EESC 9945
Geodesy with the Global Positioning System

Class 3: *The GPS Constellation*
Review-Orbits

- Keplerian orbital parameters:
  - Semimajor axis
  - Eccentricity
  - Initial anomaly
  - Longitude of ascending node
  - Inclination
  - Argument of perigee
GPS Constellation

- Six different orbital planes each with five satellites (nominal)
- Within each orbital plane, only reference anomalies differ
- Inclination of all orbits \(\sim 55^\circ\)
- Nearly circular \((e \leq 0.02)\)
- Semimajor axis \(\sim 26400\) km
- Geosynchronous
• Due to the 55° inclination of all orbits, satellite tracks as seen from ground have “hole”

• Right: Ground track for 24 hours for site with latitude 43° N
GPS Signals

- GPS satellite transmit signals at two L-band carrier frequencies:
  - $L1 \ f = 1575.42 \ \text{MHz} \ \lambda = 190 \ \text{mm}$
  - $L2 \ f = 1227.60 \ \text{MHz} \ \lambda = 244 \ \text{mm}$
- Both frequencies are integer multiples of GPS fundamental frequency of $10.23 \ \text{MHz}$
GPS Signals

- Both L1 and L2 signals are encoded
- The encoding is achieved by shifting the phase of the signal by 180° (binary phase shift keying or bi-phase modulation)
- The code is thus represented as a binary pulse (0 or 1)
Mono-chromatic (carrier only)

Signal (transmission)

Autocorrelation (detection)

Which peak?
Encoded Carrier

Auto-correlation

Accuracy \sim 1 / (chip rate)
Spread-spectrum signal

Carrier

Encoded Carrier
GPS Signals

• Spread-spectrum encoding for GPS enables a receiver to track multiple satellites simultaneously using the same frequency.

• This scheme is also known as Code-Division Multiple Access (CDMA): Multiple signals sharing the same frequency channel with minimum interference between signals.

• CDMA underlies mobile phone technology, wherein users share a frequency band but transmit and receive multiple signals.
GPS Signal Codes

- Coarse acquisition code (C/A code): Chip rate 1.023 MHz
- Precise positioning code (P code): Chip rate 10.23 MHz
- Y-code (Anti-spoofing, classified): Chip rate 10.23 MHz
- D-code: 50 Hz navigation code
P and C/A Codes

- P-code is 37 weeks long ($2.3 \times 10^{14}$ bits) and then repeats
- Each SV uses the same P-code, shifted by one week
- Pseudorandom, orthogonal
- The SVs are identified by their pseudorandom noise sequence number (PRN)
- C/A code repeats every 1023 bits (1 ms)
Accuracy and chip rate

- D-code: 50 Hz $\rightarrow$ 5950 km
- C/A code: 1.023 MHz $\rightarrow$ 293 m
- P-code: 1.023 MHz $\rightarrow$ 29.3 m
GPS Signals

$L_1$

\[ S_1^p(t) = A_P P^p(t) D^p(t) \cos 2\pi f_1 t + A_C C^p(t) D^p(t) \sin 2\pi f_1 t \]

$L_2$

\[ S_2^p(t) = B_P P^p(t) D^p(t) \cos 2\pi f_2 t \]

$L_k$ signal for SV $p$

Signal strengths for P, C/A

Navigation data stream

P-code

C/A-code
GPS Modernization

- Civilian (i.e., C/A) codes on L2
- C/A code on third carrier (L5, 1176.45 MHz)
- M-code: Military anti-jamming, autonomous
Satellite Acquisition and Tracking

GPS Satellite

Antenna

Receiver
GPS Satellites

Block I
(inactive)

Block II

Block III
(future)
GPS Satellite Transmission

- L-band antenna array always points towards center of Earth

- Angular half-width of transmitting beam is 21.3° at L1, 23.4° at L2
GPS Satellite Transmission

GPS L1 Half-width = 21.3°

Limb-grazing Half-width = 13.6°
GPS Satellite Frequency Standards

- As discussed earlier in course, “clocks” are highly accurate frequency standards
- The fundamental GPS frequency is 10.23 MHz
- Clock accuracy ("stability") is measured as \( \frac{\sigma_f}{f_0} = \frac{\sigma_t}{T} \)
- Typical stability for GPS onboard frequency standards over 24 hours:
  - Rb: \( 10^{-13} \) (10 nsec per day)
  - Cs: \( 10^{-14} \) (1 nsec per day)
- RINEX broadcast orbit files also provide polynomial corrections to satellite clock
Relativistic clock corrections

• Gravitational redshift:
  • Clocks in different gravitational potentials run at different rates: \( \Delta f \approx \frac{\Delta \Phi}{c^2} \)
  • GPS clocks appear to run faster
  • GPS compensates by setting the 10.23 MHz clocks at the factory to 10.229 999 999 543 MHz

