EESC 9945

Geodesy with the Global Positioning System

Class 2: *Satellite orbits*

Background

- The model for the pseudorange was

$$\rho(t) = |\vec{x}^s(t-\tau) - \vec{x}_r(t)| + c(\delta_r - \delta^s)$$

- Today, we'll develop how to calculate the vector position of the satellite
- The satellite and receiver position vectors have to be calculated in the same reference frame (TBD)

Central Force Problem

- Terminology for orbital geometry stems from the solution to the classical problem of the **central force** problem:

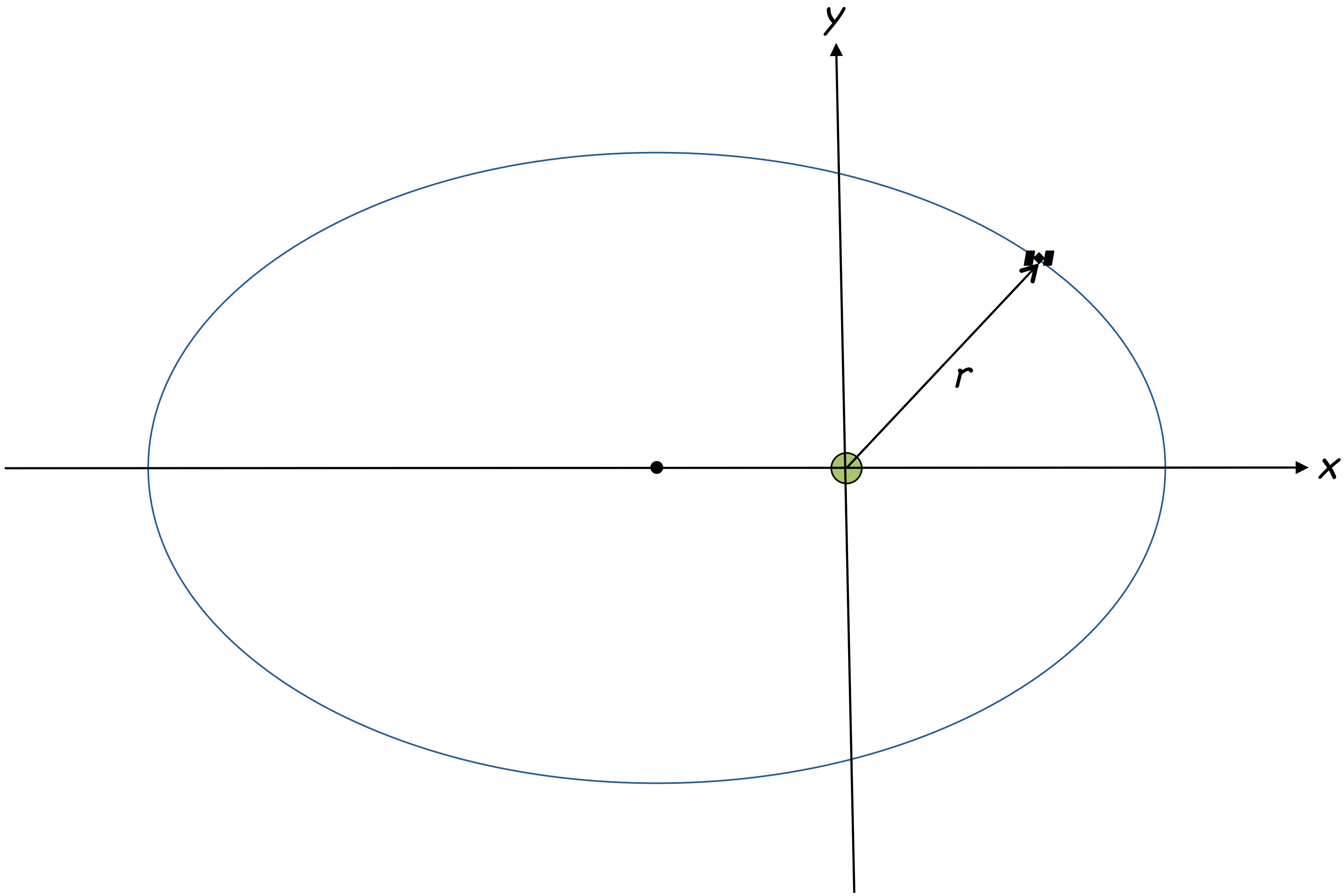
$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \qquad (1)$$

  - $\vec{F}$: force on the satellite located at $\vec{r}$
  - CoM of combined system at $\vec{r} = 0$
  - m: mass of satellite
  - $GM = 398.600415 \times 10^{12}\ \mathrm{m}^3\ \mathrm{s}^{-2}$
- Assumes Earth and satellite behave as point masses (spherically symmetric mass distribution)

