly equal, so that the Eulerian and Lagrangian infinitesimal strain tensors may be taken as equal. Thus

$$l_{ij} = \epsilon_{ij} \quad \text{or} \quad \mathbf{L} = \mathbf{E} \qquad (3.44)$$

if both the displacements and displacement gradients are sufficiently small.

3.8 RELATIVE DISPLACEMENTS. LINEAR ROTATION TENSOR. ROTATION VECTOR

In Fig. 3-3 the displacements of two neighboring particles are represented by the vectors $u_i^{(P_0)}$ and $u_i^{(Q_0)}$ (see also Fig. 3-2). The vector

$$du_i = u_i^{(Q_0)} - u_i^{(P_0)} \quad \text{or} \quad d\mathbf{u} = \mathbf{u}^{(Q_0)} - \mathbf{u}^{(P_0)} \qquad (3.45)$$

is called the *relative displacement vector* of the particle originally at Q_0 with respect to the particle originally at P_0. Assuming suitable continuity conditions on the displacement field, a Taylor series expansion for $u_i^{(P_0)}$ may be developed in the neighborhood of P_0. Neglecting higher-order terms in this expansion, the relative displacement vector can be written as

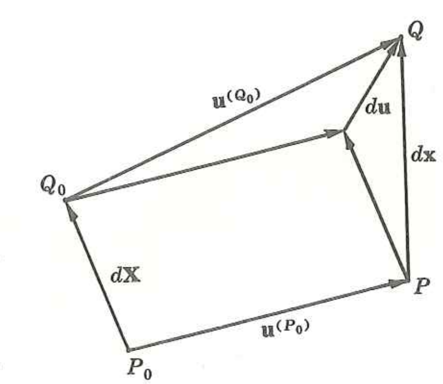

Fig. 3-3

$$du_i = (\partial u_i/\partial X_j)_{P_0}\, dX_j \quad \text{or} \quad d\mathbf{u} = (\mathbf{u}\nabla_{\mathbf{X}})_{P_0}\cdot d\mathbf{X} \qquad (3.46)$$

Here the parentheses on the partial derivatives are to emphasize the requirement that the derivatives are to be evaluated at point P_0. These derivatives are actually the components of the material displacement gradient. Equation (*3.46*) is the Lagrangian form of the relative displacement vector.

It is also useful to define the *unit relative displacement vector* du_i/dX in which dX is the magnitude of the differential distance vector dX_i. Accordingly if ν_i is a unit vector in the direction of dX_i so that $dX_i = \nu_i\, dX$, then

$$\frac{du_i}{dX} = \frac{\partial u_i}{\partial X_j}\frac{dX_j}{dX} = \frac{\partial u_i}{\partial X_j}\nu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dX} = \mathbf{u}\nabla_{\mathbf{X}}\cdot\hat{\boldsymbol{\nu}} = \mathbf{J}\cdot\hat{\boldsymbol{\nu}} \qquad (3.47)$$

Since the material displacement gradient $\partial u_i/\partial X_j$ may be decomposed uniquely into a symmetric and an antisymmetric part, the relative displacement vector du_i may be written as

$$du_i = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)\right] dX_j$$

or

$$d\mathbf{u} = [\tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) + \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} - \nabla_{\mathbf{X}}\mathbf{u})] \cdot d\mathbf{X} \qquad (3.48)$$

The first term in the square brackets in (*3.48*) is recognized as the linear Lagrangian strain tensor l_{ij}. The second term is known as the *linear Lagrangian rotation tensor* and is denoted by

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{W} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} - \nabla_{\mathbf{X}}\mathbf{u}) \qquad (3.49)$$

In a displacement for which the strain tensor l_{ij} is identically zero in the vicinity of point P_0, the relative displacement at that point will be an infinitesimal rigid body rotation. This infinitesimal rotation may be represented by the *rotation vector*

$$w_i = \tfrac{1}{2}\epsilon_{ijk}W_{kj} \quad \text{or} \quad \mathbf{w} = \tfrac{1}{2}\nabla_{\mathbf{X}} \times \mathbf{u} \qquad (3.50)$$

in terms of which the relative displacement is given by the expression

$$du_i = \epsilon_{ijk}w_j\,dX_k \quad \text{or} \quad d\mathbf{u} = \mathbf{w} \times d\mathbf{X} \qquad (3.51)$$

The development of the Lagrangian description of the relative displacement vector, the linear rotation tensor and the linear rotation vector is paralleled completely by an analogous development for the Eulerian counterparts of these quantities. Accordingly the *Eulerian description* of the *relative displacement vector* is given by

$$du_i = \frac{\partial u_i}{\partial x_j}dx_j \quad \text{or} \quad d\mathbf{u} = \mathbf{K}\cdot d\mathbf{x} \qquad (3.52)$$

and the *unit relative displacement vector* by

$$du_i = \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dx} = \frac{\partial u_i}{\partial x_j}\mu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dx} = \mathbf{u}\nabla_{\mathbf{x}}\cdot\hat{\boldsymbol{\mu}} = \mathbf{K}\cdot\hat{\boldsymbol{\mu}} \qquad (3.53)$$

Decomposition of the Eulerian displacement gradient $\partial u_i/\partial x_j$ results in the expression

$$\frac{du_i}{dx} = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dx_j$$

or

$$d\mathbf{u} = [\tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) + \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\mathbf{u})] \cdot d\mathbf{x} \qquad (3.54)$$

The first term in the square brackets of (*3.54*) is the Eulerian linear strain tensor ϵ_{ij}. The second term is the *linear Eulerian rotation tensor* and is denoted by

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{\Omega} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\mathbf{u}) \qquad (3.55)$$

From (*3.55*), the *linear Eulerian rotation vector* is defined by

$$\omega_i = \tfrac{1}{2}\epsilon_{ijk}\omega_{kj} \quad \text{or} \quad \boldsymbol{\omega} = \tfrac{1}{2}\nabla_{\mathbf{x}} \times \mathbf{u} \qquad (3.56)$$

in terms of which the relative displacement is given by the expression

$$du_i = \epsilon_{ijk}\omega_j\,dx_k \quad \text{or} \quad d\mathbf{u} = \boldsymbol{\omega} \times d\mathbf{x} \qquad (3.57)$$

3.9 INTERPRETATION OF THE LINEAR STRAIN TENSORS

For small deformation theory, the finite Lagrangian strain tensor L_{ij} in (*3.36*) may be replaced by the linear Lagrangian strain tensor l_{ij}, and that expression may now be written

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = 2l_{ij}\,dX_i\,dX_j$$

or

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = d\mathbf{X}\cdot 2\mathbf{L}\cdot d\mathbf{X} \qquad (3.58)$$

Since $dx \approx dX$ for small deformations, this equation may be put into the form

$$\frac{dx - dX}{dX} = l_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} = l_{ij}\nu_i\nu_j \quad \text{or} \quad \frac{dx - dX}{dX} = \hat{\boldsymbol{\nu}}\cdot\mathbf{L}\cdot\hat{\boldsymbol{\nu}} \qquad (3.59)$$

The left-hand side of (*3.59*) is recognized as the change in length per unit original length of the differential element and is called the *normal strain* for the line element originally having direction cosines dX_i/dX.

When (*3.59*) is applied to the differential line element P_0Q_0, located with respect to the set of local axes at P_0 as shown in Fig. 3-4, the result will be the normal strain for that element. Because P_0Q_0 here lies along the X_2 axis,

$$dX_1/dX = dX_3/dX = 0, \quad dX_2/dX = 1$$

and therefore (*3.59*) becomes

$$\frac{dx - dX}{dX} = l_{22} = \frac{\partial u_2}{\partial X_2} \qquad (3.60)$$

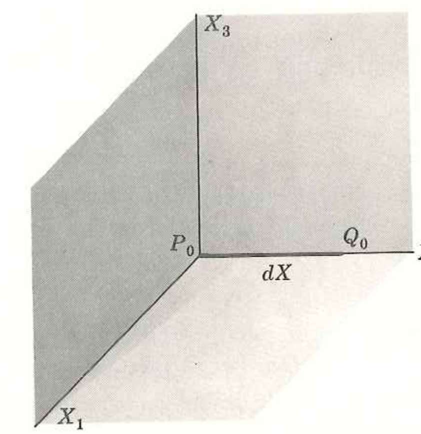

Fig. 3-4

Thus the normal strain for an element originally along the X_2 axis is seen to be the component l_{22}. Likewise for elements originally situated along the X_1 and X_3 axes, (*3.59*) yields normal strain values l_{11} and l_{33} respectively. In general, therefore, the diagonal terms of the linear strain tensor represent normal strains in the coordinate directions.

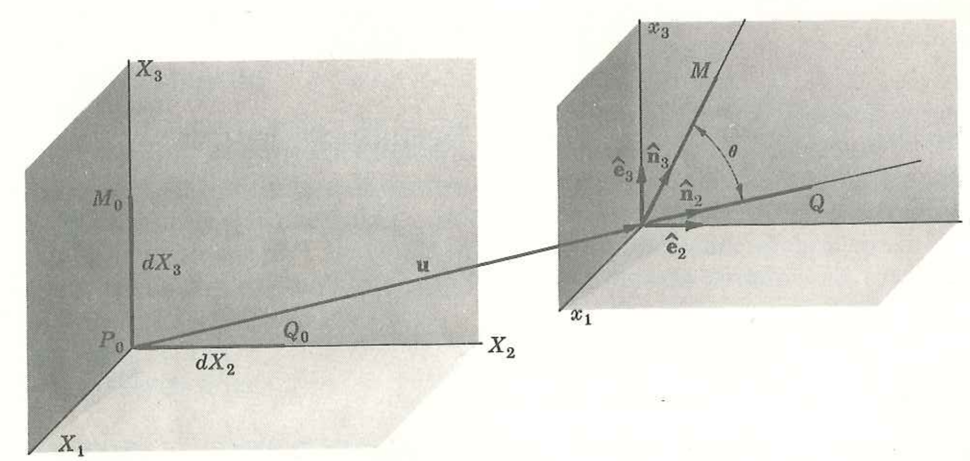

Fig. 3-5

The physical interpretation of the off-diagonal terms of l_{ij} may be obtained by a consideration of the line elements originally located along two of the coordinate axes. In Fig. 3-5 the line elements P_0Q_0 and P_0M_0 originally along the X_2 and X_3 axes, respectively, become after deformation the line elements PQ and PM with respect to the parallel set of local axes with origin at P. The original right angle between the line elements becomes the angle θ. From (*3.46*) and the assumption of small deformation theory, a first order approximation gives the unit vector at P in the direction of Q as

$$\hat{\mathbf{n}}_2 = \frac{\partial u_1}{\partial X_2}\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \frac{\partial u_3}{\partial X_2}\hat{\mathbf{e}}_3 \tag{3.61}$$

and, for the unit vector at P in the direction of M, as

$$\hat{\mathbf{n}}_3 = \frac{\partial u_1}{\partial X_3}\hat{\mathbf{e}}_1 + \frac{\partial u_2}{\partial X_3}\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \tag{3.62}$$

Therefore

$$\cos\theta = \hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3 = \frac{\partial u_1}{\partial X_3}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \tag{3.63}$$

or, neglecting the product term which is of higher order,

$$\cos\theta = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} = 2l_{23} \tag{3.64}$$

Furthermore, taking the change in the right angle between the elements as $\gamma_{23} = \pi/2 - \theta$, and remembering that for the linear theory γ_{23} is very small, it follows that

$$\gamma_{23} \approx \sin\gamma_{23} = \sin(\pi/2-\theta) = \cos\theta = 2l_{23} \tag{3.65}$$

Therefore the off-diagonal terms of the linear strain tensor represent one-half the angle change between two line elements originally at right angles to one another. These strain components are called *shearing strains*, and because of the factor 2 in (*3.65*) these tensor components are equal to one-half the familiar "engineering" shearing strains.

A development, essentially paralleling the one just presented for the interpretation of the components of l_{ij}, may also be made for the linear Eulerian strain tensor ϵ_{ij}. The essential difference in the derivations rests in the choice of line elements, which in the Eulerian description must be those that lie along the coordinate axes after deformation. The diagonal terms of ϵ_{ij} are the normal strains, and the off-diagonal terms the shearing strains. For those deformations in which the assumption $l_{ij} = \epsilon_{ij}$ is valid, no distinction is made between the Eulerian and Lagrangian interpretations.

3.10 STRETCH RATIO. FINITE STRAIN INTERPRETATION

An important measure of the extensional strain of a differential line element is the ratio dx/dX, known as the *stretch* or *stretch ratio*. This quantity may be defined at either the point P_0 in the undeformed configuration or at the point P in the deformed configuration. Thus from (*3.34*) the squared stretch at point P_0 for the line element along the unit vector $\hat{\mathbf{m}} = d\mathbf{X}/dX$, is given by

$$\left(\frac{dx}{dX}\right)^2_{P_0} = \Lambda^2_{(\hat{\mathbf{m}})} = G_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} \quad \text{or} \quad \Lambda^2_{(\hat{\mathbf{m}})} = \hat{\mathbf{m}}\cdot\mathbf{G}\cdot\hat{\mathbf{m}} \tag{3.66}$$

Similarly, from (*3.30*) the reciprocal of the squared stretch for the line element at P along the unit vector $\hat{\mathbf{n}} = d\mathbf{x}/dx$ is given by

$$\left(\frac{dX}{dx}\right)^2_{P} = \frac{1}{\lambda^2_{(\hat{\mathbf{n}})}} = C_{ij}\frac{dx_i}{dx}\frac{dx_j}{dx} \quad \text{or} \quad \frac{1}{\lambda^2_{(\hat{\mathbf{n}})}} = \hat{\mathbf{n}}\cdot\mathbf{C}\cdot\hat{\mathbf{n}} \tag{3.67}$$

For an element originally along the local X_2 axis shown in Fig. 3-4, $\hat{\mathbf{m}} \equiv \hat{\mathbf{e}}_2$ and therefore $dX_1/dX = dX_3/dX = 0$, $dX_2/dX = 1$ so that (*3.66*) yields for such an element

$$\Lambda^2_{(\hat{\mathbf{e}}_2)} = G_{22} = 1 + 2L_{22} \tag{3.68}$$

Similar results may be determined for $\Lambda^2_{(\hat{\mathbf{e}}_1)}$ and $\Lambda^2_{(\hat{\mathbf{e}}_3)}$.