• Impact of eccentricity:
  • Clock rate depends on speed in satellite orbit
  • Satellite clock correction \( \Delta t_r = -\frac{2\sqrt{GMa}}{c^2}e\sin E = -2\frac{\vec{v} \cdot \vec{r}}{c^2} \)
  • This correction can be \( \sim 45 \) nsec
GPS Satellite clock corrections

- The ground segment of the Global Positioning System is used to calculate satellite clock errors.
- These errors are modeled as second-order polynomials in time, and uploaded to the GPS satellites.
- The GPS satellites broadcast the clock-correction coefficients.
- These are the first line of each RINEX data block in the broadcast orbit file.
GPS Satellite clock corrections

• The satellite clock correction must include the (eccentricity) relativistic correction also

• The satellite clock correction is therefore

\[ \delta^s(t) = a_0 + a_1(t - t_c) + a_2(t - t_c)^2 + \Delta t_r \]

• \( t_c \) is the “time of clock” (see RINEX documentation)
Summary: Pseudorange model

- The pseudorange model (Class 1) was
  \[ \rho(t) = |\mathbf{x}^s(t - \tau) - \mathbf{x}_r(t)| + c(\delta_r - \delta^s) \]

- In Class 2, we developed the expression for the satellite position vector

- In this class, we presented the satellite clock correction

- The remaining unknown parameters are:
  - The receiver position vector (3 unknowns: x, y, z)
  - The receiver clock error (1 unknown)
Least-squares overview

- We’ll review linear least squares, which we’ll use to estimate the unknown parameters.
- This is the class of problems in which the model can be written as:
  \[ y = Ax + \epsilon \]
- Here, \( y \) is an \( n \times 1 \) vector of observations, \( x \) is an \( m \times 1 \) vector of parameters, \( \epsilon \) is an \( n \times 1 \) vector of errors, and the \( n \times m \) matrix \( A \) is the *design matrix* or *partials matrix*. 
Linearization

- Often our problem will be of the more general and possibly nonlinear form

\[ y = f(x) + \epsilon \]

- \( f(x) \) is a vector of functions
- In this case we linearize around the prior value \( x_\circ \)
- The design matrix is \( A = \left. \frac{\partial f}{\partial x} \right|_{x=x_\circ} \)
- And the linearized observation equation

\[ \Delta y = A \Delta x + \epsilon \]

- Here \( \Delta y = y - f(x_\circ) \) is the vector of prefit residuals and the parameter adjustments are \( \Delta x = x - x_\circ \)
Linearization

• Recall is $f(x)$ a vector

• Then really

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

• Then $A = \frac{\partial f}{\partial x}$ is shorthand for $A_{ij} = \frac{\partial f_i}{\partial x_j}$
Least-squares solution

- Given that the errors are unknown, there is no unique solution (value for the parameters) that satisfies the observation equation.

- Instead, we look for a solution that minimizes the sum of the squared errors, $\epsilon^T \epsilon$.

- This solution for the adjustments is

$$\Delta \hat{x} = \left( A^T \Lambda_\epsilon^{-1} A \right)^{-1} A^T \Lambda_\epsilon^{-1} \Delta y$$

- $\Lambda_\epsilon$ is the covariance matrix of the errors (assumed known):

$$\Lambda_\epsilon = \langle \epsilon \epsilon^T \rangle$$

- The least squares estimate of the parameters is $\hat{x} = x_0 + \Delta \hat{x}$.
Data and parameter uncertainties

- The covariance matrix of the errors in the parameter estimates is $\Lambda_x = (A^T \Lambda^{-1}_\varepsilon A)^{-1}$

- The data error covariance matrix $\Lambda_\varepsilon$ is usually taken to be diagonal with $[\Lambda_\varepsilon]_{ij} = \sigma_i^2 \delta_{ij}$

- In the absence of better information we often take $\Lambda_\varepsilon = \sigma^2 I$

- In this case $\Delta \hat{x} = (A^T A)^{-1} A^T \Delta y$ and $\Lambda_x = \sigma^2 (A^T A)^{-1}$
Fit Statistics

- Postfit residuals: $\hat{e} = y - f(\hat{x})$
- Normalized $\chi^2$: $\chi^2 = \left(\frac{1}{n-m}\right)\sum_{i=1}^{n} \frac{\hat{e}_i^2}{\sigma_i^2}$
- Normalized root-mean-square residual: $N RMS = \sqrt{\chi^2}$
- Weighted root-mean-square residual: $WRMS = \sqrt{\left(\frac{n}{n-m}\right)\sum_{i=1}^{n} \frac{\hat{e}_i^2}{\sigma_i^2} / \sum_{i=1}^{n} 1/\sigma_i^2}$
Fit statistics

- Plot postfit residuals to look for systematic error(s)
- NRMS: Nominal value of 1. Significantly greater than one may indicate systematic error(s) or underestimate of sigmas
- NRMS is often used to scale sigmas
- NRMS significantly less than one may indicate overestimation of sigmas or over-parametrization
- If sigma is unknown, can assume sigma of one and scale all uncertainties by NRMS.