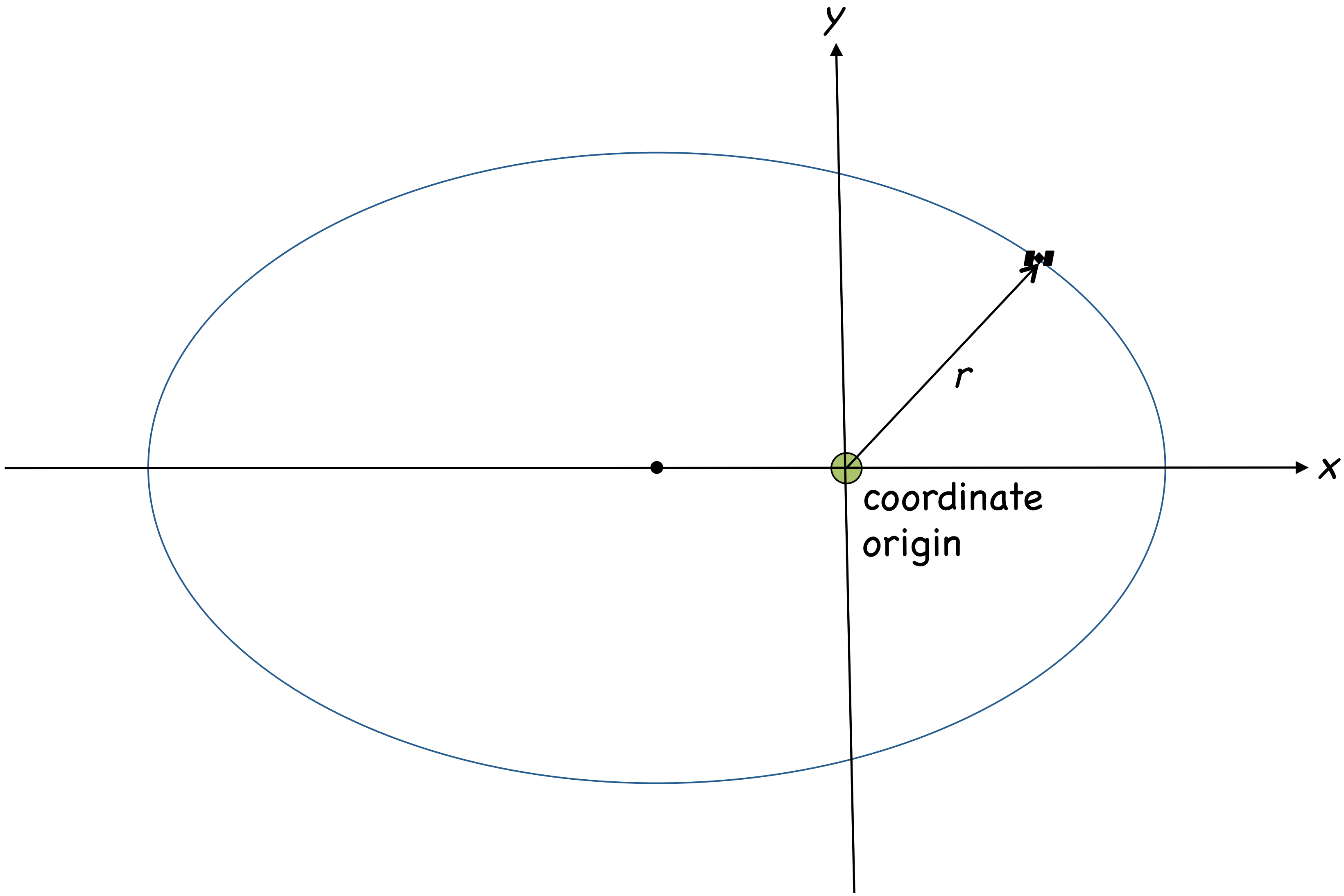
Solution to Central Force Problem

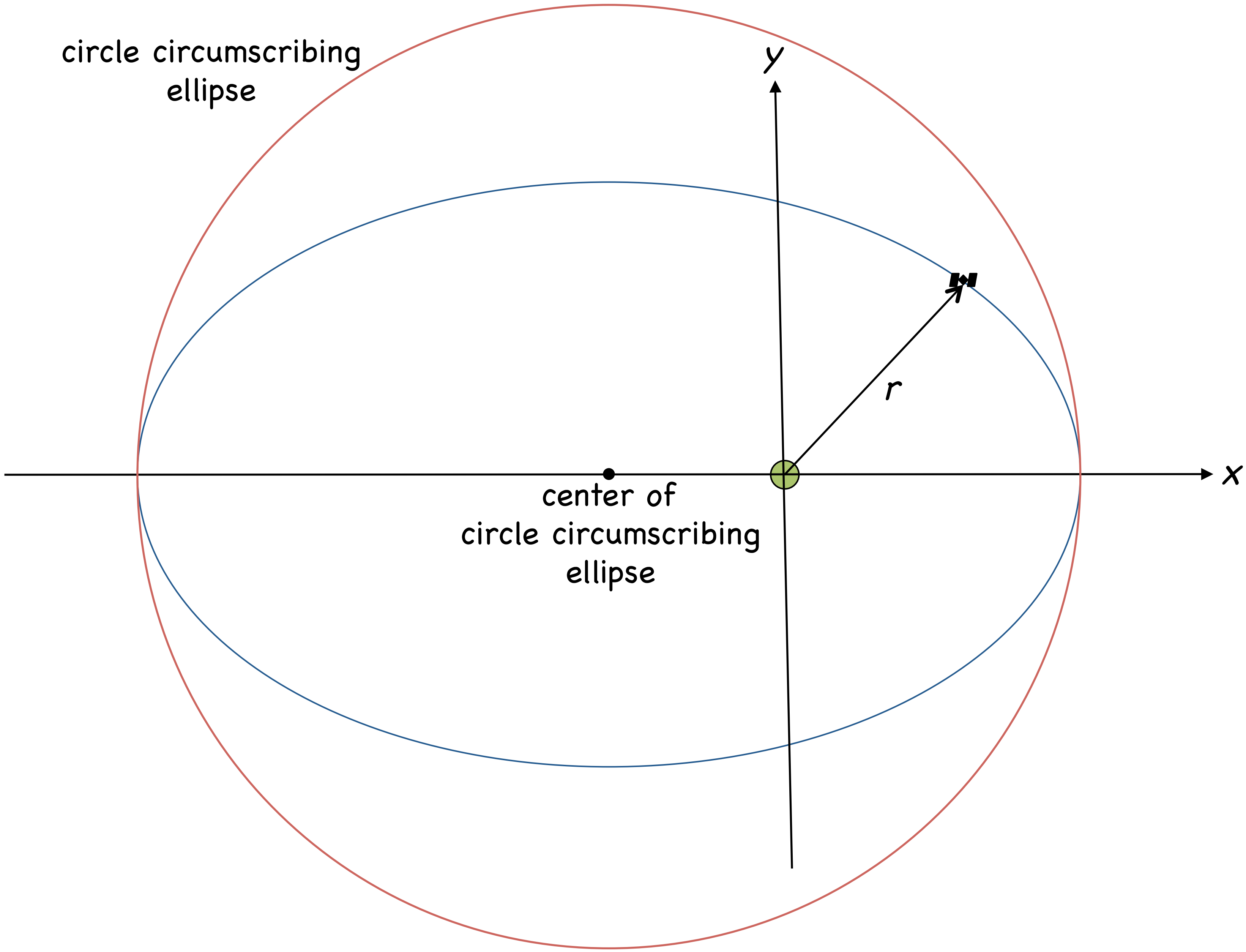
- Solution to (1) leads to condition wherein the masses **orbit** each other
- Depending on initial conditions, one object could also simply make a close pass to the other and have no sustained orbit.
- In planetary motion, the mass of one body (the **primary**) far exceeds the mass of the other (the **satellite**)
- CoM of system (coordinate origin) assumed to be at center of primary
- Shape of the orbit is an **ellipse** with primary at one of the foci and **satellite** moving along the ellipse (**Kepler's 1st Law**)
- Satellite remains in **orbital plane** containing primary and orbital ellipse

