Crustal Deformation:
Course notes on deformation of the lithosphere

Ben Holtzman

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Chapter 1

Deformation of the Lithosphere and Asthenosphere

1.1 Plate Boundaries

1.2 Stable plates

1.3 The asthenosphere...

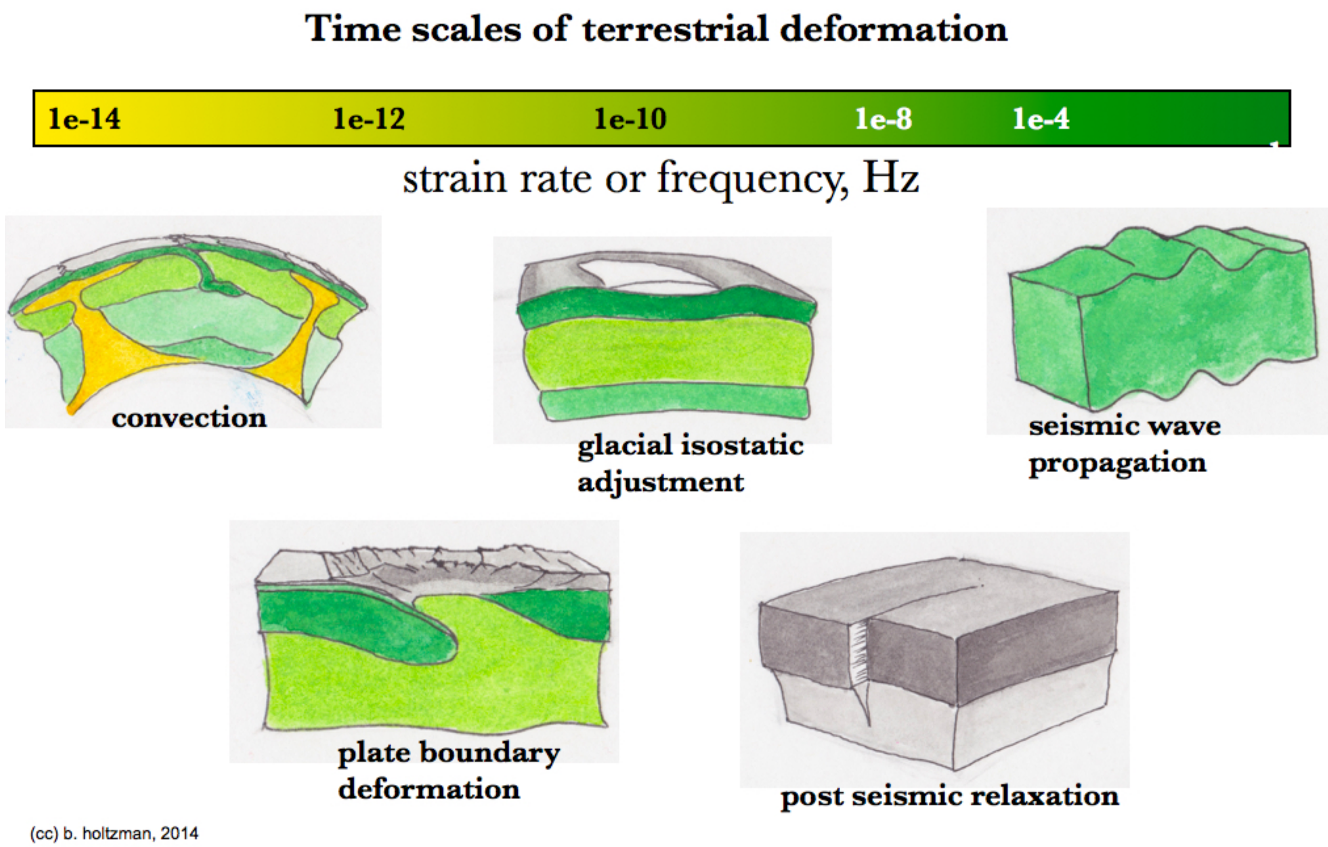
Time scales of terrestrial deformation
1e-14
1e-12
1e-10
1e-8
1e-4
strain rate or frequency, Hz
convection
glacial isostatic adjustment
seismic wave propagation
plate boundary deformation
post seismic relaxation
(cc) b. holtzman, 2014

Figure 1.1: **(a)** **(b)**

Chapter 2

Strain

In this chapter, we start with the question of what strain and deformation are, and then why we care. Then we go to the mathematical description of strain. Some people develop the ideas and math of stress and then followed with strain, which is the consequence of stress. Physically, this is the correct way to think about it, because there can be no strain without some force driving it. However, in the Earth, in engineered structures and in deformation experiments on materials in the laboratory, we cannot measure stress directly; we infer it from other measurements. We can measure displacement over time directly (with a strain or displacement gauge in an experiment, or with GPS on the Earth's surface), so we can calculate strain and velocity fields with little error. We "measure" stress by building some sort of device that will passively deform (a small gauge glued to the rock, for example), but the deformation causes a precisely calibrated stress in a well known material (a load cell).

We could start with stress or strain, as both can be thought of as thermodynamic state variables for deformation, related to each other by an invertible "constitutive relation", $\sigma(\varepsilon)$ or $\varepsilon(\sigma)$. Here we will start with strain because it is more tangible and measurable than stress, though mathematically more complex. In the following chapter, we will then work on stress.

There are many approaches to the mathematical description and calculation of strain. Here, I am using a combination of Mase 1970, Chapter 3, and Win Means' book "Stress and Strain" (Means, 1976), and André Chrysochoos' course notes on thermomechanics from the Ecole Polytechnique in Montpellier, France (based on Coirier and Nadot-Martin, 2001). Each (and seemingly all texts) use different notation and levels of detail/generality in their derivations. Here, I try to keep it as simple as is appropriate to the rest of the course notes. We will start with the notion of a deformation or flow field, and then the deformation and displacement gradients, followed by the various ways of defining a strain.

2.1 Why do we care?

As you will see, all materials are viscoelastic– some combination of elastic viscous behavior (Chapter 4). In detail, though, the behavior depends on the length and time scales of observation, and therefore, on how we best describe that behavior in the simplest way possible (that still captures the interesting stuff). Strain and strain rate are the important variables for characterizing the elastic and viscous responses respectively. Strain in natural rocks is very difficult to measure. Many many studies have been done to develop methods to measure strain, but in many of the rocks in which that is possible, the behavior is predominantly viscous, so the strain does not mean much mechanically. However, the relative strain, such as a gradient approaching and passing through a shear zone, can give an indication of the relative strain rate, or viscosity if the system was experiencing a constant stress across it. We will get into this later, but I mention it here as an example of the value of a strain measurement in a rock that was deforming at high temperature and is now rigid at the surface.