For an element parallel to the x_2 axis after deformation, (*3.67*) yields the result

$$\frac{1}{\lambda^2_{(\hat{\mathbf{e}}_2)}} = 1 - 2E_{22} \tag{3.69}$$

with similar expressions for the quantities $1/\lambda^2_{(\hat{\mathbf{e}}_1)}$ and $1/\lambda^2_{(\hat{\mathbf{e}}_3)}$. In general, $\Lambda_{(\hat{\mathbf{e}}_2)}$ is not equal to $\lambda_{(\hat{\mathbf{e}}_2)}$ since the element originally along the X_2 axis will not likely lie along the x_2 axis after deformation.

The stretch ratio provides a basis for interpretation of the finite strain tensors. Thus the change of length per unit of original length is

$$\frac{dx - dX}{dX} = \frac{dx}{dX} - 1 = \Lambda_{(\hat{\mathbf{m}})} - 1 \tag{3.70}$$

and for the element P_0Q_0 along the X_2 axis (of Fig. 3-4), the *unit extension* is therefore

$$L_{(2)} = \Lambda_{(\hat{\mathbf{e}}_2)} - 1 = \sqrt{1+2L_{22}} - 1 \tag{3.71}$$

This result may also be derived directly from (*3.36*). For small deformation theory, (*3.71*) reduces to (*3.60*). Also, the unit extensions $L_{(1)}$ and $L_{(3)}$ are given by analogous equations in terms of L_{11} and L_{33} respectively.

For the two differential line elements shown in Fig. 3-5, the change in angle $\gamma_{23} = \pi/2 - \theta$ is given in terms of $\Lambda_{(\hat{\mathbf{e}}_2)}$ and $\Lambda_{(\hat{\mathbf{e}}_3)}$ by

$$\sin\gamma_{23} = \frac{2L_{23}}{\Lambda_{(\hat{\mathbf{e}}_2)}\Lambda_{(\hat{\mathbf{e}}_3)}} = \frac{2L_{23}}{\sqrt{1+2L_{22}}\sqrt{1+2L_{33}}} \tag{3.72}$$

When deformations are small, (*3.72*) reduces to (*3.65*).

3.11 STRETCH TENSORS. ROTATION TENSOR

The so-called *polar decomposition* of an arbitrary, nonsingular, second-order tensor is given by the product of a positive symmetric second-order tensor with an orthogonal second-order tensor. When such a multiplicative decomposition is applied to the deformation gradient $\mathbf{F}$, the result may be written

$$F_{ij} \equiv \frac{\partial x_i}{\partial X_j} = R_{ik}S_{kj} = T_{ik}R_{kj} \quad \text{or} \quad \mathbf{F} = \mathbf{R}\cdot\mathbf{S} = \mathbf{T}\cdot\mathbf{R} \tag{3.73}$$

in which $\mathbf{R}$ is the *orthogonal rotation tensor*, and $\mathbf{S}$ and $\mathbf{T}$ are positive symmetric tensors known as the *right stretch tensor* and *left stretch tensor* respectively.

The interpretation of (*3.73*) is provided through the relationship $dx_i = (\partial x_i/\partial X_j)\,dX_j$ given by (*3.33*). Inserting the inner products of (*3.73*) into (*3.33*) results in the equations

$$dx_i = R_{ik}S_{kj}\,dX_j = T_{ik}R_{kj}\,dX_j \quad \text{or} \quad d\mathbf{x} = \mathbf{R}\cdot\mathbf{S}\cdot d\mathbf{X} = \mathbf{T}\cdot\mathbf{R}\cdot d\mathbf{X} \tag{3.74}$$

From these expressions the deformation of dX_i into dx_i as illustrated in Fig. 3-2 may be given either of two physical interpretations. In the first form of the right hand side of (*3.74*), the deformation consists of a sequential stretching (by $\mathbf{S}$) and rotation to be followed by a rigid body displacement to the point P. In the second form, a rigid body translation to P is followed by a rotation and finally the stretching (by $\mathbf{T}$). The translation, of course, does not alter the vector components relative to the axes X_i and x_i.

3.12 TRANSFORMATION PROPERTIES OF STRAIN TENSORS

The various strain tensors L_{ij}, E_{ij}, l_{ij} and ϵ_{ij} defined respectively by (3.37), (3.39), (3.42) and (3.43) are all second-order Cartesian tensors as indicated by the two free indices in each. Accordingly for a set of rotated axes X'_i having the transformation matrix $[b_{ij}]$ with respect to the set of local unprimed axes X_i at point P_0 as shown in Fig. 3-6(*a*), the components of L'_{ij} and l'_{ij} are given by

$$L'_{ij} = b_{ip}b_{jq}L_{pq} \quad \text{or} \quad \mathbf{L}'_G = \mathbf{B}\cdot\mathbf{L}_G\cdot\mathbf{B}_c \tag{3.75}$$

and

$$l'_{ij} = b_{ip}b_{jq}l_{pq} \quad \text{or} \quad \mathbf{L}' = \mathbf{B}\cdot\mathbf{L}\cdot\mathbf{B}_c \tag{3.76}$$

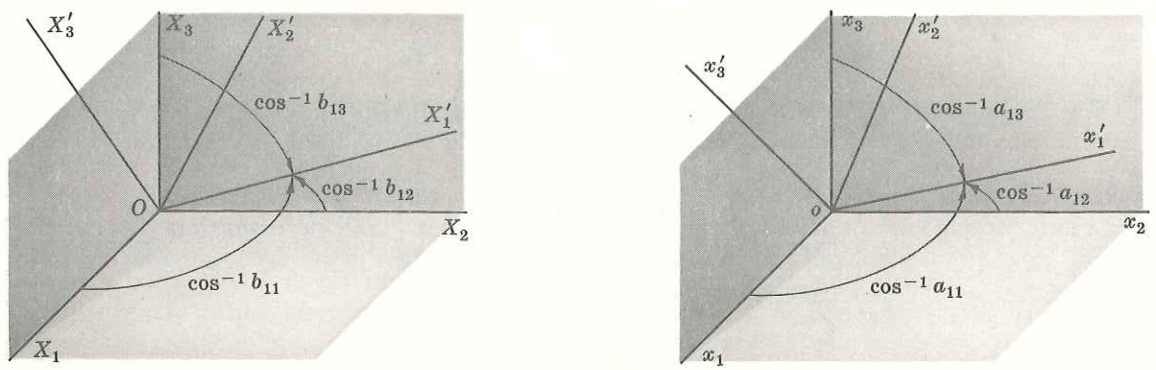

Fig. 3-6

Likewise, for the rotated axes x'_i having the transformation matrix $[a_{ij}]$ in Fig. 3-6(*b*), the components of E'_{ij} and e'_{ij} are given by

$$E'_{ij} = a_{ip}a_{jq}E_{pq} \quad \text{or} \quad \mathbf{E}'_A = \mathbf{A}\cdot\mathbf{E}_A\cdot\mathbf{A}_c \tag{3.77}$$

and

$$\epsilon'_{ij} = a_{ip}a_{jq}\epsilon_{pq} \quad \text{or} \quad \mathbf{E}' = \mathbf{A}\cdot\mathbf{E}\cdot\mathbf{A}_c \tag{3.78}$$

By analogy with the stress quadric described in Section 2.9, page 50, the *Lagrangian* and *Eulerian linear strain quadrics* may be given with reference to local Cartesian coordinates η_i and ζ_i at the points P_0 and P respectively as shown in Fig. 3-7. Thus the equation of the *Lagrangian strain quadric* is given by

$$l_{ij}\eta_i\eta_j = \pm h^2 \quad \text{or} \quad \boldsymbol{\eta}\cdot\mathbf{L}\cdot\boldsymbol{\eta} = \pm h^2 \tag{3.79}$$

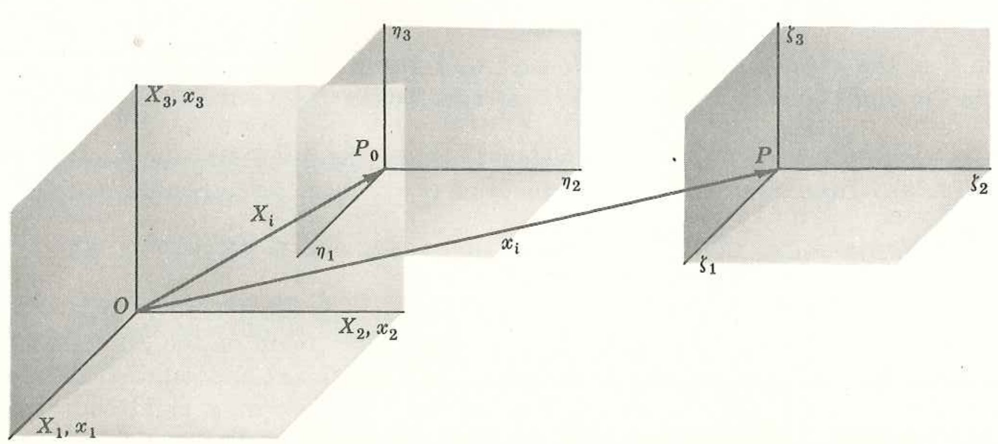

Fig. 3-7

and the equation of the *Eulerian strain quadric* is given by

$$\epsilon_{ij}\zeta_i\zeta_j = \pm g^2 \quad \text{or} \quad \boldsymbol{\zeta}\cdot\mathbf{E}\cdot\boldsymbol{\zeta} = \pm g^2 \tag{3.80}$$

Two important properties of the Lagrangian {Eulerian} linear strain quadric are:

1. The normal strain with respect to the original {final} length of a line element is inversely proportional to the distance squared from the origin of the quadric P_0 $\{P\}$ to a point on its surface.
2. The relative displacement of the neighboring particle located at Q_0 $\{Q\}$ per unit original {final} length is parallel to the normal of the quadric surface at the point of intersection with the line through P_0Q_0 $\{PQ\}$.

Additional insight into the nature of local deformations in the neighborhood of P_0 is provided by defining the *strain ellipsoid* at that point. Thus for the undeformed continuum, the equation of the bounding surface of an infinitesimal sphere of radius R is given in terms of local material coordinates by (3.28) as

$$(dX)^2 = \delta_{ij}\,dX_i\,dX_j = R^2 \quad \text{or} \quad (dX)^2 = d\mathbf{X}\cdot\mathbf{I}\cdot d\mathbf{X} = R^2 \tag{3.81}$$

After deformation, the equation of the surface of the same material particles is given by (3.30) as

$$(dX)^2 = C_{ij}\,dx_i\,dx_j = R^2 \quad \text{or} \quad (dX)^2 = d\mathbf{x}\cdot\mathbf{C}\cdot d\mathbf{x} = R^2 \tag{3.82}$$

which describes an ellipsoid, known as the *material strain ellipsoid.* Therefore a spherical volume of the continuum in the undeformed state is changed into an ellipsoid at P_0 by the deformation. By comparison, an infinitesimal spherical volume at P in the deformed continuum began as an ellipsoidal volume element in the undeformed state. For a sphere of radius r at P, the equations for these surfaces in terms of local coordinates are given by (3.32) for the sphere as

$$(dx)^2 = \delta_{ij}\,dx_i\,dx_j = r^2 \quad \text{or} \quad (dx)^2 = d\mathbf{x}\cdot\mathbf{I}\cdot d\mathbf{x} = r^2 \tag{3.83}$$

and by (3.34) for the ellipsoid as

$$(dx)^2 = G_{ij}\,dX_i\,dX_j = r^2 \quad \text{or} \quad (dx)^2 = d\mathbf{X}\cdot\mathbf{G}\cdot d\mathbf{X} = r^2 \tag{3.84}$$

The ellipsoid of (3.84) is called the *spatial strain ellipsoid.* Such strain ellipsoids as described here are frequently known as *Cauchy strain ellipsoids.*

3.13 PRINCIPAL STRAINS. STRAIN INVARIANTS. CUBICAL DILATATION

The Lagrangian and Eulerian linear strain tensors are symmetric second-order Cartesian tensors, and accordingly the determination of their principal directions and principal strain values follows the standard development presented in Section 1.19, page 20. Physically, a principal direction of the strain tensor is one for which the orientation of an element at a given point is not altered by a pure strain deformation. The principal strain value is simply the unit relative displacement (normal strain) that occurs in the principal direction.

For the Lagrangian strain tensor l_{ij}, the unit relative displacement vector is given by (3.47), which may be written

$$\frac{du_i}{dX} = (l_{ij} + W_{ij})\nu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dX} = (\mathbf{L}+\mathbf{W})\cdot\hat{\boldsymbol{\nu}} \tag{3.85}$$

Calling $l_i^{(\hat{n})}$ the normal strain in the direction of the unit vector n_i, (3.85) yields for pure strain ($W_{ij} \equiv 0$) the relation

$$l_i^{(\hat{n})} = l_{ij}n_j \quad \text{or} \quad \mathbf{l}^{(\hat{n})} = \mathbf{L}\cdot\hat{\mathbf{n}} \tag{3.86}$$

If the direction n_i is a principal direction with a principal strain value l, then

$$l_i^{(\hat{n})} = l n_i = l \delta_{ij} n_j \quad \text{or} \quad \mathbf{l}^{(\hat{n})} = l \hat{\mathbf{n}} = l \mathbf{I} \cdot \hat{\mathbf{n}} \tag{3.87}$$

Equating the right-hand sides of (3.86) and (3.87) leads to the relationship

$$(l_{ij} - \delta_{ij} l) n_j = 0 \quad \text{or} \quad (\mathbf{L} - l\mathbf{I}) \cdot \hat{\mathbf{n}} = 0 \tag{3.88}$$

which together with the condition $n_i n_i = 1$ on the unit vectors n_i provide the necessary equations for determining the principal strain value l and its direction cosines n_i. Nontrivial solutions of (3.88) exist if and only if the determinant of coefficients vanishes. Therefore

$$|l_{ij} - \delta_{ij} l| = 0 \quad \text{or} \quad |\mathbf{L} - l\mathbf{I}| = 0 \tag{3.89}$$

which upon expansion yields the characteristic equation of l_{ij}, the cubic

$$l^3 - \mathrm{I}_L l^2 + \mathrm{II}_L l - \mathrm{III}_L = 0 \tag{3.90}$$

where $\mathrm{I}_L = l_{ii} = \operatorname{tr} \mathbf{L}, \quad \mathrm{II}_L = \tfrac{1}{2}(l_{ii} l_{jj} - l_{ij} l_{ij}), \quad \mathrm{III}_L = |l_{ij}| = \det \mathbf{L}$ (3.91)

are the first, second and third *Lagrangian strain invariants* respectively. The roots of (3.90) are the principal strain values denoted by $l_{(1)}$, $l_{(2)}$ and $l_{(3)}$.