Orbital plane

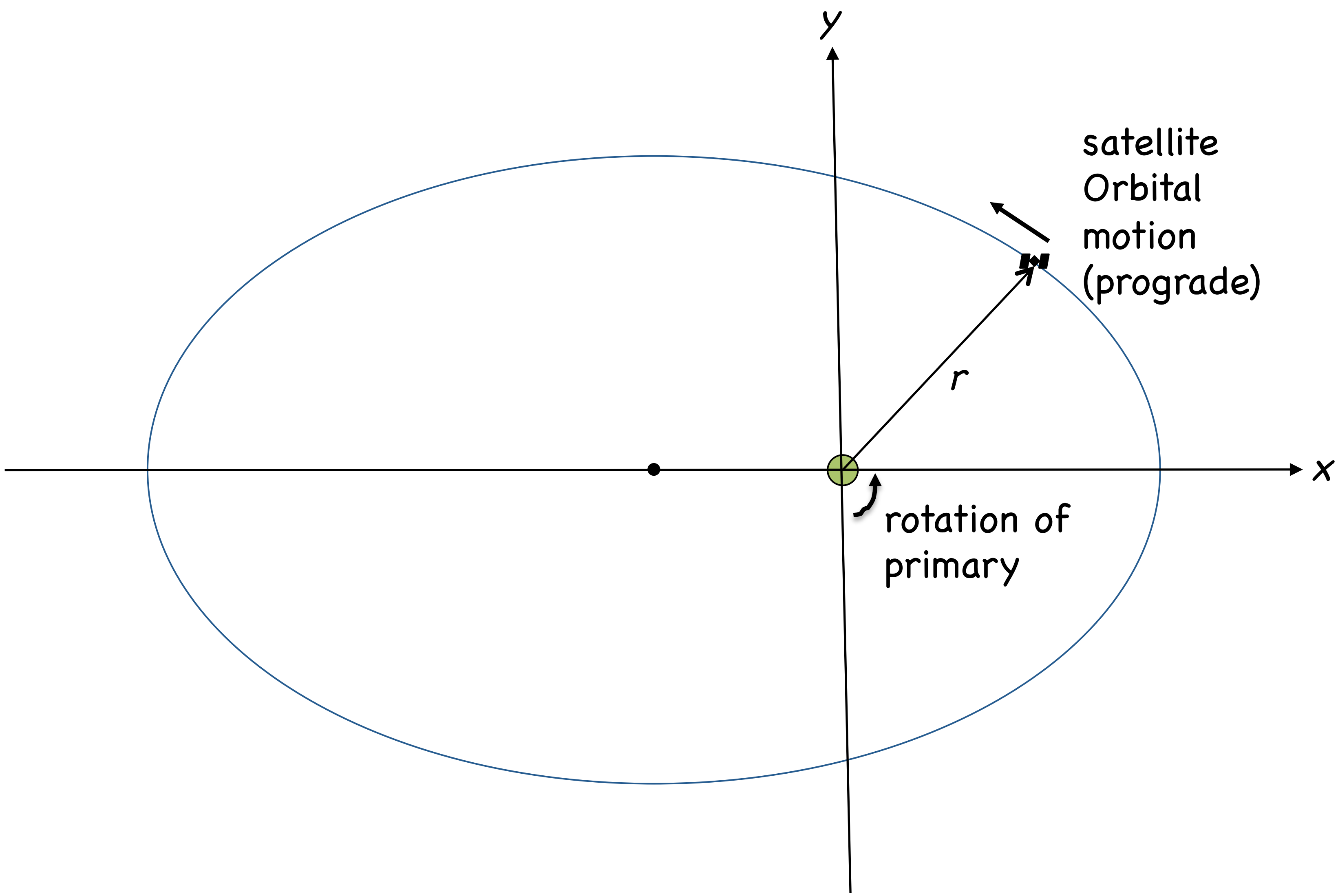


Coordinate Origin

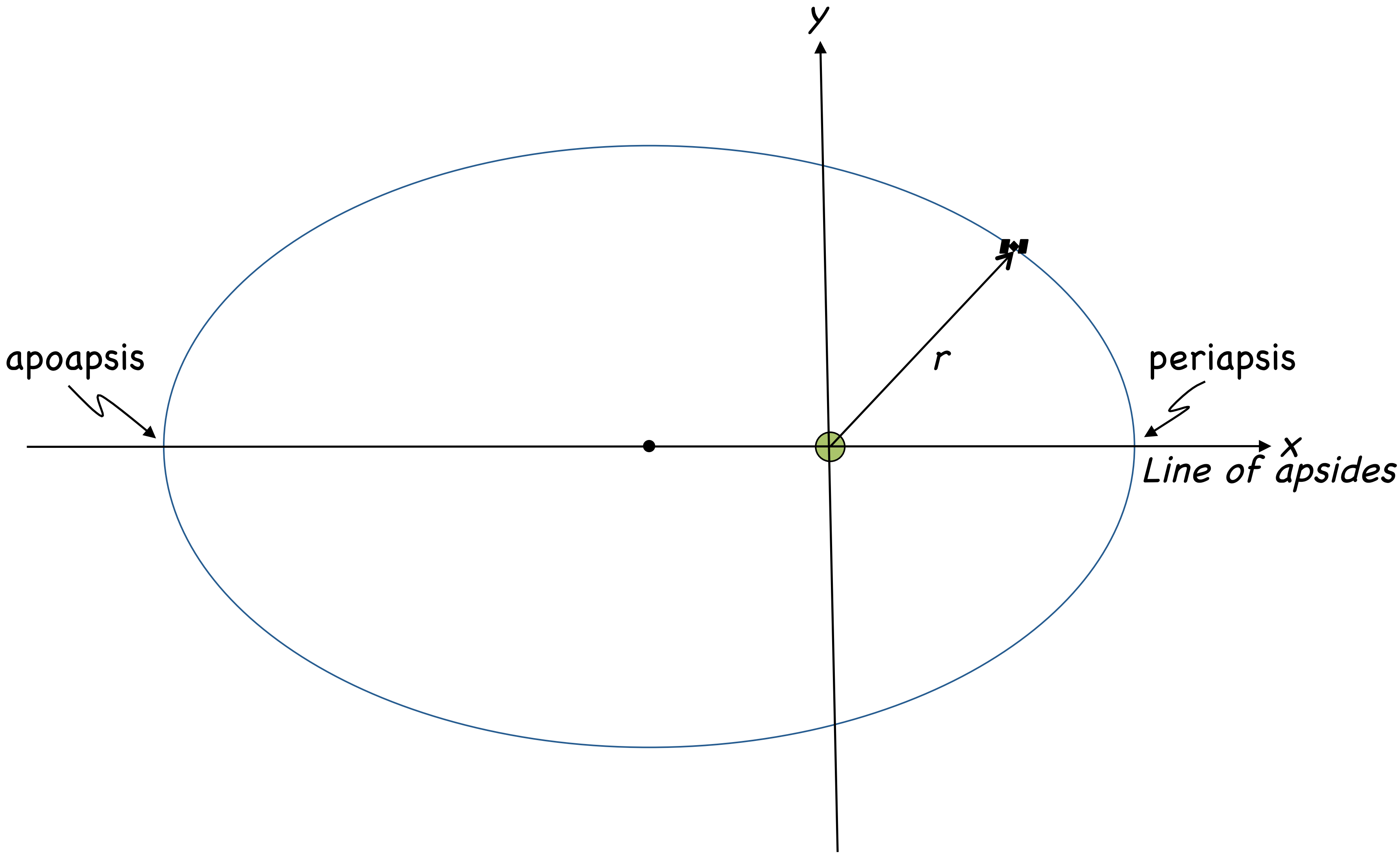



circle circumscribing
ellipse
y
r
x
center of
circle circumscribing
ellipse

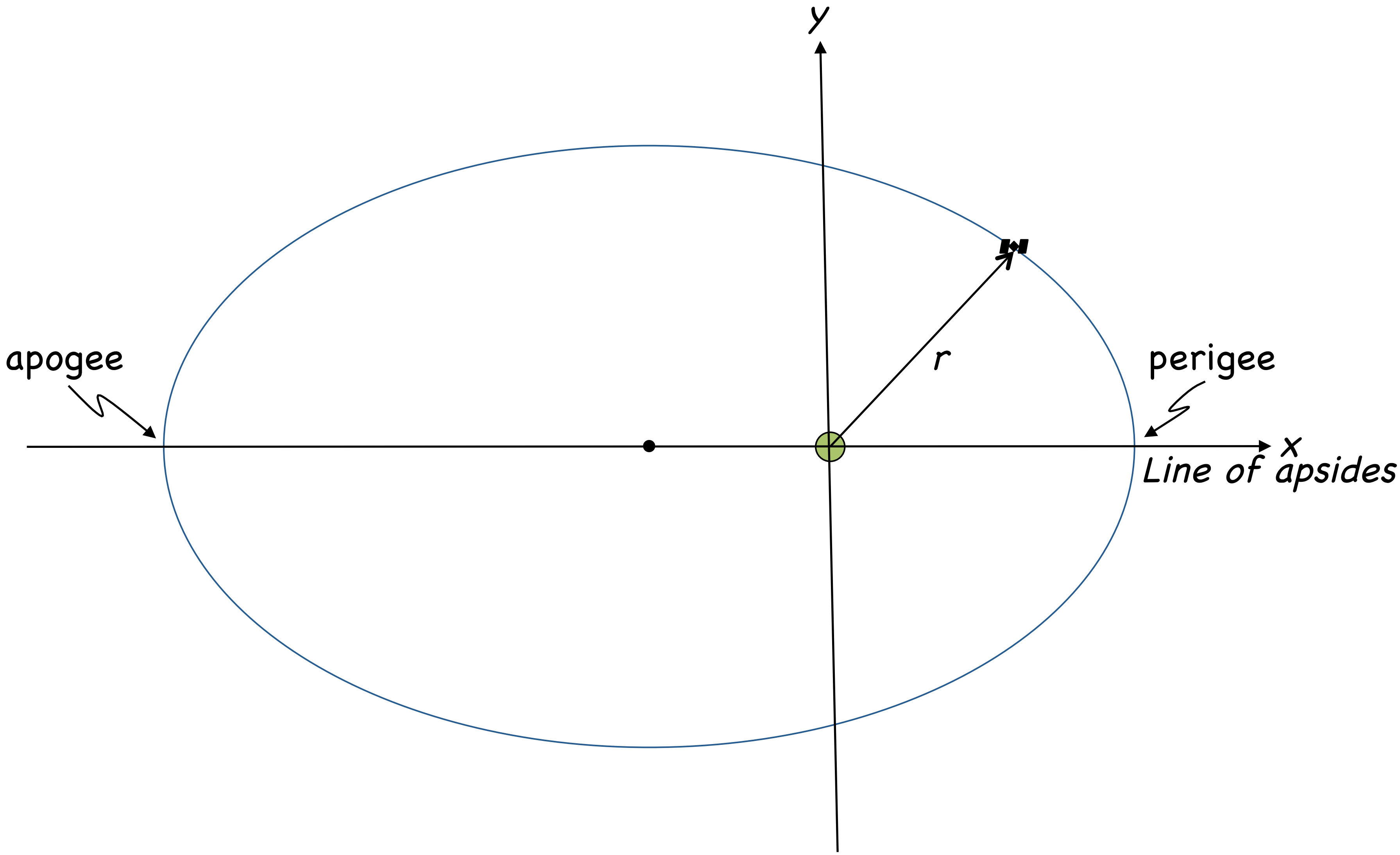
View of orbital plane



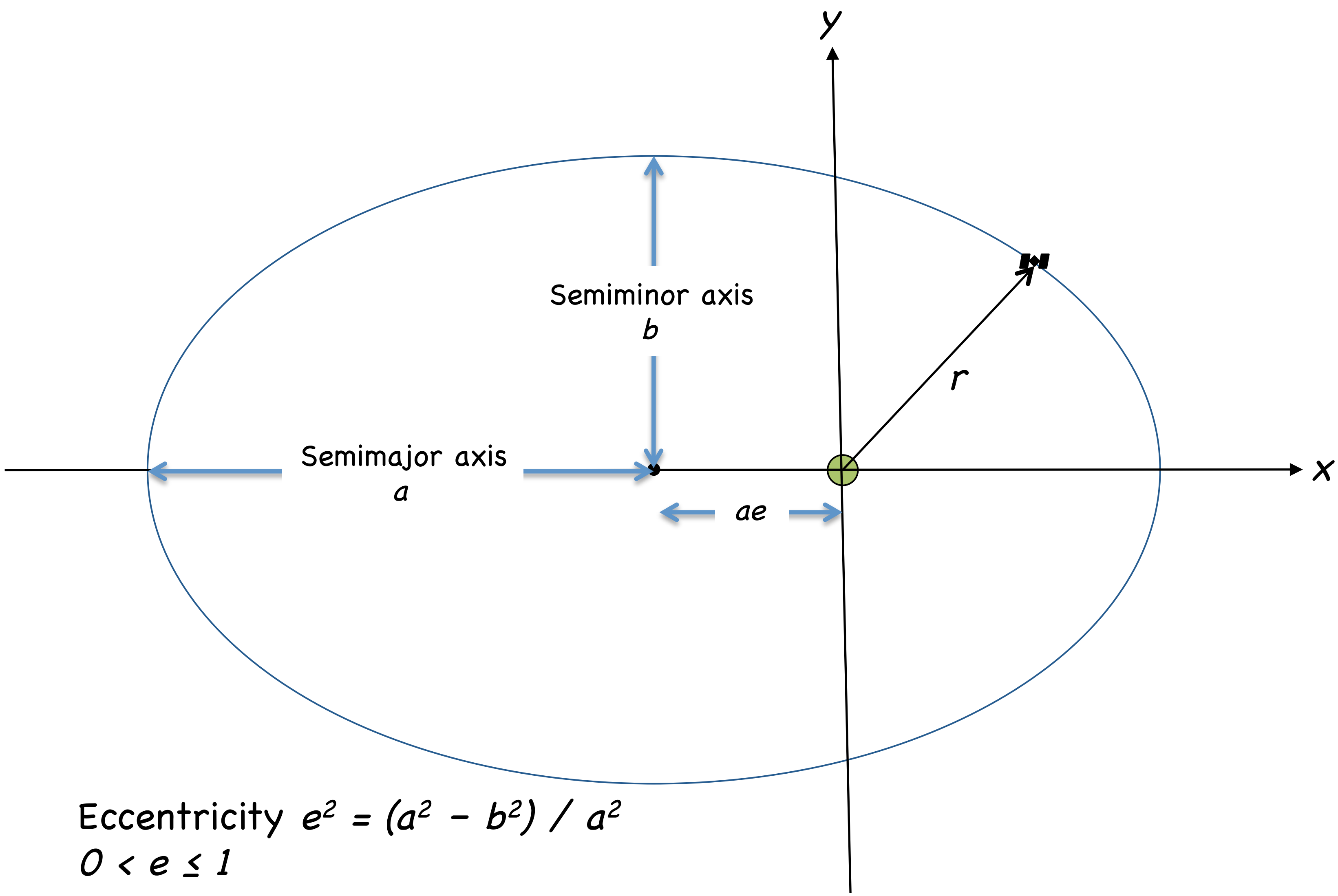
Orbital plane: Geometry



If primary is Earth…



Shape of Orbital Ellipse



Eccentricity $e^2 = (a^2 - b^2) / a^2$

$0 < e \leq 1$

Satellite orbital motion

- Satellite speed is not constant within orbit
- The line from the primary to the satellite sweeps out equal areas of space in equal spans of time (**Kepler's 2nd Law**)
- The orbital period of the satellite (**Kepler's 3rd Law**) is:

$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

Orbital motion

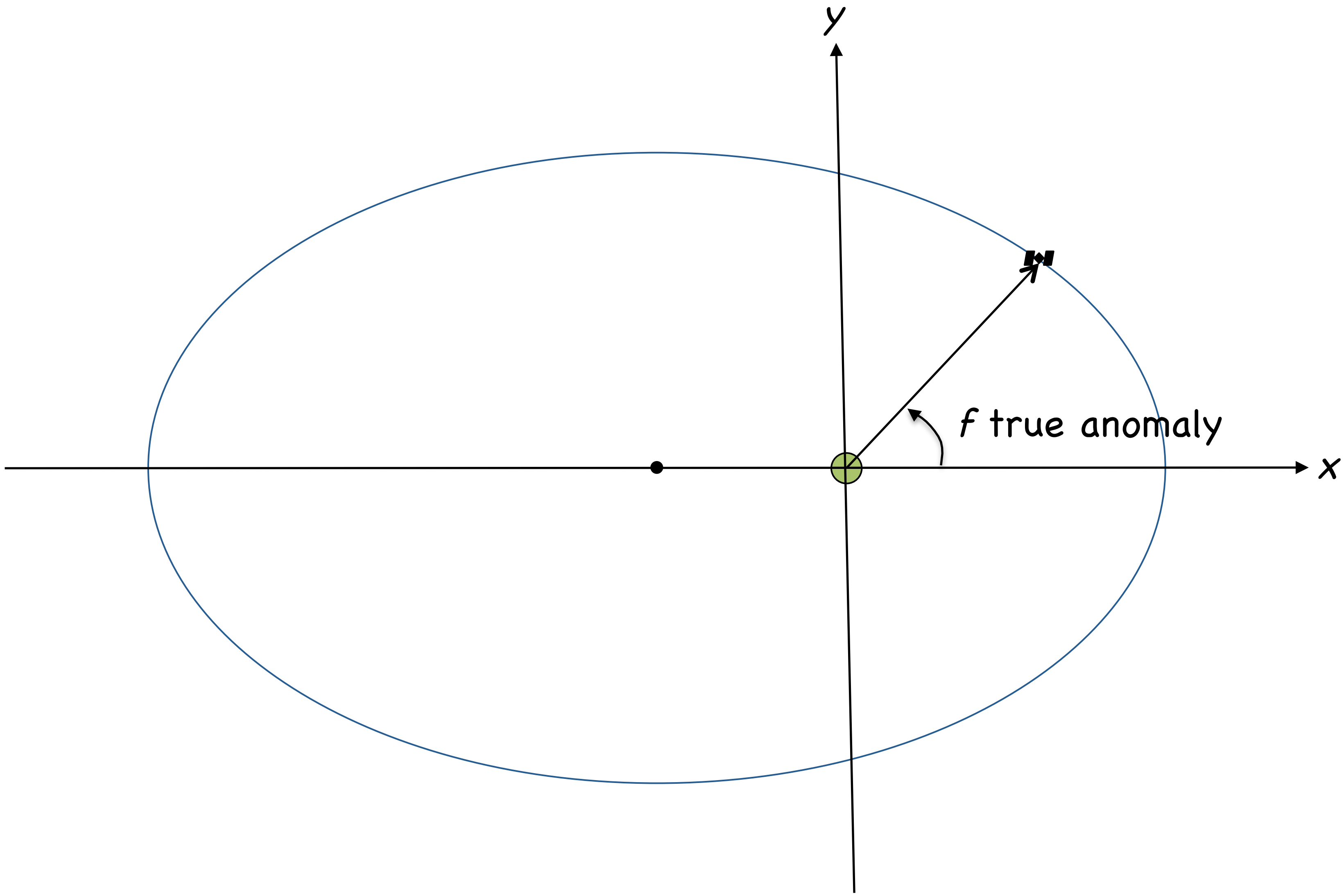
- The position of the satellite within the orbital ellipse is described by an angle with respect to the perigee
- The mean motion is the mean angular velocity of the satellite:

$$n = \frac{2\pi}{T} = \sqrt{\frac{GM}{a^3}}$$

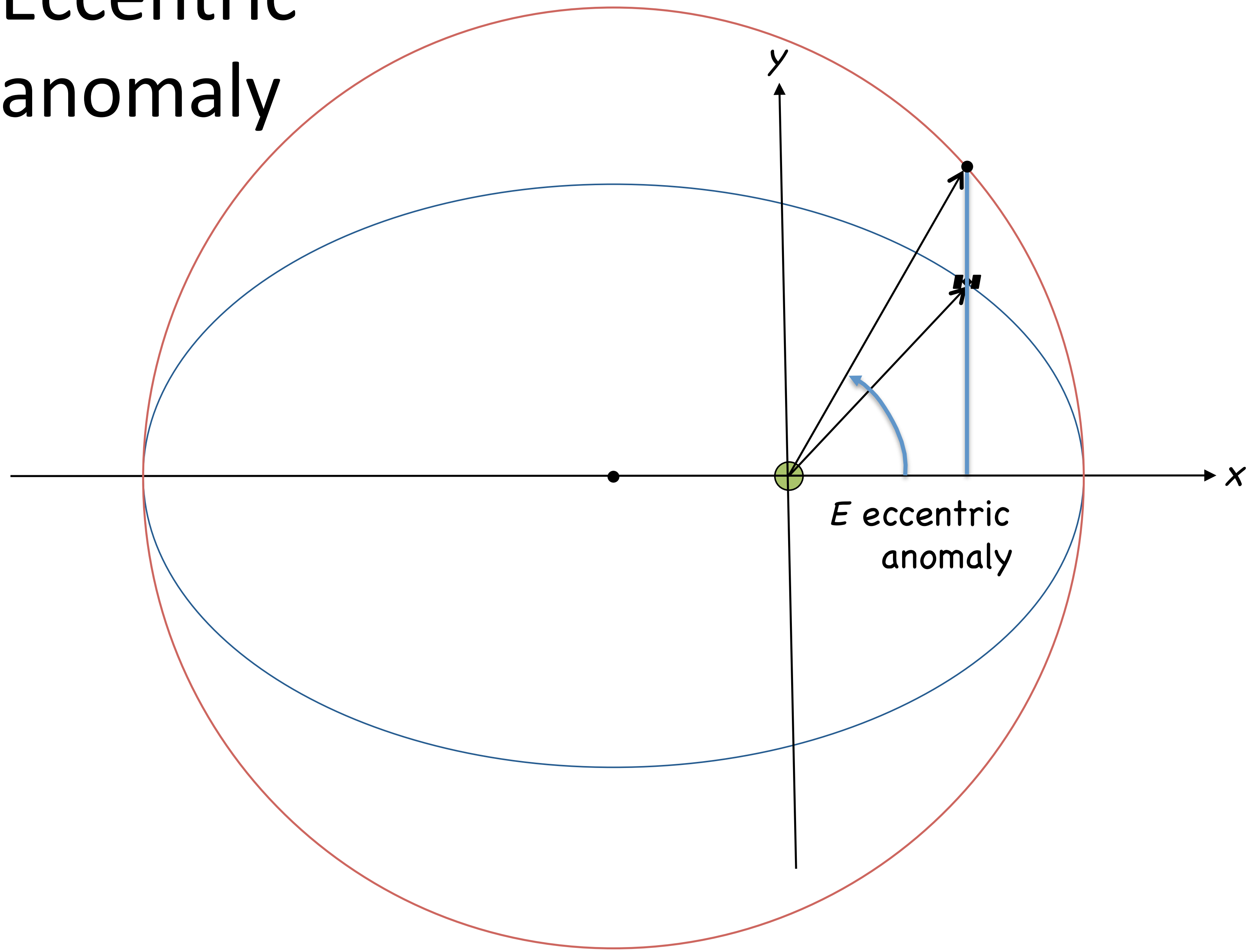
- The **mean anomaly** is the anomaly of a fictitious satellite with a constant motion equal to the mean motion of the satellite.
- The anomaly is measured from the point of perigee passing at epoch t_p:

$$\mu = n \cdot (t - t_p) = \mu_\circ + n \cdot (t - t_\circ)$$

True anomaly



Eccentric anomaly



Anomaly Relationships

- To calculate Cartesian coordinates of GPS satellite, we'll need eccentric anomaly
- To calculate eccentric anomaly at epoch *t:*
  1. Calculate mean motion *n* using Kepler's 3rd Law
  2. Calculate mean anomaly
  3. Calculate eccentric anomaly (iteratively)

$$\mu = \mu_\circ + n(t - t_\circ)$$

$$\mu = E - e \sin E$$

$$\tan f = \frac{\left(1 - e^2\right)^{1/2} \sin E}{\cos E - e}$$

Iterative solution for eccentric anomaly

- Given μ, e
- Eccentricity for GPS satellites is small, ~0.01 (nearly circular orbit)
- Zeroth iteration: $E^{(0)} = \mu$
- k^{th} iteration: $E^{(k)} = \mu + e \sin E^{(k-1)}$
- Converges rapidly for small eccentricity

Position of SV in orbital plane

- **Orbital coordinate system** (OR): origin at CoM of primary
- X-axis coincides with line of apsides
- Y-axis at $f = \pi / 2$
- Z-axis completes RH system
- Satellite position in OR system:

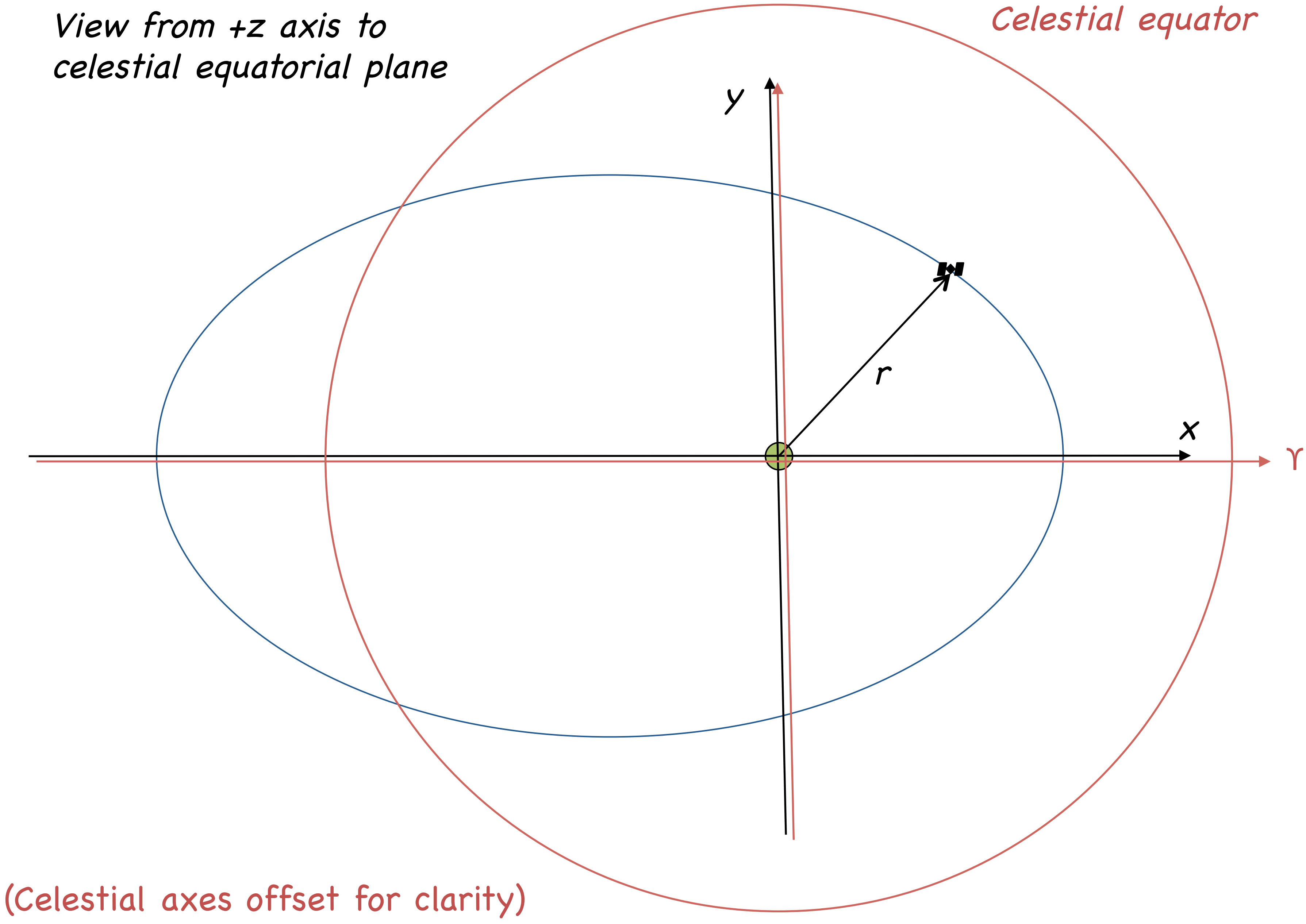
$$\vec{r}_{\mathrm{OR}} = \begin{bmatrix} x_{\mathrm{OR}} \\ y_{\mathrm{OR}} \\ z_{\mathrm{OR}} \end{bmatrix} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} = \begin{bmatrix} a(\cos E - e) \\ a(1-e^2)^{1/2}\sin E \\ 0 \end{bmatrix}$$

Position in Earth-centered system

- So far, we've used three **Keplerian orbital elements** to position the satellite in the orbital ellipse: *a*, *e*, and either t_p or μ_o
- The equation of motion was a second-order differential equation of a three-dimensional vector, and therefore we can expect six initial conditions
- The remaining three orient the orbital ellipse in three-dimensional space
- This is accomplished by three rotations