Conceptually, it is very important to understand the strain geometry, magnitudes and gradients in any system you are trying to understand. Some examples are illustrated in Fig. 2.1.

Figure 2.1: Examples of geodynamic settings and their characteristic strains geometries.

2.2 Flow and Displacement

Motion of any body or part of a body (including everything at the classical mechanical scale), can be described by "particles"– or points within that body. Motion of particles can be described by either two positions at two different times, X and x, or by the difference between these two states, the "displacement", $u_i = X - x$. The flow field can be defined by a simple analytical function or a complex numerical model; it can be homogeneous or hetergeneous, or both at different length scales. A homogeneous deformation is one that has no spatial variations in the gradients in its flow, and thus no inherent length scale in it; its form is retained when scaled (or multiplied by a scalar), referred to as "affine".

The motion can be decomposed into translation, rotation and deformation.

For example, here is function f that describes a flow of matter, in the form of $\vec{x} = f(\vec{X})$. This set of equations describes a 3D flow field that can combine a compression or extension and a simple shear to make "general shear".

$$x_1 = X_1 + Cu_1 * X_1 + Cs_1 * X_3 + T_1 \quad (2.1)$$

$$x_2 = X_2 + Cu_2 * X_2 + T_2 \quad (2.2)$$

$$x_3 = X_3 + Cu_3 * X_3 + T_3 \quad (2.3)$$

This is a relatively simple flow field function. It allows for a compression or extension in the vertical direction (3), with corresponding extensions or compressions in the (1) and (2) directions. The T_i describe the translations. The first terms describe the uniaxial compression/extension, by the relationship between the Cu constants. In the code, you will see that $Cu_3 = -a*Cu_1 = b*Cu_2$. Furthermore, the deformation is constant volume if $Cu_1 + Cu_2 + Cu_3 = 0$, that is, if $a + b = 1$. If not, the element volume will change with time. If $Cu_3 < 0$, the element will get flatter. For shear, $Cs_1 > 0$ will give you a simple shear in the 1-direction. Thus, you can control the geometry of the flow by playing with Cu_3 and Cs_1. Most geologic situations can be described by rotating this flow field into the appropriate geographic/geospatial reference frame for a given problem, as shown in Ch. 1 and explored in the exercises.

2.2.1 Eulerian vs Lagrangian Descriptions

In general, the Lagrangian point of view is to always keep track of the initial state, X_0, and all motions are calculated relative to that initial state, i.e. $x(X_0)$. This perspective is generally used for solids because the elasticity always "remembers" the unstrained state.

In contrast, the Eulerian view is $X(x)$: the past state as a function of the present state. This is quite confusing. How could the initial or previous state be a function of the current state? In practice, for example, it means that to calculate the previous position you just need to know the present position and the displacement increment or velocity, and you do not care about the initial state, just the previous state. This difference is nicely illustrated in these websites [1] This perspective is used more in fluid mechanics, where you generally do not need to keep track of the initial state of the material (because all flow is irreversible). There are various ways of mixing these in numerical computations.

We will use the Lagrangian view, without any rigid body translation or rotation, because we are starting with elasticity, and because we only care about the deformation, not the position of the element in space. For now, we are primarily interested in the volume element, and describing its internal mechanical behavior, not the deformation and flow of a larger body (as in convection or the study of plate motion and mountain belts).

[1] http://folk.ntnu.no/stoylen/strainrate/ and more specifically http://folk.ntnu.no/stoylen/strainrate/Basic_concepts.html#Lagrangian_and_Eulerian_strain_

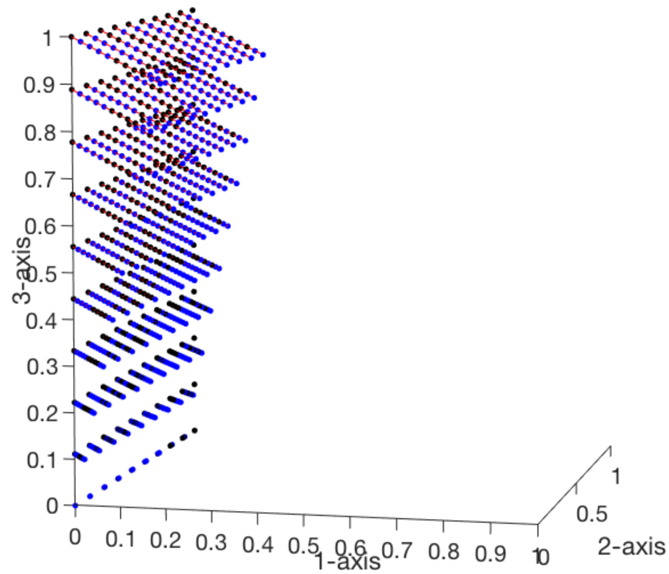

Figure 2.2: Flow field showing some compression and some shear.

2.3 Deformation and strain

"Deformation gradient tensors" relate one position to another, and can be defined from Lagrangian and Eulerian perspectives. The deformation gradient for the Lagrangian view (present with respect to the past) is

$$F_{ij} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{2.4}$$

which is equivalent to $F_{ij} = \frac{\partial f(X_i)}{\partial X_j}$.

The Eulerian deformation gradient tensor is $H = \frac{\partial X_i}{\partial x_j}$, the past with respect to the present. [2]

According to Wikipedia: *In vector calculus, the Jacobian matrix is the matrix of all first-order partial derivatives of a vector-valued function. When the matrix is a square matrix, both the matrix and its determinant are referred to as the Jacobian in literature.*

Large vs. small strain

To demonstrate the error that can be introduced in how one calculates strain, look at pulling on a one dimensional bar of initial length l_0. This is a Lagrangian view because we are referencing every step to the initial state l_0. The strain increment is defined as $d\varepsilon = dl/l_0$ (which is sometimes called the "engineering strain"). To integrate to larger strain, $\varepsilon = \int d\varepsilon dl = \int (dl/l_o)dl = ln(\frac{l_0+dl}{l_0})$. For small displacement increment dl, this reduces to the so-called "engineering" [3] strain (dl/l_0) . However, at larger strain, the engineering strain and the logarithmic strain will diverge.

