EESC 9945

Geodesy with the Global Positioning System

Class 4: *The pseudorange and phase observables*

- In previous classes we had presented the equation for the pseudorange as the true range biased by the satellite and clock errors

- The GPS receiver makes discrete measurements of the pseudorange at the observation epochs t_j, so the pseudorange R can be written as

  $$R(t_j) = \rho(t_j) + c[\delta_r(t_j) - \delta^s(t_j - \tau)]$$

- Here r and s refer to receiver and satellite and $\rho = c\tau$ is the true range of the point of transmission of the signal to the point of reception

  $$\rho(t) = |\vec{x}^s(t - \tau) - \vec{x}_r(t)|$$

- Replacing every appearance of τ with ρ/c we have

$$R(t_j) = \left|\vec{x}^s(t - \tfrac{\rho}{c}) - \vec{x}_r(t)\right| + c\left[\delta_r(t_j) - \delta^s(t_j - \tfrac{\rho}{c})\right]$$

- As we've said, $\vec{x}^s$ and δ^s can be calculated using information in the RINEX navigation file, but it appears as though we need to know the true range just to calculate the correct epoch to evaluate these values

- The GPS satellite orbit at an altitude of $\sim$20,000 km, however the maximum range (when the satellite is on the horizon) is $\sim$25,000 km

- Thus $\frac{\rho}{c} \leq 0.085$ seconds

- Typical clock rates are 10^{-12} sec/sec, so an error of 0.085 sec leads to a clock error of $\sim 10^{-13}$ sec and a range error of < 0.1 mm.

- This is negligible compared to a pseudorange measurement error of $\sim$10 m, so we can neglect the ρ/c term in the satellite clock error

- We still need to deal with the range itself, where we have

  $$\rho(t) = \left| \vec{x}^s(t - \tfrac{\rho}{c}) - \vec{x}_r(t) \right|$$

- ρ is on both sides

- GPS satellites move with a speed of $\sim$3 km/sec, so neglecting the 0.085 sec signal propagation time leads to a satellite position error of $\sim$250 m, too large to ignore

- However, the acceleration (using $a = GM/r^2$) is $\sim$0.6 m/s^2, contributing to a second-order error (if neglected in ρ) of $\sim$2 mm

- Thus, a first-order expansion should suffice

- We have

$$\rho(t) = \left| \vec{x}^s(t - \tfrac{\rho}{c}) - \vec{x}_r(t) \right|$$

- First-order expansion for $\vec{x}^s(t)$:

$$\vec{x}^s(t + \Delta t) = \vec{x}^s(t) + \vec{v}^s(t)\Delta t$$

where $\vec{v}^s$ is satellite velocity (we'll talk about how to calculate later)

- Expression for ρ becomes

$$\rho(t) = \left| \vec{x}^s(t) - \vec{v}^s(t)\tfrac{\rho}{c} - \vec{x}_r(t) \right|$$

- Let $\vec{\rho}_\circ(t) = \vec{x}^s(t) - \vec{x}_r(t)$

- $\vec{\rho}_\circ$ is instantaneous topocentric range vector from the site to the satellite. Then using $|\vec{A}| = \left[\vec{A}\cdot\vec{A}\right]^{1/2}$

$$
\begin{aligned}
\rho(t) &= \left[\left(\vec{\rho}_\circ(t) - \vec{v}^s(t)\frac{\rho}{c}\right)\cdot\left(\vec{\rho}_\circ(t) - \vec{v}^s(t)\frac{\rho}{c}\right)\right]^{1/2} \\
&= \left[\rho_\circ^2(t) - 2\vec{v}^s(t)\cdot\vec{\rho}_\circ(t)\frac{\rho}{c} + \left(v^s\frac{\rho}{c}\right)^2\right]^{1/2} \\
&\simeq \rho_\circ(t) - \vec{v}^s(t)\cdot\hat{\rho}_\circ(t)\frac{\rho}{c}
\end{aligned}
$$

- We used $\hat{\rho}_\circ = \vec{\rho}_\circ/\rho_\circ$ and $v^s/c \ll 1$ to neglect higher order terms in expansion of $(1+x)^{1/2}$

- Solving for $\rho(t)$ yields

$$\rho(t) = \rho_\circ(t) / \left[1 + \vec{v}^s(t) \cdot \widehat{\rho}_\circ(t)/c\right]$$