Position in Earth-centered system

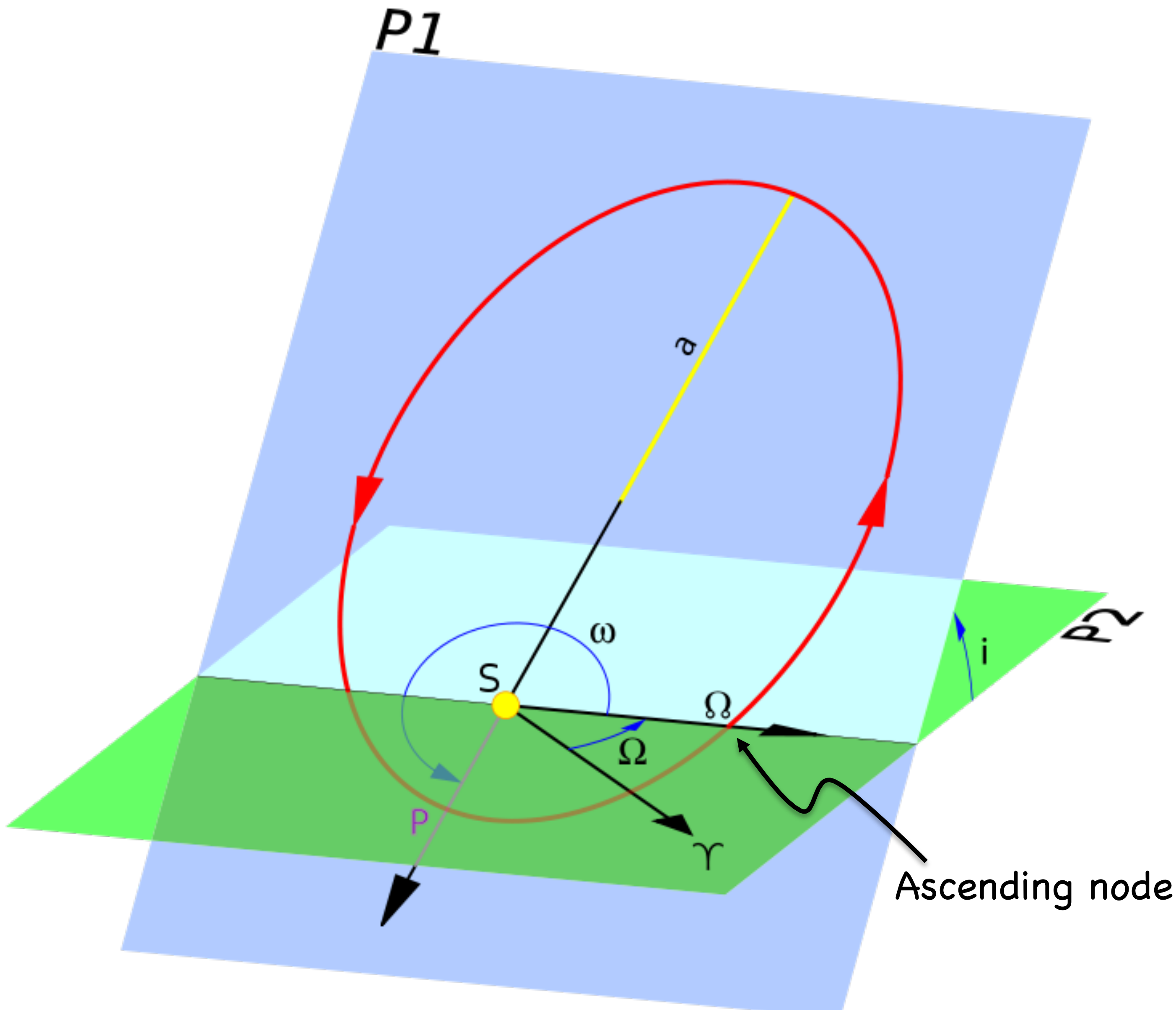
- We begin by positioning the orbital ellipse in the equatorial plane of an Earth-centered celestial sphere
  - X-axis: +X aligned with first point of Ares (♈), celestial right-ascension (like longitude) origin
  - Z-axis: +Z aligned with Conventional International Origin (mean spin axis of Earth)
  - Y-axis: Completes RH system
- We will then rotate the orbital ellipse into position



View from +z axis to
celestial equatorial plane
Celestial equator
y
r
x
ϒ
(Celestial axes offset for clarity)

Orbit Nodes

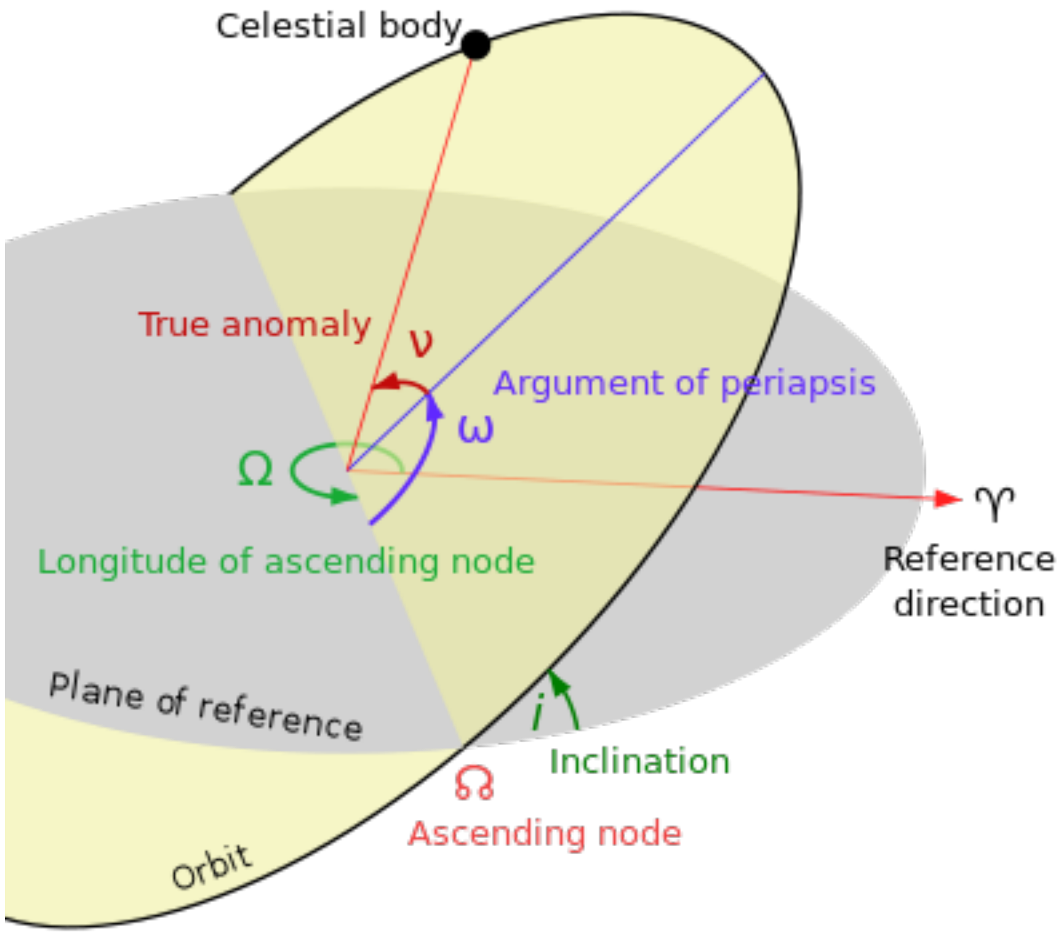
- In the final rotation, the orbital ellipse will be inclined with respect to the celestial equator
- The ellipse will intersect the celestial equatorial plane at two points
- These points are the **nodes**
- Where the satellite travels from negative z to positive z is the **ascending node**
- From positive to negative, **descending node**