[2] Mase refers to this as the "material deformation gradient", whereas he refers to the Eulerian version as the "spatial deformation gradient". I have not seen this terminology elsewhere. Coirier has yet another set of terminology. I will integrate Coirier's most likely. Also check Win Means' book.

[3] I don't know why this is called the engineering strain– must look it up– but here's a guess: perhaps engineers really only care about small strains– if the strains are large, the building or bridge is already on its way to collapsing?

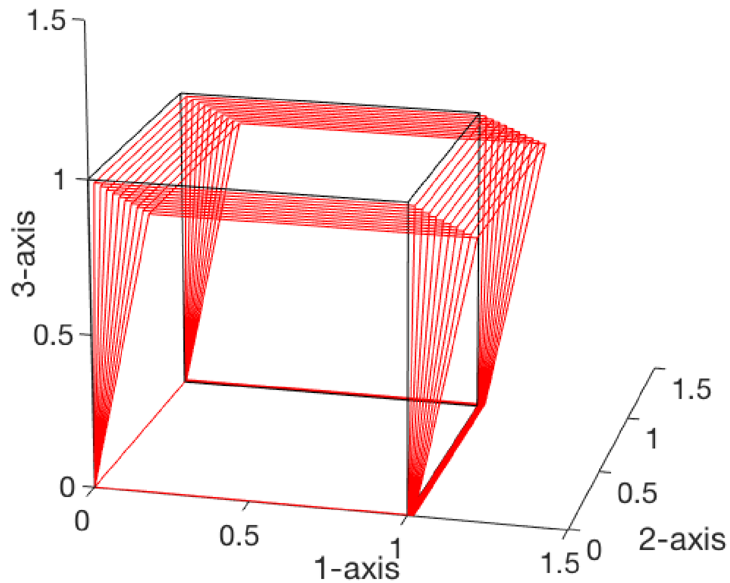

Figure 2.3: A unit box deformed by the flow field in Fig. 2.2.

(In the code, the finite strain is calculated relative to the initial configuration. See if the calculation for the logarithmic strain produces different invariants. Find the strain increment that is small enough to make these difference disappear (between linear and logarithmic calculation of finite strain). Compare this problem in uniaxial compression and simple shear: does one end-member accumulate error faster than the other?)

2.3.1 Strain tensors and the small strain hypothesis

A strain tensor is not the same as a deformation gradient tensor. Strain is a measurement of the change in length of material lines, with all the rigid-body translation and rotations removed, and is therefore a symmetric tensor. The deformation gradient tensor is not intrinsically symmetric.

To derive the strain tensors, Mase [4] progresses from the gradient and displacement gradient tensors, through the Cauchy and Green deformation tensors, which are the Lagrangian and Eulerian descriptions respectively and then to the Lagrangian and Eulerian finite strain tensors. The distinctions among them are beyond the scope of this class. When the small strain approximation is invoked (that is $du/dX << 1$), these tensors reduce to

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}(F + F^T). \tag{2.5}$$

The Eulerian description is the same but replace X with $x-$ everything is calculated in terms of the present state, with no reference to the past. If the displacements themselves are small, the two descriptions are equivalent, and we can use the more common strain tensor: $\varepsilon_{ij} = L_{ij}$.

The vorticity, spin, or rigid body rotation, is

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right). \tag{2.6}$$

Thus, we have decomposed the deformation gradient tensor into strain and vorticity:

$$J_{ij} = L_{ij} + W_{ij}. \tag{2.7}$$

Hereon, I will use ε_{ij} for the strain tensor.

[4] Mase, 1970, "Continuum Mechanics")

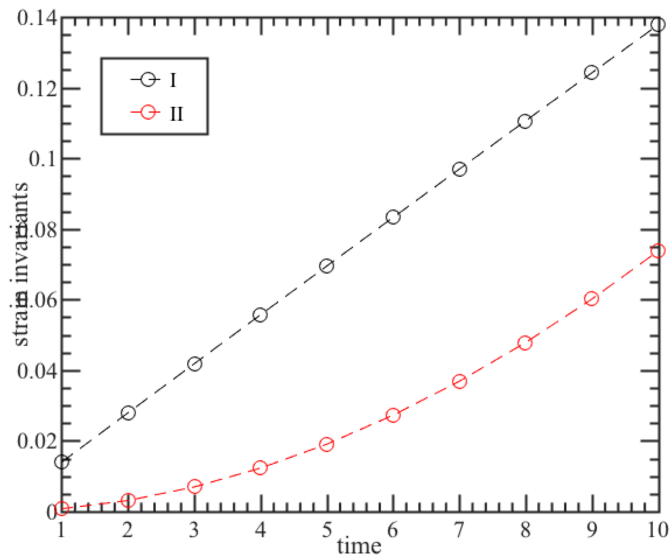

Figure 2.4: Invariants of the strain produced by the flow field in Fig. 2.2.

2.3.2 Strain invariants

The principal strains $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are the eigenvalues of a finite strain tensor. These principal strains are the values along the diagonal of the tensor when it is rotated so that there are no off-diagonal components. These are related to the "stretches", λ, and the strain ellipses, closer to what geologists can sometimes measure directly in the field [5]

The strain invariants are different; they are *not* the eigenvalues of ε_{ij}. They are defined as

$$I = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad (2.8)$$

$$II = \frac{1}{2}\left(\varepsilon_{ij}\varepsilon_{ij} - (\varepsilon_{kk})^2\right) \quad (2.9)$$

$$III = |\varepsilon_{ij}| = det(\varepsilon_{ij}) \quad (2.10)$$

$$(2.11)$$

The physical meaning of the first and second scalar invariants are useful; the first invariant represents the volumetric strain, while the second represents the intensity of the non-volumetric deformation– the change of shape. The third, the determinant, does not have as clear a physical meaning, so I do not plot it.

(The strain invariants can also be written in terms of the eigenvalues add this !)

2.3.3 Velocity and strain rate

The whole path above can be derived using velocity, v (displacement rate [m/s]), instead of displacement u. One way to derive this is by starting with $\dot{f}(X)$... These paths gives us the strain rate tensor,

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right) \quad (2.12)$$

with units of [1/s], etc. However, the strain rate is associated with viscosity, and so is usually seen in an Eulerian frame,

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \quad (2.13)$$

[5] For more on this, see any structural geology textbook, but "Microtectonics", by Passchier and Truow is a particularly good one.

When we discuss viscoelasticity, we will calculate both the strain and strain rate as variables to be tracked.