The first invariant of the Lagrangian strain tensor may be expressed in terms of the principal strains as

$$\mathrm{I}_L = l_{ii} = l_{(1)} + l_{(2)} + l_{(3)} \tag{3.92}$$

and has an important physical interpretation. To see this, consider a differential rectangular parallelepiped whose edges are parallel to the principal strain directions as shown in Fig. 3-8. The change in volume per unit original volume of this element is called the *cubical dilatation* and is given by

$$D_0 = \frac{\Delta V_0}{V_0} = \frac{dX_1(1+l_{(1)})\,dX_2(1+l_{(2)})\,dX_3(1+l_{(3)}) - dX_1\,dX_2\,dX_3}{dX_1\,dX_2\,dX_3} \tag{3.93}$$

For small strain theory, the first-order approximation of this ratio is the sum

$$D_0 = l_{(1)} + l_{(2)} + l_{(3)} = \mathrm{I}_L \tag{3.94}$$

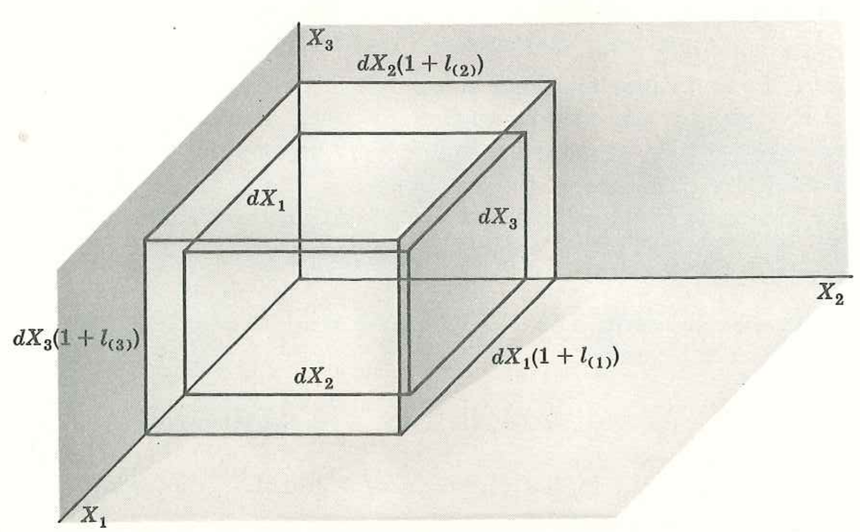

Fig. 3-8

With regard to the Eulerian strain tensor ϵ_{ij} and its associated unit relative displacement vector $\epsilon_i^{(\hat{n})}$, the principal directions and principal strain values $\epsilon_{(1)}, \epsilon_{(2)}, \epsilon_{(3)}$ are determined in exactly the same way as their Lagrangian counterparts. The Eulerian strain invariants may be expressed in terms of the principal strains as

$$\begin{aligned} \mathrm{I}_E &= \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)} \\ \mathrm{II}_E &= \epsilon_{(1)}\epsilon_{(2)} + \epsilon_{(2)}\epsilon_{(3)} + \epsilon_{(3)}\epsilon_{(1)} \\ \mathrm{III}_E &= \epsilon_{(1)}\epsilon_{(2)}\epsilon_{(3)} \end{aligned} \tag{3.95}$$

The cubical dilatation for the Eulerian description is given by

$$\Delta V/V = D = \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)} \tag{3.96}$$

3.14 SPHERICAL AND DEVIATOR STRAIN TENSORS

The Lagrangian and Eulerian *linear* strain tensors may each be split into a *spherical* and *deviator* tensor in the same manner in which the stress tensor decomposition was carried out in Chapter 2. As before, if Lagrangian and Eulerian deviator tensor components are denoted by d_{ij} and e_{ij} respectively, the resolution expressions are

$$l_{ij} = d_{ij} + \delta_{ij}\frac{l_{kk}}{3} \quad \text{or} \quad \mathbf{L} = \mathbf{L}_D + \frac{\mathbf{I}(\operatorname{tr}\mathbf{L})}{3} \tag{3.97}$$

and

$$\epsilon_{ij} = e_{ij} + \delta_{ij}\frac{\epsilon_{kk}}{3} \quad \text{or} \quad \mathbf{E} = \mathbf{E}_D + \frac{\mathbf{I}(\operatorname{tr}\mathbf{E})}{3} \tag{3.98}$$

The deviator tensors are associated with shear deformation for which the cubical dilatation vanishes. Therefore it is not surprising that the first invariants d_{ii} and e_{ii} of the deviator strain tensors are identically zero.

3.15 PLANE STRAIN. MOHR'S CIRCLES FOR STRAIN

When one and only one of the principal strains at a point in a continuum is zero, a state of *plane strain* is said to exist at that point. In the Eulerian description (the Lagrangian description follows exactly the same pattern), if x_3 is taken as the direction of the zero principal strain, a state of plane strain parallel to the x_1x_2 plane exists and the linear strain tensor is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.99}$$

When x_1 and x_2 are also principal directions, the strain tensor has the form

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{(1)} & 0 & 0 \\ 0 & \epsilon_{(2)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{(1)} & 0 & 0 \\ 0 & \epsilon_{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.100}$$

In many books on "Strength of Materials" and "Elasticity", plane strain is referred to as *plane deformation* since the deformation field is identical in all planes perpendicular to the direction of the zero principal strain. For plane strain perpendicular to the x_3 axis, the displacement vector may be taken as a function of x_1 and x_2 only. The appropriate displacement components for this case of plane strain are designated by

$$u_1 = u_1(x_1, x_2)$$
$$u_2 = u_2(x_1, x_2) \qquad (3.101)$$
$$u_3 = C \text{ (a constant, usually taken as zero)}$$

Inserting these expressions into the definition of ϵ_{ij} given by (*3.43*) produces the plane strain tensor in the same form shown in (*3.99*).

A graphical description of the state of strain at a point is provided by the *Mohr's circles for strain* in a manner exactly like that presented in Chapter 2 for the Mohr's circles for stress. For this purpose the strain tensor is often displayed in the form

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \epsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \epsilon_{33} \end{pmatrix} \qquad (3.102)$$

Here the γ_{ij} (with $i \neq j$) are the so-called "engineering" shear strain components, which are twice the tensorial shear strain components.

The state of strain at an unloaded point on the bounding surface of a continuum body is locally plane strain. Frequently in experimental studies involving strain measurements at such a surface point, Mohr's strain circles are useful for reporting the observed data. Usually three normal strains are measured at the given point by means of a strain rosette, and the Mohr's circles diagram constructed from these. Corresponding to the plane stress Mohr's circles, a typical case of plane strain diagram is shown in Fig. 3-9. The principal normal strains are labeled as such in the diagram, and the maximum shear strain values are represented by points D and E.

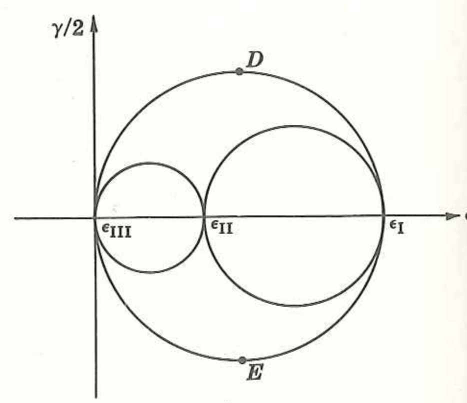

Fig. 3-9

3.16 COMPATIBILITY EQUATIONS FOR LINEAR STRAINS

If the strain components ϵ_{ij} are given explicitly as functions of the coordinates, the six independent equations (*3.43*)

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

may be viewed as a system of six partial differential equations for determining the three displacement components u_i. The system is over-determined and will not, in general, possess a solution for an arbitrary choice of the strain components ϵ_{ij}. Therefore if the displacement components u_i are to be single-valued and continuous, some conditions must be imposed upon the strain components. The necessary and sufficient conditions for such a displacement field are expressed by the equations

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_m} + \frac{\partial^2 \epsilon_{km}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_m} - \frac{\partial^2 \epsilon_{jm}}{\partial x_i \partial x_k} = 0 \qquad (3.103)$$

There are eighty-one equations in all in (*3.103*) but only six are distinct. These six written in explicit and symbolic form appear as

$$\left.\begin{array}{lll}
1. & \dfrac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \epsilon_{22}}{\partial x_1^2} & = 2\dfrac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \\
2. & \dfrac{\partial^2 \epsilon_{22}}{\partial x_3^2} + \dfrac{\partial^2 \epsilon_{33}}{\partial x_2^2} & = 2\dfrac{\partial^2 \epsilon_{23}}{\partial x_2 \partial x_3} \\
3. & \dfrac{\partial^2 \epsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \epsilon_{11}}{\partial x_3^2} & = 2\dfrac{\partial^2 \epsilon_{31}}{\partial x_3 \partial x_1} \\
4. & \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \epsilon_{23}}{\partial x_1} + \dfrac{\partial \epsilon_{31}}{\partial x_2} + \dfrac{\partial \epsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2 \epsilon_{11}}{\partial x_2 \partial x_3} \\
5. & \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \epsilon_{23}}{\partial x_1} - \dfrac{\partial \epsilon_{31}}{\partial x_2} + \dfrac{\partial \epsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2 \epsilon_{22}}{\partial x_3 \partial x_1} \\
6. & \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \epsilon_{23}}{\partial x_1} + \dfrac{\partial \epsilon_{31}}{\partial x_2} - \dfrac{\partial \epsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2 \epsilon_{33}}{\partial x_1 \partial x_2}
\end{array}\right\} \quad \text{or} \quad \nabla_{\mathbf{x}} \times \mathbf{E} \times \nabla_{\mathbf{x}} = 0 \qquad (3.104)$$

Compatibility equations in terms of the Lagrangian linear strain tensor l_{ij} may also be written down by an obvious correspondence to the Eulerian form given above. For plane strain parallel to the x_1x_2 plane, the six equations in (*3.104*) reduce to the single equation

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \quad \text{or} \quad \nabla_{\mathbf{x}} \times \mathbf{E} \times \nabla_{\mathbf{x}} = 0 \qquad (3.105)$$

where $\mathbf{E}$ is of the form given by (*3.99*).

Solved Problems

DISPLACEMENT AND DEFORMATION (Sec. 3.1-3.5)

3.1. With respect to superposed material axes X_i and spatial axes x_i, the displacement field of a continuum body is given by $x_1 = X_1$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$ where A is a constant. Determine the displacement vector components in both the material and spatial forms.

From (*3.13*) directly, the displacement components in material form are $u_1 = x_1 - X_1 = 0$, $u_2 = x_2 - X_2 = AX_3$, $u_3 = x_3 - X_3 = AX_2$. Inverting the given displacement relations to obtain $X_1 = x_1$, $X_2 = (x_2 - Ax_3)/(1-A^2)$, $X_3 = (x_3 - Ax_2)/(1-A^2)$, the spatial components of $\mathbf{u}$ are $u_1 = 0$, $u_2 = A(x_3 - Ax_2)/(1-A^2)$, $u_3 = A(x_2 - Ax_3)/(1-A^2)$.

From these results it is noted that the originally straight line of material particles expressed by $X_1 = 0$, $X_2 + X_3 = 1/(1+A)$ occupies the location $x_1 = 0$, $x_2 + x_3 = 1$ after displacement. Likewise the particle line $X_1 = 0$, $X_2 = X_3$ becomes after displacement $x_1 = 0$, $x_2 = x_3$. (Interpret the physical meaning of this.)

3.2. For the displacement field of Problem 3.1 determine the displaced location of the material particles which originally comprise (*a*) the plane circular surface $X_1 = 0$, $X_2^2 + X_3^2 = 1/(1-A^2)$, (*b*) the infinitesimal cube with edges along the coordinate axes of length $dX_i = dX$. Sketch the displaced configurations for (*a*) and (*b*) if $A = \frac{1}{2}$.

(*a*) By the direct substitutions $X_2 = (x_2 - Ax_3)/(1-A^2)$ and $X_3 = (x_3 - Ax_2)/(1-A^2)$, the circular surface becomes the elliptical surface $(1+A^2)x_2^2 - 4Ax_2x_3 + (1+A^2)x_3^2 = (1-A^2)$. For $A = \frac{1}{2}$, this is bounded by the ellipse $5x_2^2 - 8x_2x_3 + 5x_3^2 = 3$ which when referred to its principal axes x_i^* (at 45° with x_i, $i = 2, 3$) has the equation $x_2^{*2} + 9x_3^{*2} = 3$. Fig. 3-10 below shows this displacement pattern.

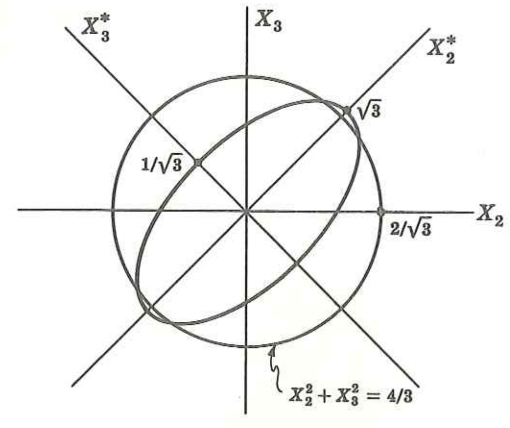

Fig. 3-10

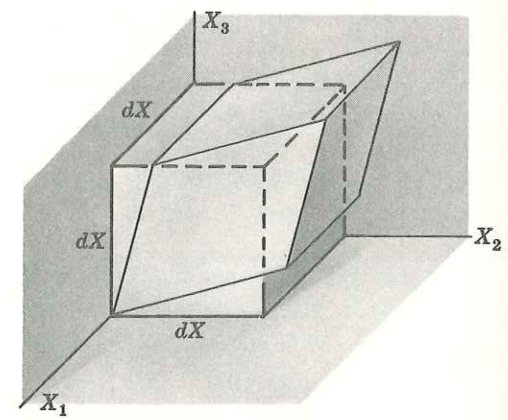

Fig. 3-11

(b) From Problem 3.1, the displacements of the edges of the cube are readily calculated. For the edge $X_1 = X_1$, $X_2 = X_3 = 0$, $u_1 = u_2 = u_3 = 0$. For the edge $X_1 = 0 = X_2$, $X_3 = X_3$, $u_1 = u_3 = 0$, $u_2 = AX_3$ and the particles on this edge are displaced in the X_2 direction proportionally to their distance from the origin. For the edge $X_1 = X_3 = 0$, $X_2 = X_2$, $u_1 = u_2 = 0$, $u_3 = AX_2$. The initial and displaced positions of the cube are shown in Fig. 3-11.