http://en.wikibooks.org/wiki/Astrodynamics/Classical_Orbit_Elements

Rotations to celestial frame

1. Rotate about the z-axis so that ascending node points in the direction of the +x-axis (♈)
   – Angle ϖ **argument of perigee (periapse)**
2. Rotate about x-axis (now also the nodal line) to incline the orbital plane with respect to the celestial equator
   – Angle ι **inclination**
3. Rotate about z-axis until the ascending node has the correct longitude
   – Angle Ω **right-ascension (or longitude) of the ascending node**



http://en.wikipedia.org/wiki/Orbital_node

Rotation matrices

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformation to Earth-centered system

- Earth-centered system with axes parallel to celestial coordinate system but centered on Earth
- This is sometimes called *apparent place* system
- This system does **not** rotate with Earth

$$\vec{r}_{\mathrm{AP}} = \begin{bmatrix} x_{\mathrm{AP}} \\ y_{\mathrm{AP}} \\ z_{\mathrm{AP}} \end{bmatrix} = R_3(-\Omega)\, R_1(-i)\, R_3(-\varpi)\, \vec{r}_{\mathrm{OR}}$$

Transformation to Earth-fixed system

- To account for the spin of the Earth, need to rotate about the z-axis by the Greenwich (apparent) sidereal time GST
- GST is an angle an not the same as UT
- The coordinates in the Earth-fixed (EF) system are

$$\vec{r}_{\mathrm{EF}} = R_3(\mathrm{GST}) R_3(-\Omega)\, R_1(-i)\, R_3(-\varpi)\, \vec{r}_{\mathrm{OR}}$$

GPS Broadcast Ephemeris

- Satellites broadcast ephemerides for constellation is the "navigation message" extracted from GPS signal
- Keplerian orbital elements, corrected for linear rate and harmonic variation to account for solar radiation and gravity perturbations
- The combined broadcast ephemerides for each UT day are available from the International GNSS Service (IGS) athttp://igscb.jpl.nasa.gov
- The files are ASCII text in Receiver Independent Exchange (RINEX) format – see http://igscb.jpl.nasa.gov/components/formats.html

GPS Navigation Message

t	Observation epoch for which we want to calculate $\vec{x}_{\mathrm{sat}}$
t_{oe}	Reference time of epoch (toe)
$M_\circ$	Mean anomaly at reference time
Δn	Mean motion difference
e	Eccentricity
$\sqrt{a}$	Square-root of semi major axis a
$\Omega_\circ$	R.A. ascending node at toe
$\dot{\Omega}$	Rate of R.A. ascending node
$i_\circ$	Inclination at toe
ϖ	Argument of perigee
idot	Rate of inclination
C_{us}/C_{uc}	Sine/cosine correction to arg of latitude
C_{rs}/C_{rc}	Sine/cosine correction to orbital radius
C_{is}/C_{ic}	Sine/cosine correction to inclination

GPS Satellite Earth-fixed position calculation (I)

$GM = 3.986005 \times 10^{14}\ \mathrm{m}^3\ \mathrm{s}^{-2}$	GM for Earth
$\Omega_e = 7.292115 \times 10^{-5}\ \mathrm{rad/s}$	Mean rotation rate of Earth
$\pi = 3.1415926535898$	GPS value of π
$a = (\sqrt{a})^2$	Semimajor axis, m
$n_\circ = \sqrt{GM/a^3}$	Mean motion, rad/s
$t_k = t - t_{oe}$	Time relative to toe
$n = n_\circ + \Delta n$	Corrected mean motion
$M_k = M_\circ + n t_k$	Mean anomaly
$M_k = E_k - e \sin E_k$	Equation for eccentric anomaly E_k
$\sin \nu_k = \sqrt{1 - e^2} \sin E_k / (1 - e \cos E_k)$	Sine of true anomaly
$\cos \nu_k = (\cos E_k - e)/(1 - e \cos E_k)$	Cosine of true anomaly

GPS Satellite Earth-fixed position calculation (II)

$\Phi_k = \nu_k + \varpi$	Argument of latitude
$\delta u_k = C_{us} \sin 2\Phi_k + C_{uc} \cos 2\Phi_k$	Argument of latitude correction
$\delta r_k = C_{rs} \sin 2\Phi_k + C_{rc} \cos 2\Phi_k$	Radius correction
$\delta i_k = C_{is} \sin 2\Phi_k + C_{ic} \cos 2\Phi_k$	Inclination correction

$u_k = \Phi_k + \delta u_k$	Corrected argument of latitude
$r_k = a(1 - e \cos E_k) + \delta r_k$	Corrected radius
$i_k = i_\circ + \delta i_k + \mathrm{idot} \cdot t_k$	Corrected inclination

GPS Satellite Earth-fixed position calculation (III)

$x'_k = r_k \cos u$	x in orbital plane
$y'_k = r_k \sin u$	y in orbital plane
$\Omega_k = \Omega_\circ + (\dot{\Omega} - \dot{\Omega}_e)t_k - \dot{\Omega}_e t_{oe}$	Corrected longitude of ascending node
$x_k = x'_k \cos \Omega_k - y'_k \cos i_k \sin \Omega_k$	Earth-fixed, Earth-centered x
$y_k = x'_k \sin \Omega_k + y'_k \cos i_k \cos \Omega_k$	Earth-fixed, Earth-centered y
$z_k = y'_k \sin i_k$	Earth-fixed, Earth-centered z

Notes

- In the RINEX broadcast ephemeris (or "navigation") data file, the ephemerides are given at particular epochs
- We should use the ephemeris closest in time to the observation time
- The main "time tag" in the RINEX file is the "time of clock" (TOC)
- The t_{oe} is calculated from the GPS week number (w) and seconds (s) of the GPS week: t_{oe}= 604800 × (w-1) + s. (w & s are given in the file)
- To calculate t_k, you need the observation epoch in GPS time

Class Project – Sub-goal 1

- Due 15 Feb: Read broadcast ephemerides in RINEX format for 24-hour block and return Cartesian coordinates for all satellites for an epoch to be provide
- See class web site by end of week for specifics
- http://www.ldeo.columbia.edu/~jdavis/eesc9945.htm