2.3.4 Strain geometries

Within a reference frame (be it static or rotating), there are terms for certain geometries of strain. "Plane strain" refers to a flow field that is essentially or effectively 2D, meaning that there are only deformation gradients in two directions, not the third, so the deformation can be well approximated by a 2D model. If these conditions are not met, then the strain is referred to as "general" or 3D.

Pure shear means that the finite strain axes are not rotating; the instantaneous strain axes are all parallel to the principal strains. Visually this is quite easy to spot. Simple shear is driven by the off diagonal components, and the instantaneous strain axes are never parallel to the finite strain axes. Finally, general shear is any combination of the above.

NEED A FIGURE HERE! Then go back to the geodynamic settings and discuss where some might be more relevant than others, and at what scale.

2.3.5 What we have not talked about

Because this course is more about how rocks deform than about continuum mechanics, we have skipped much about the various kinds of strain tensors that you would need to consider if you were doing a numerical or complex analytical model. We have not talked about rotations of reference frames in a mathematical sense, but the exercises will ask you to consider them in a qualitative sense.

2.4 TO DO:

To do in class, using the codes **s1_flowfield3D_v2.m** and **s1_flowfield3D_v2.m**:

1. Be sure that you can visualize the difference between translation and deformation. Use the code to do this. Convince yourself.
2. Again using the code, create a flow field that creates a volume change. Make sure you can see that volume change in the invariants.

To do for the Homework (Part 1 of 3):

1. Choose a geologic setting. Pick some part of that system that you want to think about, closely related to your research or not. This setting could be anything below the atmosphere: glaciers, subduction interface, lithosphere-asthenosphere boundary, the wall of a magma chamber, etc. On a page, describe this setting (using borrowed or created figures) and why you chose it, in particular, what about the flow geometry in that system raises interesting questions for you. Somewhere in that system, consider a volume that is deforming, that will be represented by a the deforming box in code **s1_flowfield3D_v2.m**. Indicate in a subfigure the representative volume that you are describing with the deforming box, where it sits in the geologic volume, and its approximate length scale.
2. Think about the motions in that volume and, by trial and error, make a flow field (using code **s1**) that fits with your conception of the flow field in 3D for a representative volume. If the flow is complex, not homogeneous, consider a smaller (or larger) volume within that setting such that the flow can be approximated as uniform (all particle motion can be described by one function). Put that flow field into geologic reference frame. Relate the material coordinates $1, 2, 3$ with the spatial coordinates X, Y, Z, where Z is vertical if you are looking at some plate boundary scale. Otherwise you can place them however you want. Then plot the deforming box in that physical/spatial reference frame. Is the strain predominantly pure shear, simple shear, plane strain or general shear? Why might there be (or not) volumetric strain?
3. Come up with reasonable numbers for displacement gradients by playing with the final shape! *There is an underlying point here: You cannot know much about the flow by looking at the final shape of a volume, and you are starting with a cube at an arbitrary moment in time. But you can think visually/qualitatively about the nature of strain in a given setting and make a movie showing that idea.*

Chapter 3

Stress and Mechanical Energy

In this chapter, we build on strain and deformation to the mathematical description of stress, and then give a very brief entry into the thermodynamics of deformation. Somewhere, I read a nice point: the math for stress is "easier" because you are only looking at a state– stress is instantaneous, while strain necessarily represents a difference or comparison of one state to another. As discussed in the introduction, we could have started with stress or strain, as both can be thought of as thermodynamic state variables for deformation, related to each other by an invertible "constitutive relation", $\sigma(\varepsilon)$ or $\varepsilon(\sigma)$.

First, we will discuss forces, tractions and stress– the meaning of forces as viewed from the interior of a material. Then we will then calculate stress for an elastically isotropic material, and then visualize the stress using a Mohr circle. Even though the focus of this class is viscosity, we will start with elasticity because it is the source of free energy that drives the microscopic deformation mechanisms. All materials are viscoelastic, but for many purposes, we can get by just fine by approximating their behavior as an end-member. Viscoelasticity is for the next chapter.

3.1 Force, tractions and stress

Force is a vector quantity. Traction is also a vector quantity, that is calculated as the force resolved on a face of a body (a crystal, a rock, etc...):

$$t_i = f_i/a \tag{3.1}$$

Stress is a representation of all the possible tractions on a material body for any plane in that material. As Heather described these relationships, I will not go into them more here. See Scholz notes on "Stress, Strain and Energy". The calculation of internal stress from forces via surface tractions: $F_i \rightarrow t_i \rightarrow \sigma_{ij}$. Add: derive the stress tensor from the tractions.

3.2 Elasticity

Any relation between stress and strain (or the inverse) is called a "constitutive model" or equation, and describes the rheological behavior of the material. The very simplest (and it ain't so simple!) is Hooke's Law, the linear elastic constitutive equation,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{3.2}$$

where σ_{ij} is defined above, and ε_{kl} is the strain tensor from the previous chapter.

The elastic tensor C is often called the stiffness; its inverse is called the compliance, and is indicated by S. Why C and S for stiffness and compliance, respectively, and not the other way around, I have no idea? So I prefer to use M instead of C, standing for the elastic Modulus tensor. Anyway, in the Wikipedia page on Hooke's Law, there is a good array of ways of writing the C. The modulus tensors are constructed from

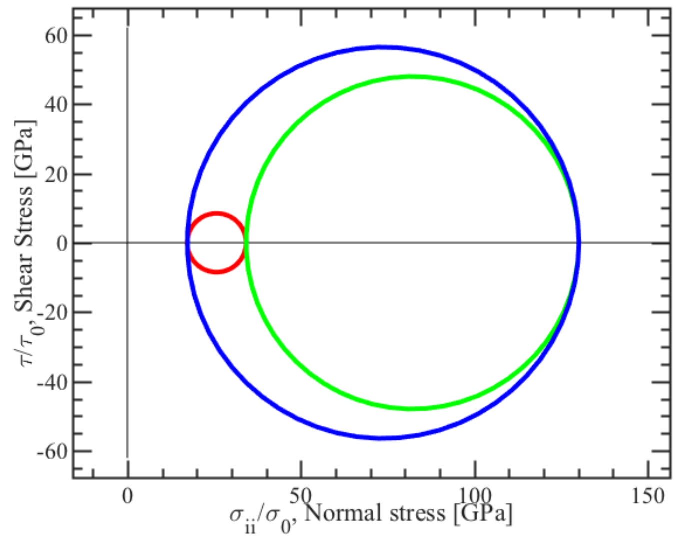

Figure 3.1: Mohr circle representation of 3D stress state.

components that are functions of the elastic constants. There are many combinations of these that are all worked out in numerous texts. In **s3**, I use the bulk modulus K and the shear modulus μ.