3.3. For superposed material and spatial axes, the displacement vector of a body is given by $\mathbf{u} = 4X_1^2\hat{\mathbf{e}}_1 + X_2X_3^2\hat{\mathbf{e}}_2 + X_1X_3^2\hat{\mathbf{e}}_3$. Determine the displaced location of the particle originally at $(1,0,2)$.

The original position vector of the particle is $\mathbf{X} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_3$. Its displacement is $\mathbf{u} = 4\hat{\mathbf{e}}_1 + 4\hat{\mathbf{e}}_3$ and since $\mathbf{x} = \mathbf{X} + \mathbf{u}$, its final position vector is $\mathbf{x} = 5\hat{\mathbf{e}}_1 + 6\hat{\mathbf{e}}_3$.

3.4. With respect to rectangular Cartesian material coordinates X_i, a displacement field is given by $U_1 = -AX_2X_3$, $U_2 = AX_1X_3$, $U_3 = 0$ where A is a constant. Determine the displacement components for cylindrical spatial coordinates x_i if the two systems have a common origin.

From the geometry of the axes (Fig. 3-12) the transformation tensor $\alpha_{pK} = \hat{\mathbf{e}}_p \cdot \hat{\mathbf{I}}_K$ is

$$\alpha_{pK} = \begin{pmatrix} \cos x_2 & \sin x_2 & 0 \\ -\sin x_2 & \cos x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and from the inverse form of (3.9) $u_p = \alpha_{pK}U_K$. Thus since Cartesian and cylindrical coordinates are related through the equations $X_1 = x_1 \cos x_2$, $X_2 = x_1 \sin x_2$, $X_3 = x_3$, equation (3.9) gives

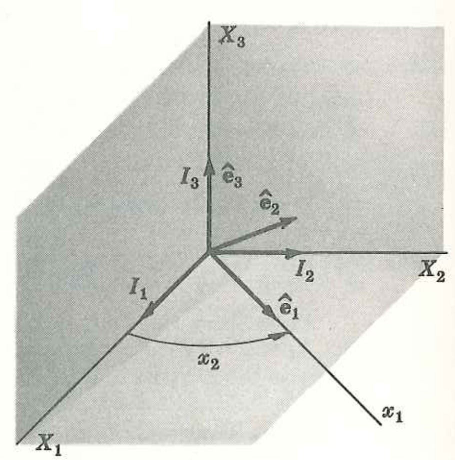

Fig. 3-12

$$\begin{aligned} u_1 &= (-\cos x_2)AX_2X_3 + (\sin x_2)AX_1X_3 \\ &= (-\cos x_2)Ax_3x_1 \sin x_2 + (\sin x_2)Ax_3x_1 \cos x_2 = 0 \\ u_2 &= (\sin x_2)AX_2X_3 + (\cos x_2)AX_1X_3 \\ &= (\sin^2 x_2)Ax_1x_3 + (\cos^2 x_2)Ax_1x_3 = Ax_1x_3 \\ u_3 &= 0 \end{aligned}$$

This displacement is that of a circular shaft in torsion.

3.5. The Lagrangian description of a deformation is given by $x_1 = X_1 + X_3(e^2-1)$, $x_2 = X_2 + X_3(e^2 - e^{-2})$, $x_3 = e^2X_3$ where e is a constant. Show that the Jacobian J does not vanish and determine the Eulerian equations describing this motion.

$$\text{From (3.16),} \quad J = \begin{vmatrix} 1 & 0 & (e^2-1) \\ 0 & 1 & (e^2-e^{-2}) \\ 0 & 0 & (e^2) \end{vmatrix} = e^2 \neq 0.$$

Inverting the equations, $X_1 = x_1 + x_3(e^{-2}-1)$, $X_2 = x_2 + x_3(e^{-4}-1)$, $X_3 = e^{-2}x_3$.

3.6. A displacement field is given by $\mathbf{u} = X_1X_3^2\hat{\mathbf{e}}_1 + X_1^2X_2\hat{\mathbf{e}}_2 + X_2^2X_3\hat{\mathbf{e}}_3$. Determine independently the material deformation gradient F and the material displacement gradient J and verify (3.24), $\mathsf{J} = \mathsf{F} - \mathsf{I}$.

From the given displacement vector $\mathbf{u}$, J is found to be

$$\frac{\partial u_i}{\partial X_j} = \begin{pmatrix} X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & X_1^2 & 0 \\ 0 & 2X_2X_3 & X_2^2 \end{pmatrix}$$

Since $\mathbf{x} = \mathbf{u} + \mathbf{X}$, the displacement field may also be described by equations $x_1 = X_1(1+X_3^2)$, $x_2 = X_2(1+X_1^2)$, $x_3 = X_3(1+X_2^2)$ from which F is readily found to be

$$\partial x_i/\partial X_j = \begin{pmatrix} 1+X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & 1+X_1^2 & 0 \\ 0 & 2X_2X_3 & 1+X_2^2 \end{pmatrix}$$

Direct substitution of the calculated tensors F and J into (3.24) verifies that the equation is satisfied.

3.7. A continuum body undergoes the displacement $\mathbf{u} = (3X_2 - 4X_3)\hat{\mathbf{e}}_1 + (2X_1 - X_3)\hat{\mathbf{e}}_2 + (4X_2 - X_1)\hat{\mathbf{e}}_3$. Determine the displaced position of the vector joining particles $A(1,0,3)$ and $B(3,6,6)$, assuming superposed material and spatial axes.

From (3.13), the spatial coordinates for this displacement are $x_1 = X_1 + 3X_2 - 4X_3$, $x_2 = 2X_1 + X_2 - X_3$, $x_3 = -X_1 + 4X_2 + X_3$. Thus the displaced position of particle A is given by $x_1 = -11$, $x_2 = -1$, $x_3 = 2$; and of particle B, $x_1 = -3$, $x_2 = 6$, $x_3 = 27$. Therefore the displaced position of the vector joining A and B may be written $\mathbf{V} = 8\hat{\mathbf{e}}_1 + 7\hat{\mathbf{e}}_2 + 25\hat{\mathbf{e}}_3$.

3.8. For the displacement field of Problem 3.7 determine the displaced position of the position vector of particle $C(2,6,3)$ which is parallel to the vector joining particles A and B. Show that the two vectors remain parallel after deformation.

By the analysis of Problem 3.7 the position vector of C becomes $\mathbf{U} = 8\hat{\mathbf{e}}_1 + 7\hat{\mathbf{e}}_2 + 25\hat{\mathbf{e}}_3$ which is clearly parallel to $\mathbf{V}$. This is an example of so-called *homogeneous deformation*.

3.9. The general formulation of homogeneous deformation is given by the displacement field $u_i = A_{ij}X_j$ where the A_{ij} are constants or at most functions of time. Show that this deformation is such that (a) plane sections remain plane, (b) straight lines remain straight.

(a) From (3.13), $\quad x_i = X_i + u_i = X_i + A_{ij}X_j = (\delta_{ij} + A_{ij})X_j$

Chapter 2

Analysis of Stress

2.1 THE CONTINUUM CONCEPT

The molecular nature of the structure of matter is well established. In numerous investigations of material behavior, however, the individual molecule is of no concern and only the behavior of the material as a whole is deemed important. For these cases the observed macroscopic behavior is usually explained by disregarding molecular considerations and, instead, by assuming the material to be continuously distributed throughout its volume and to completely fill the space it occupies. This *continuum concept* of matter is the fundamental postulate of Continuum Mechanics. Within the limitations for which the continuum assumption is valid, this concept provides a framework for studying the behavior of solids, liquids and gases alike.

Adoption of the continuum viewpoint as the basis for the mathematical description of material behavior means that field quantities such as stress and displacement are expressed as piecewise continuous functions of the space coordinates and time.

2.2 HOMOGENEITY. ISOTROPY. MASS-DENSITY

A *homogeneous* material is one having identical properties at all points. With respect to some property, a material is *isotropic* if that property is the same in all directions at a point. A material is called *anisotropic* with respect to those properties which are directional at a point.

The concept of *density* is developed from the *mass-volume ratio* in the neighborhood of a point in the continuum. In Fig. 2-1 the mass in the small element of volume ΔV is denoted by ΔM. The *average density* of the material within ΔV is therefore

$$\rho_{(\text{av})} = \frac{\Delta M}{\Delta V} \tag{2.1}$$

The *density* at some interior point P of the volume element ΔV is given mathematically in accordance with the continuum concept by the limit,

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV} \tag{2.2}$$

Mass-density ρ is a scalar quantity.

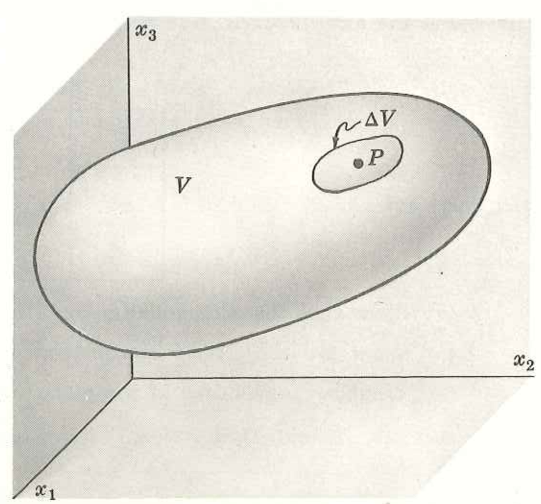

Fig. 2-1

2.3 BODY FORCES. SURFACE FORCES

Forces are vector quantities which are best described by intuitive concepts such as push or pull. Those forces which act on all elements of volume of a continuum are known as *body forces*. Examples are gravity and inertia forces. These forces are represented by the symbol b_i (force per unit mass), or as p_i (force per unit volume). They are related through the density by the equation

$$\rho b_i = p_i \quad \text{or} \quad \rho \mathbf{b} = \mathbf{p} \tag{2.3}$$

Those forces which act on a surface element, whether it is a portion of the bounding surface of the continuum or perhaps an arbitrary internal surface, are known as *surface forces*. These are designated by f_i (force per unit area). Contact forces between bodies are a type of surface forces.

2.4 CAUCHY'S STRESS PRINCIPLE. THE STRESS VECTOR

A material continuum occupying the region R of space, and subjected to surface forces f_i and body forces b_i, is shown in Fig. 2-2. As a result of forces being transmitted from one portion of the continuum to another, the material within an arbitrary volume V enclosed by the surface S interacts with the material outside of this volume. Taking n_i as the outward unit normal at point P of a small element of surface ΔS of S, let Δf_i be the resultant force exerted across ΔS upon the material within V by the material outside of V. Clearly the force element Δf_i will depend upon the choice of ΔS and upon n_i. It should also be noted that the distribution of force on ΔS is not necessarily uniform. Indeed the force distribution is, in general, equipollent to a force and a moment at P, as shown in Fig. 2-2 by the vectors Δf_i and ΔM_i.

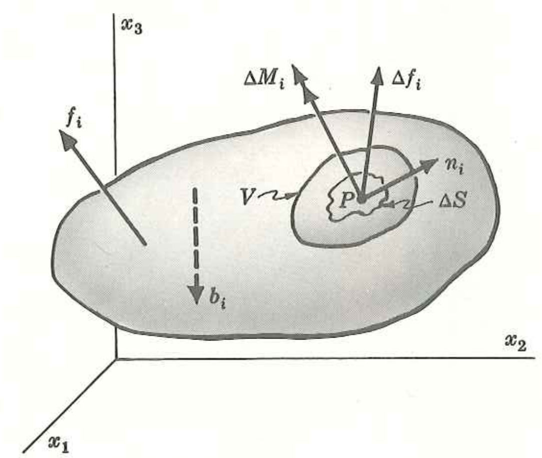

Fig. 2-2

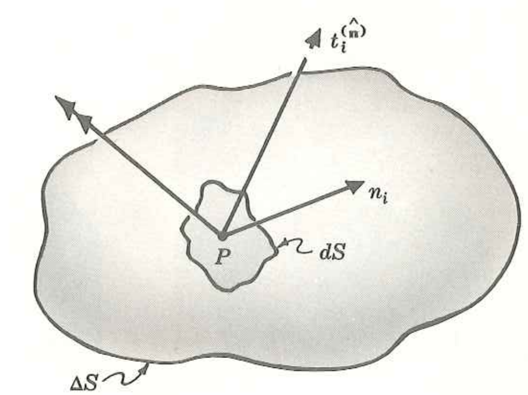

Fig. 2-3

The average force per unit area on ΔS is given by $\Delta f_i/\Delta S$. The *Cauchy stress principle* asserts that this ratio $\Delta f_i/\Delta s$ tends to a definite limit df_i/dS as ΔS approaches zero at the point P, while at the same time the moment of Δf_i about the point P vanishes in the limiting process. The resulting vector df_i/dS (force per unit area) is called the *stress vector* $t_i^{(\hat{n})}$ and is shown in Fig. 2-3. If the moment at P were not to vanish in the limiting process, a *couple-stress vector*, shown by the double-headed arrow in Fig. 2-3, would also be defined at the point. One branch of the theory of elasticity considers such couple stresses but they are not considered in this text.