Add much on the simple isotropic components, and an introduction to anisotropy– for now, see Scholz book and Heather's slides.

3.2.1 The Mohr Circle representation of stress state

How to plot the 3D Mohr circle:
Derive from the the eigenvalues of the stress tensor, as done in the code **s3_strain2stress.m**. These are the principle values of the stress tensor, finding the solution that gives the largest possible values along the trace (and the off diagonal elements are gone). For the moment, see equations Heather's lecture.

3.2.2 Stress invariants

For stress, the principle stresses are $\sigma_1, \sigma_2, \sigma_3$).

The invariants are calculated in the same way as the strain: They are defined as

$$I = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (3.3)$$

$$II = \frac{1}{2}\left(\sigma_{ij}\sigma_{ij} - (\sigma_{kk})^2\right) \quad (3.4)$$

$$III = |\sigma_{ij}| = det(\sigma_{ij}) \quad (3.5)$$

$$(3.6)$$

The differential stress, $\sigma_1 - \sigma_3$, is a commonly used value. Another is the "deviatoric stress", defined as... And the Von Mises equivalent stress...

3.3 Energy

This is very beginning of thermodynamics in these notes. In most of geology/petrology, people use the Gibbs free energy, which has state variables of [T,P]. The Helmholtz free energy has state variables of [ε, T], making

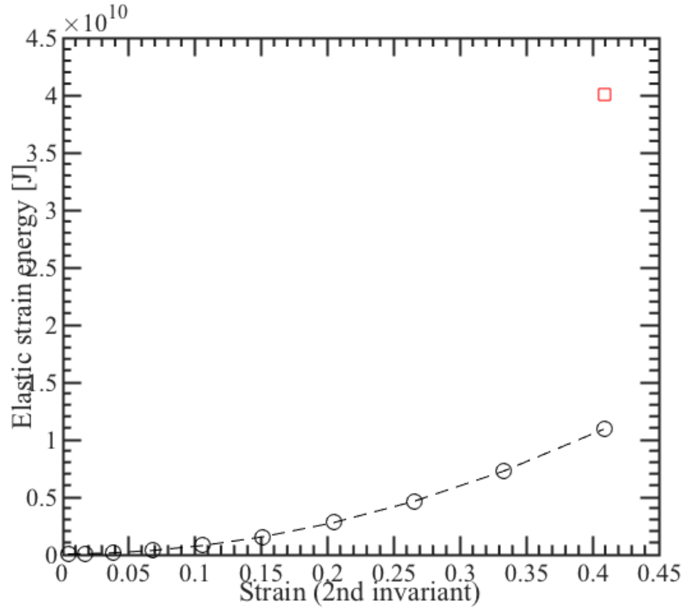

Figure 3.2: Elastic strain energy (Helmholtz Free Energy). The red square is the actual tensor summation; the circles are the scalar approximation (for the strain fields in Chapter 2). This figure will be replaced by only the tensor summation, and will be expanded to show a comparison of all of energy for different strain geometries.

it more suitable for deformation problems where strain is measured (often seen as volume as the state variable, V, but it is really the strain, when applied to solid mechanics).

The mechanical contribution to the Helmholtz free energy is defined as

$$\Psi(\varepsilon) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{3.7}$$

which gives energy in Joules $[N.m]$ per unit volume $[N.m^{-2}]$, which is equivalent to Pascals. Note that in the code at the moment I use a shortcut, for the scalar calculation of $\Psi = \frac{1}{2}\mu\varepsilon_{II}^2$, because I had already saved it as a time series. The real 3D tensor summation is also done for the final value. The difference depends on the strain geometry– it can be minimal or quite large!

The Legendre transformations allow changes between the thermodynamic potentials. Add: the figure showing the various transformations.

3.4 TO DO:

1. So, above, using the **s2** code, you calculated a strain tensor. Now calculate the stress (using the **s3** code) that is induced by that strain (or really, the stress that caused that strain!). Plot the Mohr circle for 3D stress for your geologic setting. The numbers will be completely insane (notice the units!), so flip the problem around:
   What strain gives you reasonable numbers for stress? (What *are* reasonable numbers for stress?). Let's limit σ_1 to 500 MPa. What order of magnitude strain give you this stress? Try different flow geometry; does simple shear or pure shear reach that stress at lower strain ?

2. Now consider velocities! This is a leading question for the next chapter. Come up with some constraint on the velocities and the strain rates in your setting. What do we know? Surface velocity of a glacier? subduction velocity of a slab? Plate velocity in a hot-spot reference frame? Then decide on the length dimension of your initial box. Roughly, a strain rate component is v/l.

Chapter 4

Viscoelasticity

All materials are viscoelastic, in some sense, under some condition. Viscoelasticity is the admission that under certain conditions, all the mechanical energy that is put into a material will not be stored as elastic energy. As discussed in the previous chapter, elastic strain is entirely reversible: when the stress is removed, the material will return instantaneously to its initial state. We benefit from this property all the time; for example, all the suspension of vehicles relies on elastic springs (bridges, sound waves, tendons). However, the counterpart to reversible strain, irreversible strain, is equally important. The simplest (among many) mechanical description of an irreversible behavior is viscosity, which is the proportionality between stress and strain *rate*, $\sigma = \eta\dot{\varepsilon}$. The important point at the moment is that because it relates the strain rate, it does not ever require a knowledge of the initial configuration because all the deformation is *irreversible*: when the stress is removed, the material stops flowing, but does not undo any of its strain. This end-member approximation is the basis for fluid mechanics; recall the Eulerian description of strain that does not require a knowledge of the initial state.

There are endless examples of materials that will exhibit both reversible and irreversible deformation simultaneously. Air and water will both transmit sound waves (compressional waves) because of elastic collisions between molecules, but the water and air could both be flowing. Sound will travel through wind– the wind motion is irreversible but the sound wave propagation is mostly reversible. Rocks in the Earth's mantle convect on the time scale of tens of millions of years, but appear elastic to a seismic wave passing through in seconds to minutes.

However, in both, the elastic waves will not travel forever– the collisions are not perfectly elastic. Some energy is lost with each cycle of the sound wave, and more energy is lost for lower frequency waves. So, the apparent behavior of the material depends on the time scale as well as, in some cases the amplitude of the strain. In class, we will demonstrate these properties with silly putty, cheese, chia pudding and other edible and non-edible stuff.

4.1 Phenomenological models

In this chapter, we will use "phenomenological models" that describe the behavior we observe or expect without specifying the mechanism of deformation.