Mathematically the stress vector is defined by

$$t_i^{(\hat{n})} = \lim_{\Delta S \to 0} \frac{\Delta f_i}{\Delta S} = \frac{df_i}{dS} \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{f}}{\Delta S} = \frac{d\mathbf{f}}{dS} \tag{2.4}$$

The notation $t_i^{(\hat{n})}$ (or $\mathbf{t}^{(\hat{n})}$) is used to emphasize the fact that the stress vector at a given point P in the continuum depends explicitly upon the particular surface element ΔS chosen there, as represented by the unit normal n_i (or $\hat{\mathbf{n}}$). For some differently oriented surface element, having a different unit normal, the associated stress vector at P will also be different. The stress vector arising from the action across ΔS at P of the material within V upon the material outside is the vector $-t_i^{(\hat{n})}$. Thus by Newton's law of action and reaction,

$$-t_i^{(\hat{n})} = t_i^{(-\hat{n})} \quad \text{or} \quad -\mathbf{t}^{(\hat{n})} = \mathbf{t}^{(-\hat{n})} \tag{2.5}$$

The *stress vector* is very often referred to as the *traction vector*.

2.5 STATE OF STRESS AT A POINT. STRESS TENSOR

At an arbitrary point P in a continuum, Cauchy's stress principle associates a stress vector $t_i^{(\hat{n})}$ with each unit normal vector n_i, representing the orientation of an infinitesimal surface element having P as an interior point. This is illustrated in Fig. 2-3. The totality of all possible pairs of such vectors $t_i^{(\hat{n})}$ and n_i at P defines the *state of stress* at that point. Fortunately it is not necessary to specify every pair of stress and normal vectors to completely describe the state of stress at a given point. This may be accomplished by giving the stress vector on each of three mutually perpendicular planes at P. Coordinate transformation equations then serve to relate the stress vector on any other plane at the point to the given three.

Adopting planes perpendicular to the coordinate axes for the purpose of specifying the state of stress at a point, the appropriate stress and normal vectors are shown in Fig. 2-4.

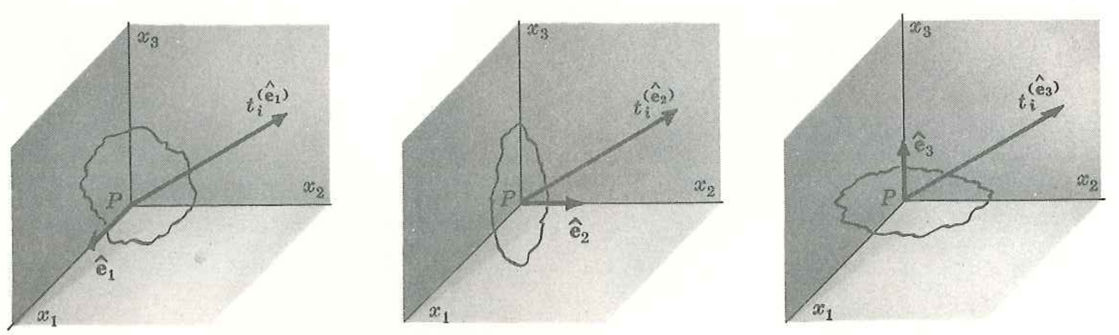

Fig. 2-4

For convenience, the three separate diagrams in Fig. 2-4 are often combined into a single schematic representation as shown in Fig. 2-5 below.

Each of the three coordinate-plane stress vectors may be written according to (*1.69*) in terms of its Cartesian components as

$$\begin{aligned}
\mathbf{t}^{(\hat{e}_1)} &= t_1^{(\hat{e}_1)}\hat{\mathbf{e}}_1 + t_2^{(\hat{e}_1)}\hat{\mathbf{e}}_2 + t_3^{(\hat{e}_1)}\hat{\mathbf{e}}_3 = t_j^{(\hat{e}_1)}\hat{\mathbf{e}}_j \\
\mathbf{t}^{(\hat{e}_2)} &= t_1^{(\hat{e}_2)}\hat{\mathbf{e}}_1 + t_2^{(\hat{e}_2)}\hat{\mathbf{e}}_2 + t_3^{(\hat{e}_2)}\hat{\mathbf{e}}_3 = t_j^{(\hat{e}_2)}\hat{\mathbf{e}}_j \\
\mathbf{t}^{(\hat{e}_3)} &= t_1^{(\hat{e}_3)}\hat{\mathbf{e}}_1 + t_2^{(\hat{e}_3)}\hat{\mathbf{e}}_2 + t_3^{(\hat{e}_3)}\hat{\mathbf{e}}_3 = t_j^{(\hat{e}_3)}\hat{\mathbf{e}}_j
\end{aligned} \tag{2.6}$$

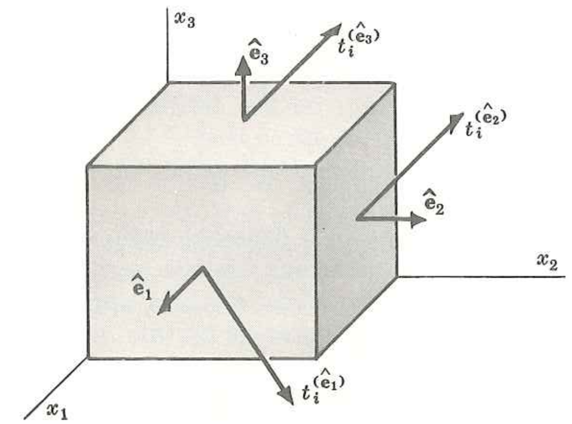

Fig. 2-5

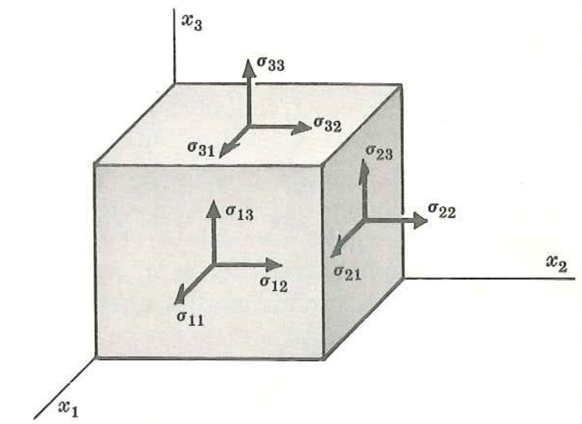

Fig. 2-6

The nine stress vector components,

$$t_j^{(\hat{e}_i)} \equiv \sigma_{ij} \tag{2.7}$$

are the components of a second-order Cartesian tensor known as the *stress tensor*. The equivalent stress dyadic is designated by Σ, so that explicit component and matrix representations of the stress tensor, respectively, take the forms

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \text{or} \quad [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.8}$$

Pictorially, the stress tensor components may be displayed with reference to the coordinate planes as shown in Fig. 2-6. The components perpendicular to the planes ($\sigma_{11}, \sigma_{22}, \sigma_{33}$) are called *normal stresses*. Those acting in (tangent to) the planes ($\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \sigma_{32}$) are called *shear stresses*. A stress component is *positive* when it acts in the positive direction of the coordinate axes, and on a plane whose outer normal points in one of the positive coordinate directions. The component σ_{ij} acts in the direction of the jth coordinate axis and on the plane whose outward normal is parallel to the ith coordinate axis. The stress components shown in Fig. 2-6 are all positive.

2.6 THE STRESS TENSOR — STRESS VECTOR RELATIONSHIP

The relationship between the stress tensor σ_{ij} at a point P and the stress vector $t_i^{(\hat{n})}$ on a plane of arbitrary orientation at that point may be established through the force equilibrium or momentum balance of a small tetrahedron of the continuum, having its vertex at P. The base of the tetrahedron is taken perpendicular to n_i, and the three faces are taken perpendicular to the coordinate planes as shown by Fig. 2-7. Designating the area of the base ABC as dS, the areas of the faces are the projected areas, $dS_1 = dS\, n_1$ for face CPB, $dS_2 = dS\, n_2$ for face APC, $dS_3 = dS\, n_3$ for face BPA or

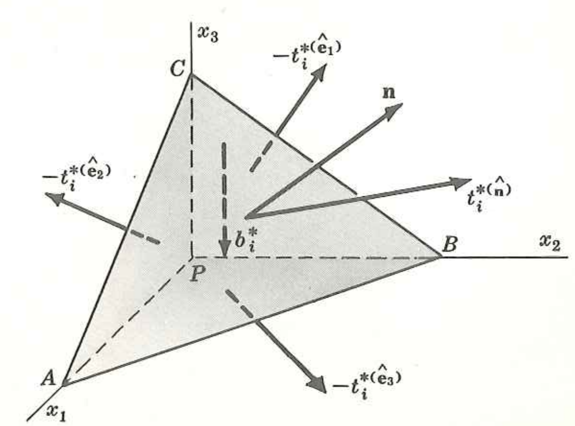

Fig. 2-7

$$dS_i = dS(\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i) = dS\cos(\hat{\mathbf{n}}, \hat{\mathbf{e}}_i) = dS\,n_i \tag{2.9}$$

The average traction vectors $-t_i^{*(\hat{\mathbf{e}}_j)}$ on the faces and $t_i^{*(\hat{\mathbf{n}})}$ on the base, together with the average body forces (including inertia forces, if present), acting on the tetrahedron are shown in the figure. Equilibrium of forces on the tetrahedron requires that

$$t_i^{*(\hat{\mathbf{n}})}\,dS - t_i^{*(\hat{\mathbf{e}}_1)}\,dS_1 - t_i^{*(\hat{\mathbf{e}}_2)}\,dS_2 - t_i^{*(\hat{\mathbf{e}}_3)}\,dS_3 + \rho b_i^*\,dV = 0 \tag{2.10}$$

If now the linear dimensions of the tetrahedron are reduced in a constant ratio to one another, the body forces, being an order higher in the small dimensions, tend to zero more rapidly than the surface forces. At the same time, the average stress vectors approach the specific values appropriate to the designated directions at P. Therefore by this limiting process and the substitution (*2.9*), equation (*2.10*) reduces to

$$t_i^{(\hat{\mathbf{n}})}\,dS = t_i^{(\hat{\mathbf{e}}_1)}n_1\,dS + t_i^{(\hat{\mathbf{e}}_2)}n_2\,dS + t_i^{(\hat{\mathbf{e}}_3)}n_3\,dS = t_i^{(\hat{\mathbf{e}}_j)}n_j\,dS \tag{2.11}$$

Cancelling the common factor dS and using the identity $t_i^{(\hat{\mathbf{e}}_j)} \equiv \sigma_{ji}$, (*2.11*) becomes

$$t_i^{(\hat{\mathbf{n}})} = \sigma_{ji}n_j \quad \text{or} \quad \mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \mathbf{\Sigma} \tag{2.12}$$

Equation (*2.12*) is also often expressed in the matrix form

$$[t_{1j}^{(\hat{\mathbf{n}})}] = [n_{1k}][\sigma_{kj}] \tag{2.13}$$

which is written explicitly

$$[t_1^{(\hat{\mathbf{n}})}, t_2^{(\hat{\mathbf{n}})}, t_3^{(\hat{\mathbf{n}})}] = [n_1, n_2, n_3]\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.14}$$

The matrix form (*2.14*) is equivalent to the component equations

$$\begin{aligned} t_1^{(\hat{\mathbf{n}})} &= n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31} \\ t_2^{(\hat{\mathbf{n}})} &= n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32} \\ t_3^{(\hat{\mathbf{n}})} &= n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33} \end{aligned} \tag{2.15}$$

2.7 FORCE AND MOMENT EQUILIBRIUM. STRESS TENSOR SYMMETRY

Equilibrium of an arbitrary volume V of a continuum, subjected to a system of surface forces $t_i^{(\hat{\mathbf{n}})}$ and body forces b_i (including inertia forces, if present) as shown in Fig. 2-8, requires that the resultant force and moment acting on the volume be zero.

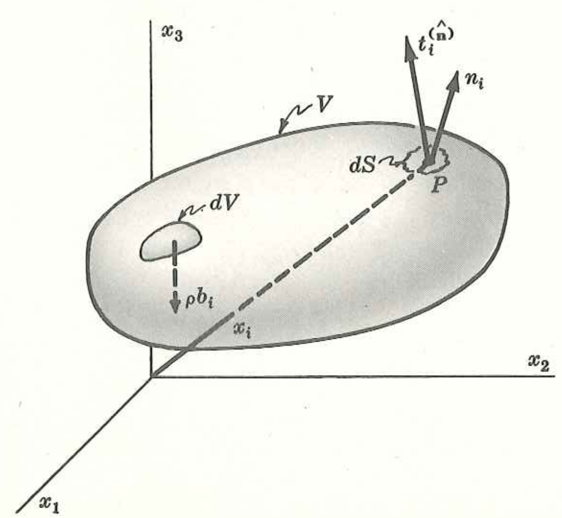

Fig. 2-8

Summation of surface and body forces results in the integral relation,

$$\int_S t_i^{(\hat{\mathbf{n}})}\,dS + \int_V \rho b_i\,dV = 0$$

or

$$\int_S \mathbf{t}^{(\hat{\mathbf{n}})}\,dS + \int_V \rho\mathbf{b}\,dV = 0 \tag{2.16}$$

Replacing $t_i^{(\hat{\mathbf{n}})}$ here by $\sigma_{ji}n_j$ and converting the resulting surface integral to a volume integral by the divergence theorem of Gauss (*1.157*), equation (*2.16*) becomes

$$\int_V (\sigma_{ji,j} + \rho b_i)\,dV = 0 \quad \text{or} \quad \int_V (\nabla \cdot \mathbf{\Sigma} + \rho\mathbf{b})\,dV = 0 \tag{2.17}$$

Since the volume V is arbitrary, the integrand in (*2.17*) must vanish, so that

$$\sigma_{ji,j} + \rho b_i = 0 \quad \text{or} \quad \nabla \cdot \mathbf{\Sigma} + \rho\mathbf{b} = 0 \tag{2.18}$$

which are called the *equilibrium equations*.

In the absence of distributed moments or couple-stresses, the equilibrium of moments about the origin requires that

$$\int_S \epsilon_{ijk}x_j t_k^{(\hat{\mathbf{n}})}\,dS + \int_V \epsilon_{ijk}x_j\rho b_k\,dV = 0$$

or

$$\int_S \mathbf{x} \times \mathbf{t}^{(\hat{\mathbf{n}})}\,dS + \int_V \mathbf{x} \times \rho\mathbf{b}\,dV = 0 \tag{2.19}$$

in which x_i is the position vector of the elements of surface and volume. Again, making the substitution $t_i^{(\hat{\mathbf{n}})} = \sigma_{ji}n_j$, applying the theorem of Gauss and using the result expressed in (*2.18*), the integrals of (*2.19*) are combined and reduced to

$$\int_V \epsilon_{ijk}\sigma_{jk}\,dV = 0 \quad \text{or} \quad \int_V \mathbf{\Sigma}_v\,dV = 0 \tag{2.20}$$

For the arbitrary volume V, (*2.20*) requires

$$\epsilon_{ijk}\sigma_{jk} = 0 \quad \text{or} \quad \mathbf{\Sigma}_v = 0 \tag{2.21}$$

Equation (*2.21*) represents the equations $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$, or in all

$$\sigma_{ij} = \sigma_{ji} \tag{2.22}$$

which shows that the *stress tensor is symmetric*. In view of (*2.22*), the equilibrium equations (*2.18*) are often written

$$\sigma_{ij,j} + \rho b_i = 0 \tag{2.23}$$

which appear in expanded form as

$$\begin{aligned} \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} + \rho b_1 &= 0 \\ \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} + \rho b_2 &= 0 \\ \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + \rho b_3 &= 0 \end{aligned} \tag{2.24}$$

2.8 STRESS TRANSFORMATION LAWS

At the point P let the rectangular Cartesian coordinate systems $Px_1x_2x_3$ and $Px_1'x_2'x_3'$ of Fig. 2-9 be related to one another by the table of direction cosines

	x_1	x_2	x_3
x_1'	a_{11}	a_{12}	a_{13}
x_2'	a_{21}	a_{22}	a_{23}
x_3'	a_{31}	a_{32}	a_{33}

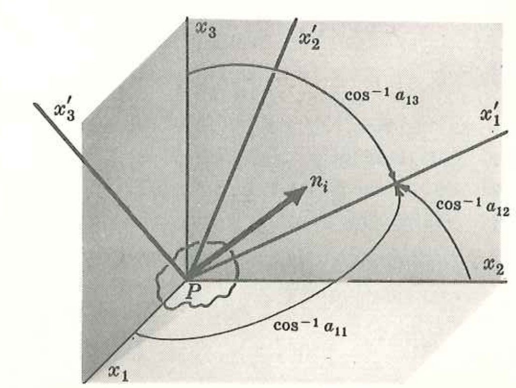

Fig. 2-9

or by the equivalent alternatives, the transformation matrix $[a_{ij}]$, or the transformation dyadic

$$\mathbf{A} = a_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{2.25}$$

According to the transformation law for Cartesian tensors of order one (*1.93*), the components of the stress vector $t_i^{(\hat{n})}$ referred to the unprimed axes are related to the primed axes components $t_i'^{(\hat{n})}$ by the equation

$$t_i'^{(\hat{n})} = a_{ij}t_j^{(\hat{n})} \quad \text{or} \quad \mathbf{t}'^{(\hat{n})} = \mathbf{A}\cdot\mathbf{t}^{(\hat{n})} \tag{2.26}$$

Likewise, by the transformation law (*1.102*) for second-order Cartesian tensors, the stress tensor components in the two systems are related by

$$\sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq} \quad \text{or} \quad \mathbf{\Sigma}' = \mathbf{A}\cdot\mathbf{\Sigma}\cdot\mathbf{A}_c \tag{2.27}$$

In matrix form, the stress vector transformation is written

$$[t_{i1}'^{(\hat{n})}] = [a_{ij}][t_{j1}^{(\hat{n})}] \tag{2.28}$$

and the stress tensor transformation as

$$[\sigma_{ij}] = [a_{ip}][\sigma_{pq}][a_{qj}] \tag{2.29}$$

Explicitly, the matrix multiplications in (*2.28*) and (*2.29*) are given respectively by

$$\begin{bmatrix} t_1'^{(\hat{n})} \\ t_2'^{(\hat{n})} \\ t_3'^{(\hat{n})} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} t_1^{(\hat{n})} \\ t_2^{(\hat{n})} \\ t_3^{(\hat{n})} \end{bmatrix} \tag{2.30}$$

and

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{2.31}$$

2.9 STRESS QUADRIC OF CAUCHY

At the point P in a continuum, let the stress tensor have the values σ_{ij} when referred to directions parallel to the local Cartesian axes $P\zeta_1\zeta_2\zeta_3$ shown in Fig. 2-10. The equation

$$\sigma_{ij}\zeta_i\zeta_j = \pm k^2 \quad \text{(a constant)} \tag{2.32}$$

represents geometrically similar quadric surfaces having a common center at P. The plus or minus choice assures the surfaces are real.

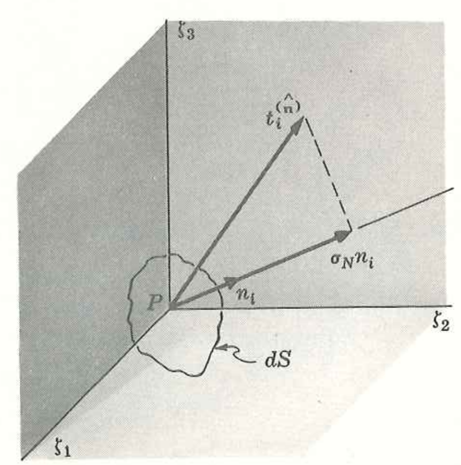

Fig. 2-10

The position vector $\mathbf{r}$ of an arbitrary point lying on the quadric surface has components $\zeta_i = rn_i$, where n_i is the unit normal in the direction of $\mathbf{r}$. At the point P the normal component $\sigma_N n_i$ of the stress vector $t_i^{(\hat{n})}$ has a magnitude

$$\sigma_N = t_i^{(\hat{n})}n_i = \mathbf{t}^{(\hat{n})}\cdot\mathbf{n} = \sigma_{ij}n_in_j \tag{2.33}$$

Accordingly if the constant k^2 of (*2.32*) is set equal to $\sigma_N r^2$, the resulting quadric

$$\sigma_{ij}\zeta_i\zeta_j = \pm\sigma_N r^2 \tag{2.34}$$

is called the *stress quadric of Cauchy*. From this definition it follows that the magnitude σ_N of the normal stress component on the surface element dS perpendicular to the position vector $\mathbf{r}$ of a point on Cauchy's stress quadric, is inversely proportional to r^2, i.e. $\sigma_N = \pm k^2/r^2$. Furthermore it may be shown that the stress vector $t_i^{(\hat{n})}$ acting on dS at P is parallel to the normal of the tangent plane of the Cauchy quadric at the point identified by $\mathbf{r}$.

2.10 PRINCIPAL STRESSES. STRESS INVARIANTS. STRESS ELLIPSOID

At the point P for which the stress tensor components are σ_{ij}, the equation (*2.12*), $t_i^{(\hat{n})} = \sigma_{ji}n_j$, associates with each direction n_i a stress vector $t_i^{(\hat{n})}$. Those directions for which $t_i^{(\hat{n})}$ and n_i are collinear as shown in Fig. 2-11 are called *principal stress directions*. For a principal stress direction,

$$t_i^{(\hat{n})} = \sigma n_i \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \sigma\hat{\mathbf{n}} \tag{2.35}$$

in which σ, the magnitude of the stress vector, is called a *principal stress value*. Substituting (*2.35*) into (*2.12*) and making use of the identities $n_i = \delta_{ij}n_j$ and $\sigma_{ij} = \sigma_{ji}$, results in the equations

$$(\sigma_{ij} - \delta_{ij}\sigma)n_j = 0 \quad \text{or} \quad (\mathbf{\Sigma} - \mathbf{I}\sigma)\cdot\hat{\mathbf{n}} = 0 \tag{2.36}$$

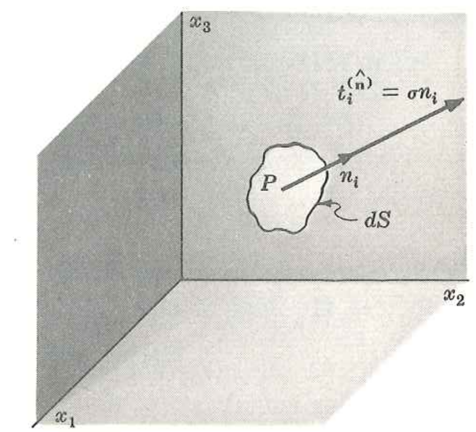

Fig. 2-11

In the three equations (*2.36*), there are four unknowns, namely, the three direction cosines n_i and the principal stress value σ.

For solutions of (*2.36*) other than the trivial one $n_j = 0$, the determinant of coefficients, $|\sigma_{ij} - \delta_{ij}\sigma|$, must vanish. Explicitly,

$$|\sigma_{ij} - \delta_{ij}\sigma| = 0 \quad \text{or} \quad \begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \tag{2.37}$$

which upon expansion yields the cubic polynomial in σ,

$$\sigma^3 - \mathrm{I}_\Sigma\sigma^2 + \mathrm{II}_\Sigma\sigma - \mathrm{III}_\Sigma = 0 \tag{2.38}$$

where

$$\mathrm{I}_\Sigma = \sigma_{ii} = \operatorname{tr}\mathbf{\Sigma} \tag{2.39}$$

$$\mathrm{II}_\Sigma = \tfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \tag{2.40}$$

$$\mathrm{III}_\Sigma = |\sigma_{ij}| = \det\mathbf{\Sigma} \tag{2.41}$$

are known respectively as the *first*, *second* and *third stress invariants*.

The three roots of (*2.38*), $\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}$ are the three principal stress values. Associated with each principal stress $\sigma_{(k)}$, there is a principal stress direction for which the direction cosines $n_i^{(k)}$ are solutions of the equations

$$(\sigma_{ij} - \sigma_{(k)}\delta_{ij})n_j^{(k)} = 0 \quad \text{or} \quad (\mathbf{\Sigma} - \sigma_{(k)}\mathbf{I})\cdot\hat{\mathbf{n}}^{(k)} = 0 \quad (k = 1,2,3) \tag{2.42}$$

In (*2.42*) letter subscripts or superscripts enclosed by parentheses are merely labels and as such do not participate in any summation process. The expanded form of (*2.42*) for the *second* principal direction, for example, is therefore

$$\begin{aligned}(\sigma_{11}-\sigma_{(2)})n_1^{(2)}+\sigma_{12}n_2^{(2)}+\sigma_{13}n_3^{(2)} &= 0\\ \sigma_{21}n_1^{(2)}+(\sigma_{22}-\sigma_{(2)})n_2^{(2)}+\sigma_{23}n_3^{(2)} &= 0\\ \sigma_{31}n_1^{(2)}+\sigma_{32}n_2^{(2)}+(\sigma_{33}-\sigma_{(2)})n_3^{(2)} &= 0\end{aligned} \qquad (2.43)$$

Because the stress tensor is real and symmetric, the principal stress values are also real.

When referred to principal stress directions, the stress matrix $[\sigma_{ij}]$ is diagonal,

$$[\sigma_{ij}] \equiv \begin{bmatrix}\sigma_{(1)} & 0 & 0\\ 0 & \sigma_{(2)} & 0\\ 0 & 0 & \sigma_{(3)}\end{bmatrix} \quad \text{or} \quad [\sigma_{ij}] = \begin{bmatrix}\sigma_{\rm I} & 0 & 0\\ 0 & \sigma_{\rm II} & 0\\ 0 & 0 & \sigma_{\rm III}\end{bmatrix} \qquad (2.44)$$

in the second form of which Roman numeral subscripts are used to show that the principal stresses are ordered, i.e. $\sigma_{\rm I} > \sigma_{\rm II} > \sigma_{\rm III}$. Since the principal stress directions are coincident with the principal axes of Cauchy's stress quadric, the principal stress values include both the maximum and minimum normal stress components at a point.

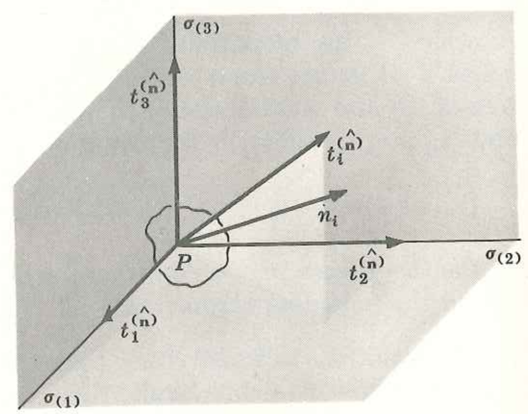

Fig. 2-12

In a *principal stress space,* i.e. a space whose axes are in the principal stress directions and whose coordinate unit of measure is stress $(t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})})$ as shown in Fig. 2-12, the arbitrary stress vector $t_i^{(\hat{n})}$ has components

$$t_1^{(\hat{n})} = \sigma_{(1)}n_1, \qquad t_2^{(\hat{n})} = \sigma_{(2)}n_2, \qquad t_3^{(\hat{n})} = \sigma_{(3)}n_3 \qquad (2.45)$$

according to (*2.12*). But inasmuch as $(n_1)^2+(n_2)^2+(n_3)^2 = 1$ for the unit vector n_i, (*2.45*) requires the stress vector $t_i^{(\hat{n})}$ to satisfy the equation

$$\frac{(t_1^{(\hat{n})})^2}{(\sigma_{(1)})^2}+\frac{(t_2^{(\hat{n})})^2}{(\sigma_{(2)})^2}+\frac{(t_3^{(\hat{n})})^2}{(\sigma_{(3)})^2} = 1 \qquad (2.46)$$

in stress space. This equation is an ellipsoid known as the *Lamé stress ellipsoid.*

2.11 MAXIMUM AND MINIMUM SHEAR STRESS VALUES

If the stress vector $t_i^{(\hat{n})}$ is resolved into orthogonal components normal and tangential to the surface element dS upon which it acts, the magnitude of the normal component may be determined from (*2.33*) and the magnitude of the tangential or *shearing component* is given by

$$\sigma_S^2 = t_i^{(\hat{n})}t_i^{(\hat{n})} - \sigma_N^2 \qquad (2.47)$$

This resolution is shown in Fig. 2-13 where the axes are chosen in the principal stress directions and it is assumed the principal stresses are ordered according to $\sigma_{\rm I} > \sigma_{\rm II} > \sigma_{\rm III}$. Hence from (*2.12*), the components of $t_i^{(\hat{n})}$ are

$$\begin{aligned}t_1^{(\hat{n})} &= \sigma_{\rm I}n_1\\ t_2^{(\hat{n})} &= \sigma_{\rm II}n_2\\ t_3^{(\hat{n})} &= \sigma_{\rm III}n_3\end{aligned} \qquad (2.48)$$

and from (*2.33*), the normal component magnitude is

$$\sigma_N = \sigma_{\rm I}n_1^2+\sigma_{\rm II}n_2^2+\sigma_{\rm III}n_3^2 \qquad (2.49)$$