A variety of models can be derived by assembling a collection of elementary blocks either in series or in parallel and connected by rigid bars, as illustrated in Fig. 4.1. In each of theses blocks the internal stress and strain variables are indexed as $\sigma_1, \ldots$ and $\varepsilon_1, \ldots$ while the external stress and strain are denoted by σ and ε respectively. The mechanical equilibrium in a system composed of a number of N blocks follows the rules given below.

$$\text{Blocks in series: } \begin{cases} \sigma = \sigma_1 = \cdots = \sigma_N, \\ \varepsilon = \varepsilon_1 + \cdots + \varepsilon_N. \end{cases} \qquad \text{Blocks in parallel: } \begin{cases} \sigma = \sigma_1 + \cdots + \sigma_N, \\ \varepsilon = \varepsilon_1 = \cdots = \varepsilon_N. \end{cases}$$

In the rest of this chapter, we will now switch to just using scalars for stress and strain ($\sigma = \eta\dot{\varepsilon}$), though all of what follows can be generalized to tensors.

4.2 Viscosity

Viscosity is the proportionality between the stress and the strain *rate*, so, like elasticity, it is also a 4th rank tensor:

$$\sigma_{ij} = \eta_{ijkl}\dot{\varepsilon}_{kl}. \tag{4.1}$$

Symbolically in a phenomenological model, the viscosity is represented by a "dashpot", as illustrated in Fig. 4.1, which is some device (theoretical or practical) that is designed to slow down or "absorb" energy of some deformation. Its rate is determined by a fluid inside it that is forced through some constriction such that the viscosity predictably slows down some motion. Common examples are shock absorbers and those door closing things. Physically, in experiments, it is very difficult to measure anisotropy in viscosity, to determine the components of the 4th rank tensor [1].

The mechanisms of deformation that produce viscous behavior are the topic of the next X chapters (next three classes). The following chapter will develop the physics of why viscous behavior emerges in rocks at high temperature. These properties can be linear or non-linear (in various parameters), as described by "flow laws" that can be rewritten as viscosity functions. For now, trust that rocks display viscous behavior at various length and time scales that determine much of the long-term dynamics of the solid Earth. For example, the convecting mantle is doing so because it has a much lower viscosity than the plates. As we will see, viscosity alone is not sufficient to cause the emergence of plate-tectonic like behavior, so the lithosphere must have more complex constitutive behavior.

[1] It is possible in elasticity because the materials can be deformed at room temperature and to small strains, so the machines can be much more complex in the strains they can measure. Elastic anisotropy can also be measured by sending ultrasonic waves through a material, but viscosity cannot be directly measured this way.

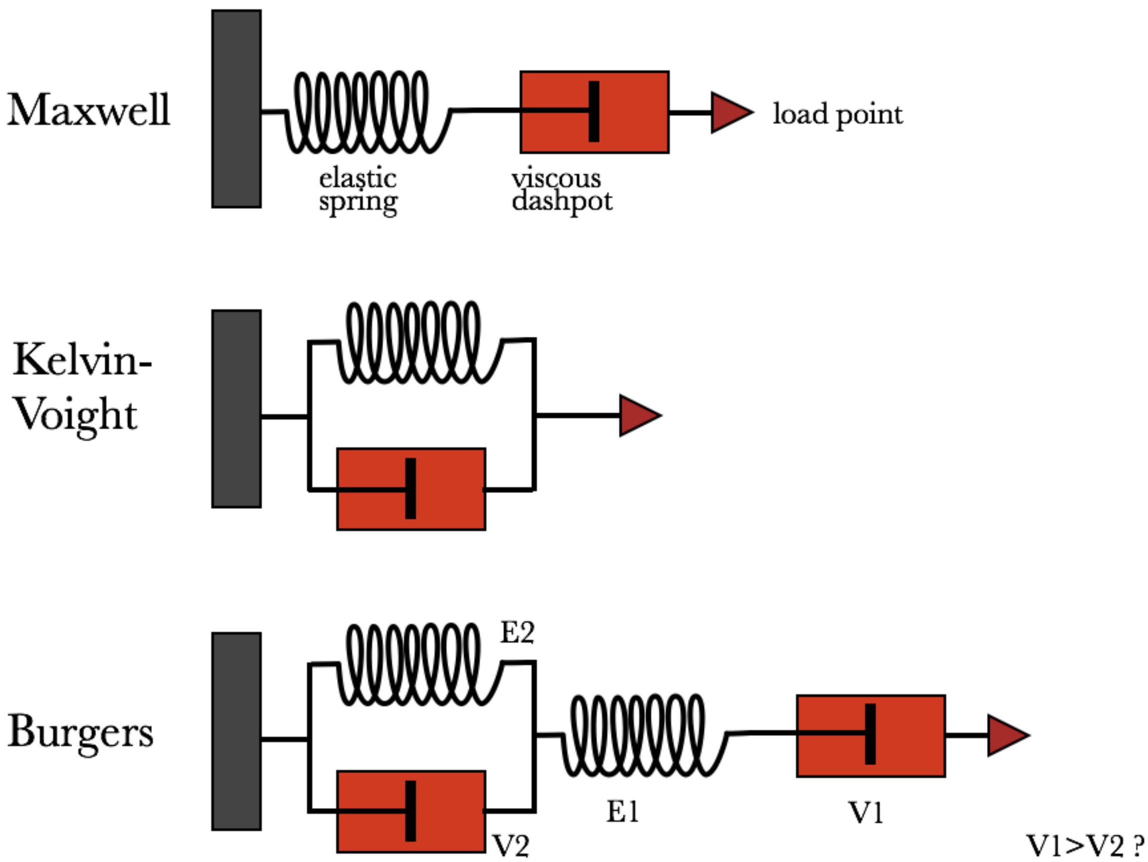

Figure 4.1: Phenomenological models for the most basic building blocks for phenomenological models of material behavior. Viscosity is entirely irreversible; elasticity is entirely reversible. In combination, they produce various mixtures of reversible and irreversible.

4.3 Maxwell Viscoelasticity

The simplest model for viscoelasticity was devised by James Clerk Maxwell (yes, the same Maxwell of electricity, magnetism and thermodynamics). Maxwell viscoelasticity is comprised of a spring ($_e$ for elastic) and a dashpot ($_v$ for viscous) in series, as shown in Fig. 4.1. The two mechanical elements have the same stress, but not the same strain. Why ?