Substituting (*2.48*) and (*2.49*) into (*2.47*), the squared magnitude of the shear stress as a function of the direction cosines n_i is given by

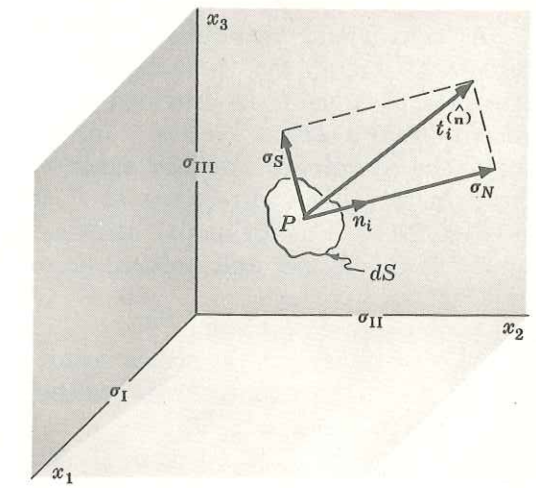

Fig. 2-13

$$\sigma_S^2 = \sigma_{\rm I}^2n_1^2+\sigma_{\rm II}^2n_2^2+\sigma_{\rm III}^2n_3^2-(\sigma_{\rm I}n_1^2+\sigma_{\rm II}n_2^2+\sigma_{\rm III}n_3^2)^2 \qquad (2.50)$$

The maximum and minimum values of σ_S may be obtained from (*2.50*) by the method of *Lagrangian multipliers.* The procedure is to construct the function

$$F = \sigma_S^2 - \lambda n_i n_i \qquad (2.51)$$

in which the scalar λ is called a Lagrangian multiplier. Equation (*2.51*) is clearly a function of the direction cosines n_i, so that the conditions for stationary (maximum or minimum) values of F are given by $\partial F/\partial n_i = 0$. Setting these partials equal to zero yields the equations

$$n_1\{\sigma_{\rm I}^2-2\sigma_{\rm I}(\sigma_{\rm I}n_1^2+\sigma_{\rm II}n_2^2+\sigma_{\rm III}n_3^2)+\lambda\} = 0 \qquad (2.52a)$$

$$n_2\{\sigma_{\rm II}^2-2\sigma_{\rm II}(\sigma_{\rm I}n_1^2+\sigma_{\rm II}n_2^2+\sigma_{\rm III}n_3^2)+\lambda\} = 0 \qquad (2.52b)$$

$$n_3\{\sigma_{\rm III}^2-2\sigma_{\rm III}(\sigma_{\rm I}n_1^2+\sigma_{\rm II}n_2^2+\sigma_{\rm III}n_3^2)+\lambda\} = 0 \qquad (2.52c)$$

which, together with the condition $n_in_i = 1$, may be solved for λ and the direction cosines n_1, n_2, n_3, conjugate to the extremum values of shear stress.

One set of solutions to (*2.52*), and the associated shear stresses from (*2.50*), are

$$n_1 = \pm 1, \quad n_2 = 0, \quad n_3 = 0; \quad \text{for which } \sigma_S = 0 \qquad (2.53a)$$

$$n_1 = 0, \quad n_2 = \pm 1, \quad n_3 = 0; \quad \text{for which } \sigma_S = 0 \qquad (2.53b)$$

$$n_1 = 0, \quad n_2 = 0, \quad n_3 = \pm 1; \quad \text{for which } \sigma_S = 0 \qquad (2.53c)$$

The shear stress values in (*2.53*) are obviously minimum values. Furthermore, since (*2.35*) indicates that shear components vanish on principal planes, the directions given by (*2.53*) are recognized as principal stress directions.

A second set of solutions to (*2.52*) may be verified to be given by

$$n_1 = 0, \quad n_2 = \pm 1/\sqrt{2}, \quad n_3 = \pm 1/\sqrt{2}; \quad \text{for which } \sigma_S = (\sigma_{\rm II}-\sigma_{\rm III})/2 \qquad (2.54a)$$

$$n_1 = \pm 1/\sqrt{2}, \quad n_2 = 0, \quad n_3 = \pm 1/\sqrt{2}; \quad \text{for which } \sigma_S = (\sigma_{\rm III}-\sigma_{\rm I})/2 \qquad (2.54b)$$

$$n_1 = \pm 1/\sqrt{2}, \quad n_2 = \pm 1/\sqrt{2}, \quad n_3 = 0; \quad \text{for which } \sigma_S = (\sigma_{\rm I}-\sigma_{\rm II})/2 \qquad (2.54c)$$

Equation (*2.54b*) gives the maximum shear stress value, which is equal to half the difference of the largest and smallest principal stresses. Also from (*2.54b*), the maximum shear stress component acts in the plane which bisects the right angle between the directions of the maximum and minimum principal stresses.

2.12 MOHR'S CIRCLES FOR STRESS

A convenient two-dimensional graphical representation of the three-dimensional state of stress at a point is provided by the well-known *Mohr's stress circles.* In developing these, the coordinate axes are again chosen in the principal stress directions at P as shown by Fig. 2-14. The principal stresses are assumed to be distinct and ordered according to

$$\sigma_I > \sigma_{II} > \sigma_{III} \tag{2.55}$$

For this arrangement the stress vector $t_i^{(\hat{n})}$ has normal and shear components whose magnitudes satisfy the equations

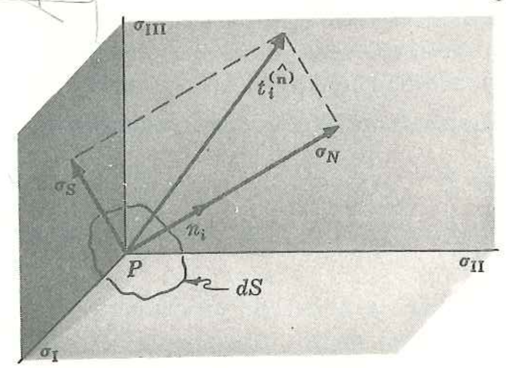

Fig. 2-14

$$\sigma_N = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2 \tag{2.56}$$

$$\sigma_N^2 + \sigma_S^2 = \sigma_I^2 n_1^2 + \sigma_{II}^2 n_2^2 + \sigma_{III}^2 n_3^2 \tag{2.57}$$

Combining these two expressions with the identity $n_i n_i = 1$ and solving for the direction cosines n_i, results in the equations

$$(n_1)^2 = \frac{(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\sigma_S)^2}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})} \tag{2.58a}$$

$$(n_2)^2 = \frac{(\sigma_N - \sigma_{III})(\sigma_N - \sigma_I) + (\sigma_S)^2}{(\sigma_{II} - \sigma_{III})(\sigma_{II} - \sigma_I)} \tag{2.58b}$$

$$(n_3)^2 = \frac{(\sigma_N - \sigma_I)(\sigma_N - \sigma_{II}) + (\sigma_S)^2}{(\sigma_{III} - \sigma_I)(\sigma_{III} - \sigma_{II})} \tag{2.58c}$$

These equations serve as the basis for Mohr's stress circles, shown in the "stress plane" of Fig. 2-15, for which the σ_N axis is the abscissa, and the σ_S axis is the ordinate.

In (*2.58a*), since $\sigma_I - \sigma_{II} > 0$ and $\sigma_I - \sigma_{III} > 0$ from (*2.55*), and since $(n_1)^2$ is non-negative, the numerator of the right-hand side satisfies the relationship

$$(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\sigma_S)^2 \geqq 0 \tag{2.59}$$

which represents stress points in the (σ_N, σ_S) plane that are *on* or *exterior* to the circle

$$[\sigma_N - (\sigma_{II} + \sigma_{III})/2]^2 + (\sigma_S)^2 = [(\sigma_{II} - \sigma_{III})/2]^2 \tag{2.60}$$

In Fig. 2-15, this circle is labeled C_1.

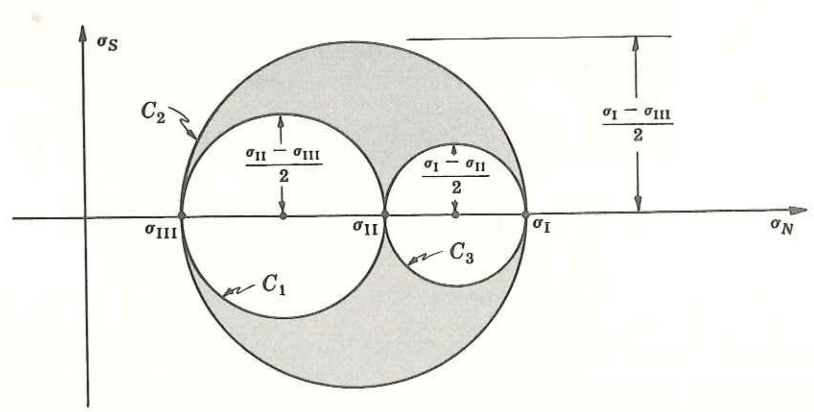

Fig. 2-15

Similarly, for (*2.58b*), since $\sigma_{II} - \sigma_{III} > 0$ and $\sigma_{II} - \sigma_I < 0$ from (*2.55*), and since $(n_2)^2$ is non-negative, the right hand numerator satisfies

$$(\sigma_N - \sigma_{III})(\sigma_N - \sigma_I) + (\sigma_S)^2 \leqq 0 \tag{2.61}$$

which represents points *on* or *interior* to the circle

$$[\sigma_N - (\sigma_{III} + \sigma_I)/2]^2 + (\sigma_S)^2 = [(\sigma_{III} - \sigma_I)/2]^2 \tag{2.62}$$

labeled C_2 in Fig. 2-15. Finally, for (*2.58c*), since $\sigma_{III} - \sigma_I < 0$ and $\sigma_{III} - \sigma_{II} < 0$ from (*2.55*), and since $(n_3)^2$ is non-negative,

$$(\sigma_N - \sigma_I)(\sigma_N - \sigma_{II}) + (\sigma_S)^2 \geqq 0 \tag{2.63}$$

which represents points *on* or *exterior* to the circle

$$[\sigma_N - (\sigma_I + \sigma_{II})/2]^2 + (\sigma_S)^2 = [(\sigma_I - \sigma_{II})/2]^2 \tag{2.64}$$

labeled C_3 in Fig. 2-15.

Since each "stress point" (pair of values of σ_N and σ_S) in the (σ_N, σ_S) plane represents a particular stress vector $t_i^{(\hat{n})}$, the state of stress at P expressed by (*2.58*) is represented in Fig. 2-15 as the shaded area bounded by the Mohr's stress circles. The diagram confirms a maximum shear stress of $(\sigma_I - \sigma_{III})/2$ as was determined analytically in Section 2.11. Frequently, because the sign of the shear stress is not of critical importance, only the top half of this symmetrical diagram is drawn.

The relationship between Mohr's stress diagram and the physical state of stress may be established through consideration of Fig. 2-16, which shows the first octant of a sphere of the continuum centered at point P. The normal n_i at the arbitrary point Q of the spherical surface ABC simulates the normal to the surface element dS at point P. Because of the symmetry properties of the stress tensor and the fact that principal stress axes are used in Fig. 2-16, the state of stress at P is completely represented through the totality of locations Q can occupy on the surface ABC. In the figure, circle arcs KD, GE and FH designate locations for Q along which one direction cosine of n_i has a constant value. Specifically,

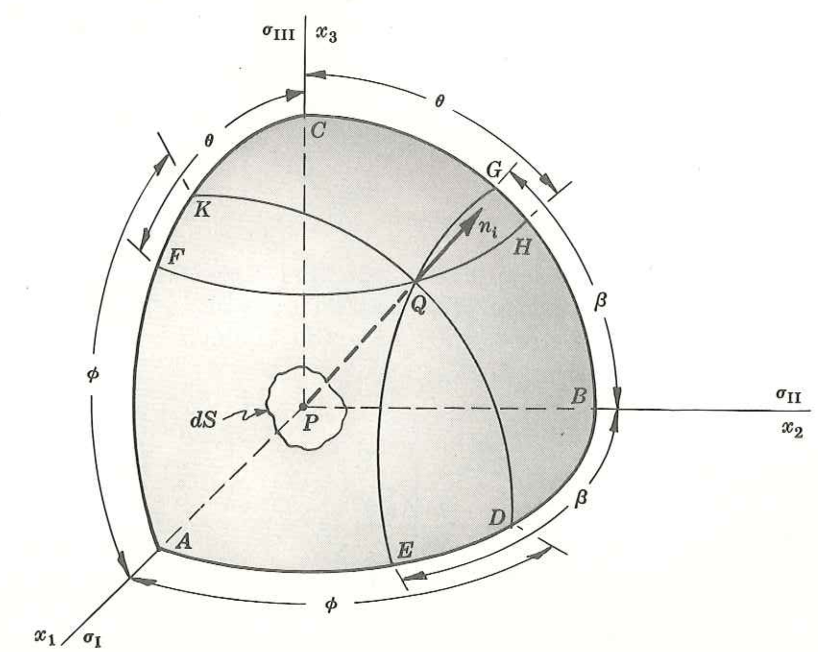

Fig. 2-16

$$n_1 = \cos\phi \text{ on } KD, \qquad n_2 = \cos\beta \text{ on } GE, \qquad n_3 = \cos\theta \text{ on } FH$$

and, on the bounding circle arcs BC, CA and AB,

$$n_1 = \cos\pi/2 = 0 \text{ on } BC, \qquad n_2 = \cos\pi/2 = 0 \text{ on } CA, \qquad n_3 = \cos\pi/2 = 0 \text{ on } AB$$

According to the first of these and the equation (*2.58a*), stress vectors for Q located on BC will have components given by stress points on the circle C_1 in Fig. 2-15. Likewise, CA in Fig. 2-16 corresponds to the circle C_2, and AB to the circle C_3 in Fig. 2-15.