The strains add: $\varepsilon_t = \varepsilon_v + \varepsilon_e$ and so do the strain rates,

$$\dot{\varepsilon}_T = \dot{\varepsilon}_e + \dot{\varepsilon}_v \tag{4.2}$$

but the stresses are equal. This may be less intuitive; the stress in the dashpot only comes from the stress in the spring. Thus the mechanical model is derived simply from Hooke's law:

$$\sigma = M\varepsilon_e = M(\varepsilon_t - \varepsilon_v) \tag{4.3}$$

And can also be written as an integral of a differential equation:

$$\int_t M\left(\dot{\varepsilon}_t - \frac{\sigma}{\eta}\right) dt \tag{4.4}$$

So, if we are going to solve for stress under strain controlled conditions (e.g. $\dot{\varepsilon}_t$ is a constant), use the second equation.

An important concept to understand is the the Maxwell relaxation time, defined as $\tau_{Mxw} = \eta/M$. It is the characteristic time of the equation, approximately indicates the time (or strain) at which the mechanical behavior transitions from elastic to viscous. In the code I use the idea of this "Maxwell" strain: $\tau_{Mxw}\dot{\varepsilon}_t$, to illustrate when the behavior starts to deviate from the elastic.

If you are solving for strain under stress controlled conditions, invert that equation:

$$\dot{\varepsilon}_t = \frac{\dot{\sigma}}{M} + \frac{\sigma}{\eta} \tag{4.5}$$

This question of whether a system stress- or strain-controlled evokes the concept of boundary conditions. In experiments, the difference is relatively clear. But how do you relate this to a geological problem ? Rock samples in an experiment are a small part of a larger system (the machine), in which simplified boundary conditions can be created; in a natural system, boundary conditions are more abstract and are a construct of the model. One way to think about it is how physically bounded the deforming body or region is (once you have defined your system). If An example of a system that is minimally bounded and is only moving by body forces, such as a landslide or glacier, constant force/stress may be a better approximation. If the bed gets weaker, the mass will move faster. For constant strain rate or velocity, consider a plate boundary with velocity gradient across it that is controlled by far field forces. If these forces will not be affected by the changes in the strength of the material inside the REV, then the boundary conditions on the plate boundary are not coupled to the changes in the viscosity in the plate boundary. No feedbacks are possible between the system and the rheological properties. We will return to this question in future chapters when we discuss feedbacks that can occur during deformation, leading to non-linear behavior and localization.

A little more foreshadowing: In Chapter X, we'll develop the thermodynamic framework for deformation of solids. In there, we will define the dissipation $\mathcal{D}$ which can be broken into intrinsic (or internal) and extrinsic (or thermal) dissipations:

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2. \tag{4.6}$$

Though there are others, the main component of the intrinsic dissipation $\mathcal{D}_1$ is the deformational work rate:

$$\mathcal{D}_1 = \sigma\dot{\varepsilon}_e \tag{4.7}$$

In this case of a Maxwell body, $\dot{\varepsilon}_e = \dot{\varepsilon}_t - \dot{\varepsilon}_v$, so $\mathcal{D}_1 = \sigma(\dot{\varepsilon}_t - \dot{\varepsilon}_v)$ and $\dot{\varepsilon}_v$ is calculated from the flow laws discussed in Chapter X. $\mathcal{D}$ becomes the part of the total entropy production and therefore the possible heat production, or feedback through the temperature.

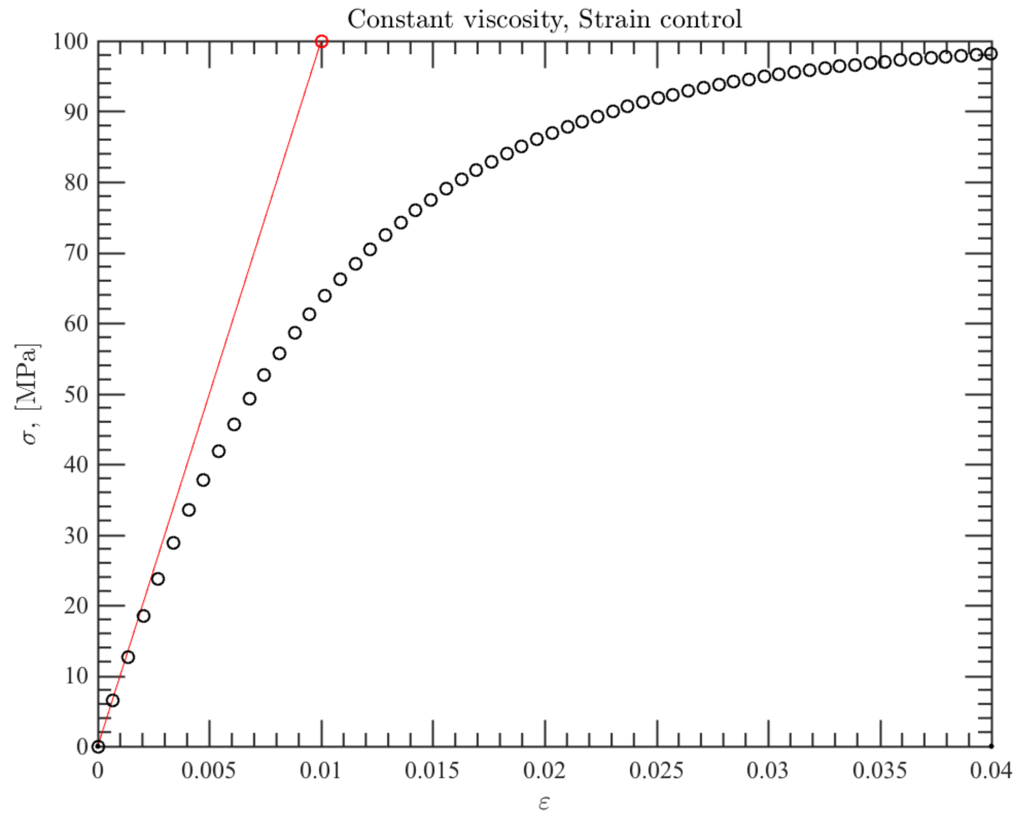

Figure 4.2: Stress-strain curve for a Maxwell material (e.g. experimental values). Solutions to ODEs in Equations 4.2. The red line shows the strain at the "Maxwell strain", defined in the text.