The stress vector components σ_N and σ_S for an arbitrary location of Q may be determined by the construction shown in Fig. 2-17. Thus point e may be located on C_3 by drawing the radial line from the center of C_3 at the angle 2β. Note that angles in the physical space of Fig. 2-16 are doubled in the stress space of Fig. 2-17 (arc AB subtends 90° in Fig. 2-16 whereas the conjugate stress points σ_I and σ_{II} are 180° apart on C_3). In the same way, points g, h and f are located in Fig. 2-17 and the appropriate pairs joined by circle arcs having their centers on the σ_N axis. The intersection of circle arcs ge and hf represents the components σ_N and σ_S of the stress vector $t_i^{(\hat{n})}$ on the plane having the normal direction n_i at Q in Fig. 2-16.

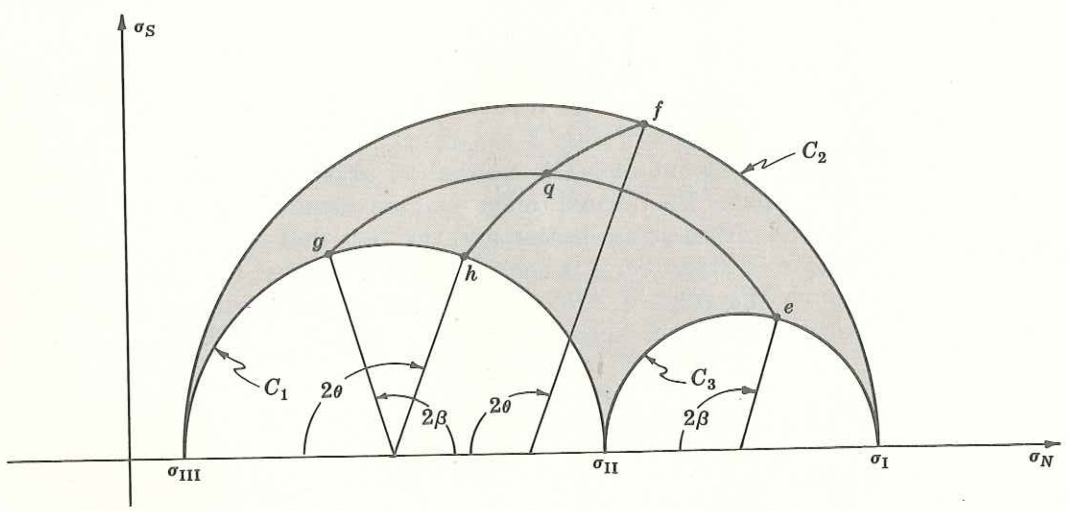

Fig. 2-17

2.13 PLANE STRESS

In the case where one and only one of the principal stresses is zero a state of *plane stress* is said to exist. Such a situation occurs at an unloaded point on the free surface bounding a body. If the principal stresses are ordered, the Mohr's stress circles will have one of the characterizations appearing in Fig. 2-18.

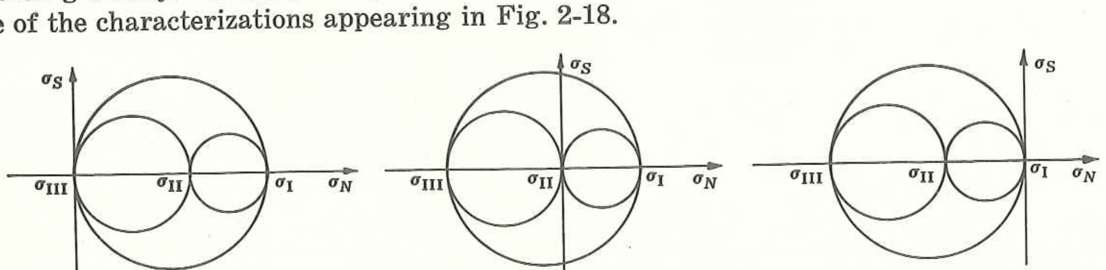

Fig. 2-18

If the principal stresses are not ordered and the direction of the zero principal stress is taken as the x_3 direction, the state of stress is termed plane stress parallel to the x_1x_2 plane. For arbitrary choice of orientation of the orthogonal axes x_1 and x_2 in this case, the stress matrix has the form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2.65)$$

The stress quadric for this plane stress is a cylinder with its base lying in the x_1x_2 plane and having the equation

$$\sigma_{11}x_1^2 + 2\sigma_{12}x_1x_2 + \sigma_{22}x_2^2 = \pm k^2 \qquad (2.66)$$

Frequently in elementary books on Strength of Materials a state of plane stress is represented by a single Mohr's circle. As seen from Fig. 2-18 this representation is necessarily incomplete since all three circles are required to show the complete stress picture. In particular, the maximum shear stress value at a point will not be given if the single circle presented happens to be one of the inner circles of Fig. 2-18. A single circle Mohr's diagram is able, however, to display the stress points for all those planes at the point P which include the zero principal stress axis. For such planes, if the coordinate axes are chosen in accordance with the stress representation given in (*2.65*), the single plane stress Mohr's circle has the equation

$$[\sigma_N - (\sigma_{11} + \sigma_{22})/2]^2 + (\sigma_S)^2 = [(\sigma_{11} - \sigma_{22})/2]^2 + (\sigma_{12})^2 \qquad (2.67)$$

The essential features in the construction of this circle are illustrated in Fig. 2-19. The circle is drawn by locating the center C at $\sigma_N = (\sigma_{11} + \sigma_{22})/2$ and using the radius $R = \sqrt{[(\sigma_{11} - \sigma_{22})/2]^2 + (\sigma_{12})^2}$ given in (*2.67*). Point A on the circle represents the stress state on the surface element whose normal is n_1 (the right-hand face of the rectangular parallelepiped shown in Fig. 2-19). Point B on the circle represents the stress state on the top surface of the parallelepiped with normal n_2. Principal stress points σ_I and σ_{II} are so labeled, and points E and D on the circle are points of maximum shear stress value.

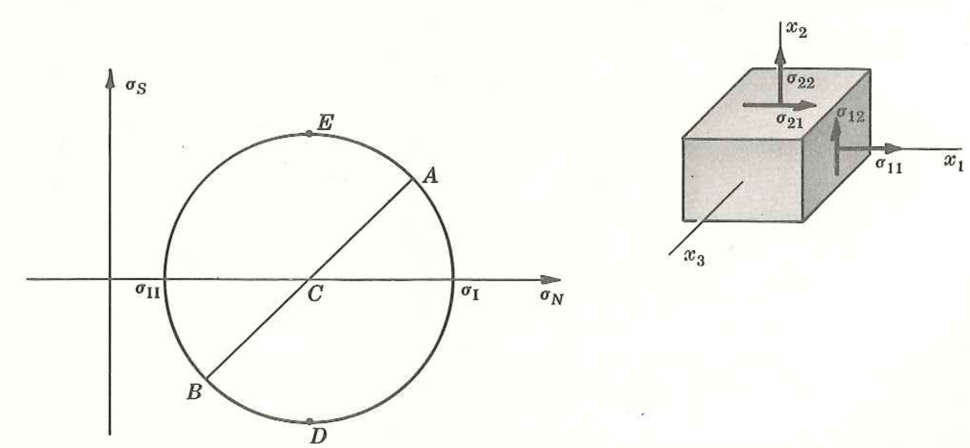

Fig. 2-19

2.14 DEVIATOR AND SPHERICAL STRESS TENSORS

It is very often useful to split the stress tensor σ_{ij} into two component tensors, one of which (the *spherical* or *hydrostatic stress tensor*) has the form

$$\Sigma_M = \sigma_M \mathbf{I} = \begin{pmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{pmatrix} \tag{2.68}$$

where $\sigma_M = -p = \sigma_{kk}/3$ is the mean normal stress, and the second (the *deviator stress tensor*) has the form

$$\Sigma_D = \begin{pmatrix} \sigma_{11}-\sigma_M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_M \end{pmatrix} \equiv \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \tag{2.69}$$

This decomposition is expressed by the equations

$$\sigma_{ij} = \delta_{ij}\sigma_{kk}/3 + s_{ij} \quad \text{or} \quad \Sigma = \sigma_M \mathbf{I} + \Sigma_D \tag{2.70}$$

The principal directions of the deviator stress tensor s_{ij} are the same as those of the stress tensor σ_{ij}. Thus *principal deviator stress* values are

$$s_{(k)} = \sigma_{(k)} - \sigma_M \tag{2.71}$$

The characteristic equation for the deviator stress tensor, comparable to (*2.38*) for the stress tensor, is the cubic

$$s^3 + \mathrm{II}_{\Sigma_D}s - \mathrm{III}_{\Sigma_D} = 0 \quad \text{or} \quad s^3 + (s_{\mathrm{I}}s_{\mathrm{II}} + s_{\mathrm{II}}s_{\mathrm{III}} + s_{\mathrm{III}}s_{\mathrm{I}})s - s_{\mathrm{I}}s_{\mathrm{II}}s_{\mathrm{III}} = 0 \tag{2.72}$$

It is easily shown that the first invariant of the deviator stress tensor I_{Σ_D} is identically zero, which accounts for its absence in (*2.72*).

Solved Problems

STATE OF STRESS AT A POINT. STRESS VECTOR. STRESS TENSOR (Sec. 2.1-2.6)

2.1. At the point P the stress vectors $t_i^{(\hat{n})}$ and $t_i^{(\hat{n}*)}$ act on the respective surface elements $n_i \Delta S$ and $n_i^* \Delta S^*$. Show that the component of $t_i^{(\hat{n})}$ in the direction of n_i^* is equal to the component of $t_i^{(\hat{n}*)}$ in the direction of n_i.

It is required to show that

$$t_i^{(\hat{n}*)} n_i = t_i^{(\hat{n})} n_i^*$$

From (*2.12*) $t_i^{(\hat{n}*)} n_i = \sigma_{ji} n_j^* n_i$, and by (*2.22*) $\sigma_{ji} = \sigma_{ij}$, so that

$$\sigma_{ji} n_j^* n_i = (\sigma_{ij} n_i) n_j^* = t_j^{(\hat{n})} n_j^*$$

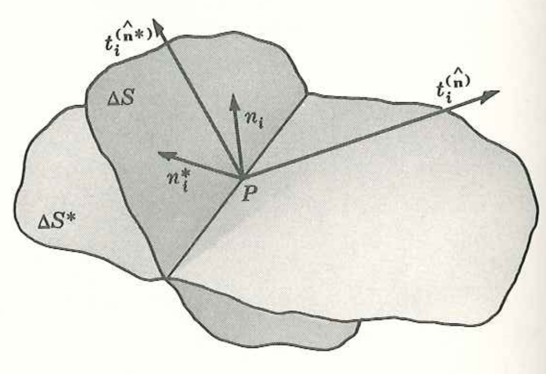

Fig. 2-20

2.2. The stress tensor values at a point P are given by the array

$$\Sigma = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Determine the traction (stress) vector on the plane at P whose unit normal is $\hat{\mathbf{n}} = (2/3)\hat{\mathbf{e}}_1 - (2/3)\hat{\mathbf{e}}_2 + (1/3)\hat{\mathbf{e}}_3$.

From (*2.12*), $\mathbf{t}^{(\hat{n})} = \hat{\mathbf{n}} \cdot \Sigma$. The multiplication is best carried out in the matrix form of (*2.13*):

$$[t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})}] = [2/3, -2/3, 1/3] \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \left[\frac{14}{3} - \frac{2}{3}, \frac{-10}{3}, \frac{-4}{3} + \frac{4}{3}\right]$$

Thus $\mathbf{t}^{(\hat{n})} = 4\hat{\mathbf{e}}_1 - \frac{10}{3}\hat{\mathbf{e}}_2$.

2.3. For the traction vector of Problem 2.2, determine (*a*) the component perpendicular to the plane, (*b*) the magnitude of $t_i^{(\hat{n})}$, (*c*) the angle between $t_i^{(\hat{n})}$ and $\hat{\mathbf{n}}$.

(*a*) $t_i^{(\hat{n})} \cdot \hat{\mathbf{n}} = (4\hat{\mathbf{e}}_1 - \frac{10}{3}\hat{\mathbf{e}}_2) \cdot (\frac{2}{3}\hat{\mathbf{e}}_1 - \frac{2}{3}\hat{\mathbf{e}}_2 + \frac{1}{3}\hat{\mathbf{e}}_3) = 44/9$

(*b*) $|t_i^{(\hat{n})}| = \sqrt{16 + 100/9} = 5.2$

(*c*) Since $t_i^{(\hat{n})} \cdot \hat{\mathbf{n}} = |t_i^{(\hat{n})}| \cos\theta$, $\cos\theta = (44/9)/5.2 = 0.94$ and $\theta = 20°$.

2.4. The stress vectors acting on the three coordinate planes are given by $t_i^{(\hat{e}_1)}$, $t_i^{(\hat{e}_2)}$ and $t_i^{(\hat{e}_3)}$. Show that the sum of the squares of the magnitudes of these vectors is independent of the orientation of the coordinate planes.

Let S be the sum in question. Then

$$S = t_i^{(\hat{e}_1)} t_i^{(\hat{e}_1)} + t_i^{(\hat{e}_2)} t_i^{(\hat{e}_2)} + t_i^{(\hat{e}_3)} t_i^{(\hat{e}_3)}$$

which from (*2.7*) becomes $S = \sigma_{1i}\sigma_{1i} + \sigma_{2i}\sigma_{2i} + \sigma_{3i}\sigma_{3i} = \sigma_{ji}\sigma_{ji}$, an invariant.

2.5. The state of stress at a point is given by the stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix}$$

where a, b, c are constants and σ is some stress value. Determine the constants a, b and c so that the stress vector on the *octahedral* plane $(\hat{\mathbf{n}} = (1/\sqrt{3})\hat{\mathbf{e}}_1 + (1/\sqrt{3})\hat{\mathbf{e}}_2 + (1/\sqrt{3})\hat{\mathbf{e}}_3)$ vanishes.

In matrix form, $t_i^{(\hat{n})} = \sigma_{ij}n_j$ must be zero for the given stress tensor and normal vector.

$$\begin{bmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{hence} \quad \begin{aligned} a + b &= -1 \\ a + c &= -1 \\ b + c &= -1 \end{aligned}$$

Solving these equations, $a = b = c = -1/2$. Therefore the solution tensor is

$$\sigma_{ij} = \begin{pmatrix} \sigma & -\sigma/2 & -\sigma/2 \\ -\sigma/2 & \sigma & -\sigma/2 \\ -\sigma/2 & -\sigma/2 & \sigma \end{pmatrix}$$