4.3.1 Calculation of ODE solutions for Maxwell materials

Time dependence (no spatial dependence)– i.e. the solution for a representative elementary volume (REV)– or an element in a continuum. The length scale is large enough that the perturbations inside the material can be considered as having single statistical properties (e.g. the mean grain size)

Strain/Strain rate controlled

For constant applied strain rate (a "strain controlled" or "displacement controlled" experiment)

$$\sigma = M(\epsilon_t - \epsilon_v) \tag{4.8}$$

$$\dot{\epsilon}_v = \sigma/\eta \tag{4.9}$$

$$\epsilon_v = \int \dot{\epsilon}_v dt \tag{4.10}$$

Or, written in "pseudo-code",

```
sigma(i_t,1)=M*(eps(i_t,1)-eps_v(i_t-1,1));
epsp_v(i_t) = \sigma(i_t-1)/eta ;
eps_v(i_t) = eps_v(i_t-1) + dt*epsp_v(i_t)
```

These will get more and more useful. Note that this is written for a situation in which the strain rate is controlled.

4.3.2 The Deborah number, De

The important question when considering any process is its relationship to the relaxation time, and whether it will cause a more viscous or elastic response in the material. The *Deborah* number, De is defined as the ratio of the relaxation time of the material to the characteristic time scale of the process. For a Maxwell material, that looks like:

$$De = \frac{\tau_{Mxw}}{t_p} \tag{4.11}$$

You can also think of this as a ratio of frequencies:

$$De = \frac{f_p}{f_{Mxw}} = \frac{\dot{\varepsilon}_p}{f_{Mxw}} = \frac{\eta\dot{\varepsilon}_p}{M} \tag{4.12}$$

So, what is the process time or frequency? How do you define it for your system ? Think of a seismic wave. Think of a non-periodic process like the flow at a mid ocean ridge, or the loading or unloading of a plate during glacial isostatic adjustment.

4.4 Anelasticity

History of the anelastic models.. Kelvin/Voight, Burgers, Maxwell... etc.

recoverable but time dependent creep. Transient Creep...

This will probably become a separate chapter, but for now briefly folded in to this one...

4.4.1 Kelvin-Voight material

Reversible strain but positive entropy production– time dependent recovery ! MECHANISMS...

4.4.2 Burgers material

combined Maxwell and K-V.

4.4.3 Others anelastic mechanical models

Andrade, extended Burgers, Zeners.. etc.. why ?

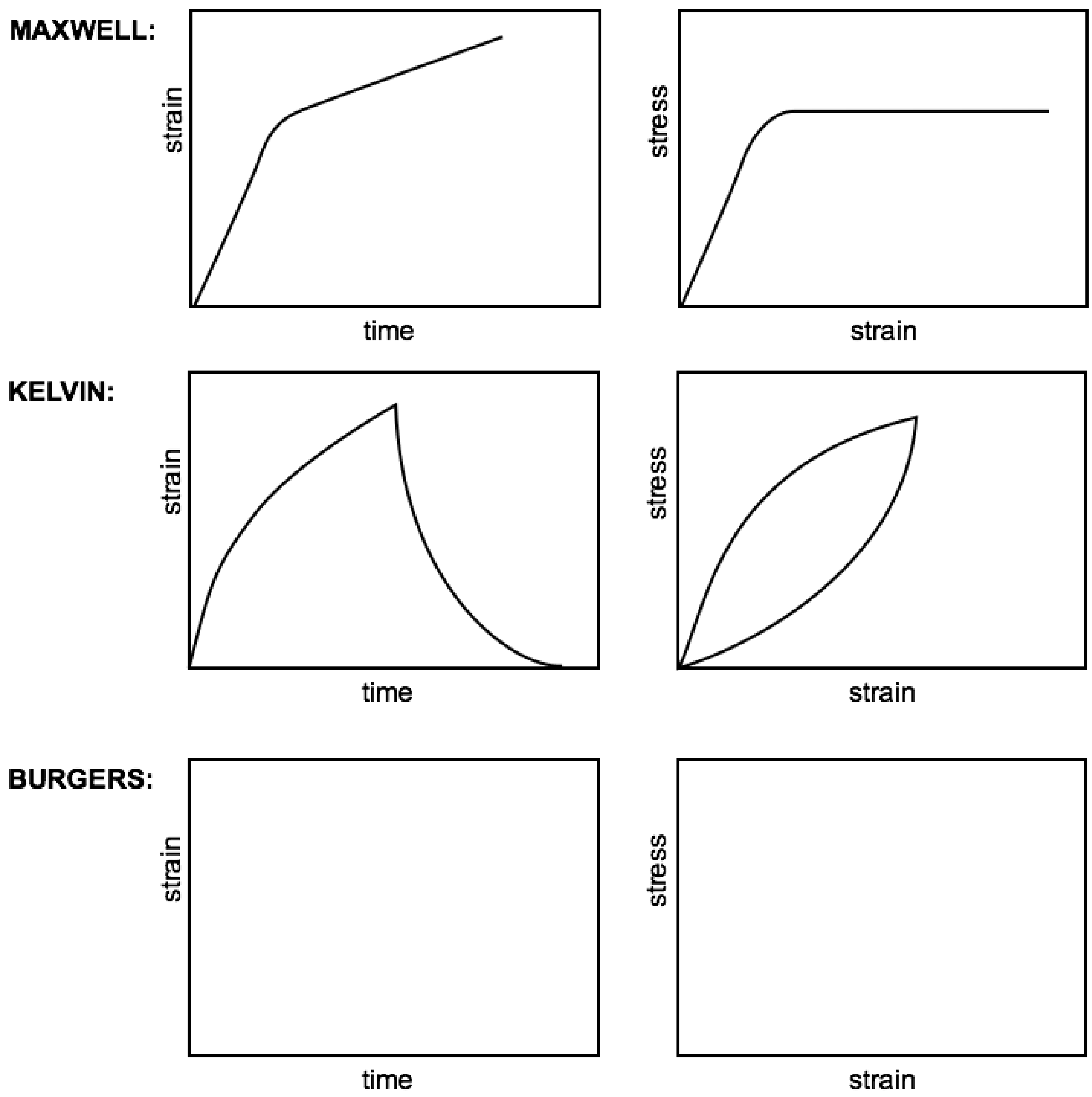

Figure 4.3: Strain-time and stress-strain curves for Maxwell and anelastic models. Please fill in the Burgers model.

4.5 PROJECT: Part 2

IN CLASS:

1. Run the Maxwell ODE code for constant viscosity. Play with it. Vary the Maxwell relaxation time by varying the viscosity. Make sure you understand what the red line is and what is happening as the black dots move off that line. No need to plot anything.

We will use these for homeworks after the sections on deformation mechanisms.