- Letting $\vec{\beta}(t) = \vec{v}^s/c$ we can write

$$\rho(t) = \rho_\circ(t) / \left[1 + \vec{\beta}(t) \cdot \widehat{\rho}_\circ(t)\right]$$

- The observable equation for the pseudorange is

$$R(t_j) = \rho_\circ(t) / \left[1 + \vec{\beta}(t_j) \cdot \widehat{\rho}_\circ(t_j)\right] + c\left[\delta_r(t_j) - \delta^s(t_j)\right]$$

- This is still for propagation in vacuum. We haven't yet considered the atmospheric propagation media. (Next time.)

Design Matrix for Pseudorange

- The least-squares solution requires a design matrix with the partial derivatives of the observation with respect to the unknown parameters

- Observation equation with explicit unknowns:

$$R = \frac{\left[(x^s - x_r)^2 + (y^s - y_r)^2 + (y^s - y_r)^2\right]^{1/2}}{1 + \vec{\beta}\cdot\hat{\rho}_\circ} + c\left[\delta_r - \delta^s\right]$$

- I've pretended that $\hat{\rho}_\circ$ doesn't depend on x_r, y_r, and z_r, because it is a weak dependence compared to numerator, so I will approximate $\partial\hat{\rho}_\circ/\partial x_r \simeq 0$ and so on

- Then the partial derivatives with respect to cartesian components of unknown receiver position are

$$\frac{\partial R}{x_r} \simeq -\frac{x^s - x_r}{\rho_\circ} \quad \frac{\partial R}{y_r} \simeq -\frac{y^s - y_r}{\rho_\circ} \quad \frac{\partial R}{z_r} \simeq -\frac{z^s - z_r}{\rho_\circ}$$

- Note that these partial derivatives are typically of order of magnitude 1 and are unit less

- The partial derivative with respect to the unknown receiver clock error is $\partial R/\partial\delta_r \simeq c$

- These partials are of order 3×10^9 if the observation is in meters and the clock error is in seconds

- Therefore it is useful to define clock error in units of length $\delta'_r = c\,\delta_r$ and then $\partial R/\partial\delta'_r \simeq 1$

- This makes the inversion much more stable (could also scale)

Nonlinear Least-squares Solutions

- The design matrix depends on the prior values for the position coordinates

- Example for x:

  $$\left.\frac{\partial R}{\partial x_r}\right|_{\vec{x}_r=\vec{x}_r^a} \simeq -\frac{x^s - x_r^a}{\rho_\circ^a}$$

- Superscript a stands for *a priori*

- In least-squares parlance, this is a *non-linear* problem

Nonlinear Least-squares Solutions

- One common way to solve this is to iterate

    1. Solve for adjustment from *a priori* parameter values, calculate *a posteriori* parameter estimates

    2. Use the *a posteriori* value from the solution as a new *a prior* values, and go back to step 1

    3. Continue until some criterion is met (e.g., RMS fit doesn't change)

- Iteration works if there are no local χ^2 minima near to the starting point, and if the problem is not "too nonlinear"

Least-squares solution with pseudoranges

- In our project, as in many situations, the site position is fixed over 24 hours

- However, the clock error may vary greatly even over 30 sec (the observation interval for the data set)

- With standard least squares, you'll have only one x_r, y_r, and z_r, but number of clock correction parameters is equal to number of observations ($\sim$2880)

- Pseudorange solution is often called *clock solution*

Dilution of Precision (DOP)

- Normalized standard deviation of an estimate

- Geometric DOP GDOP $= [\sigma_n^2 + \sigma_e^2 + \sigma_u^2 + \sigma_\delta^2]^{1/2}/\sigma_{\text{obs}}$

- Vertical DOP VDOP $= \sigma_u/\sigma_{\text{obs}}$

- Horizontal DOP HDOP $= [\sigma_n^2 + \sigma_e^2]^{1/2}/\sigma_{\text{obs}}$

Dilution of Precision (DOP)

- Position DOP $\text{PDOP} = [\sigma_n^2 + \sigma_e^2 + \sigma_u^2]^{1/2}/\sigma_{\text{obs}}$

- Time DOP $\text{TDOP} = \sigma_\delta/\sigma_{\text{obs}}$

- Good achievable DOPS are in the range < 4

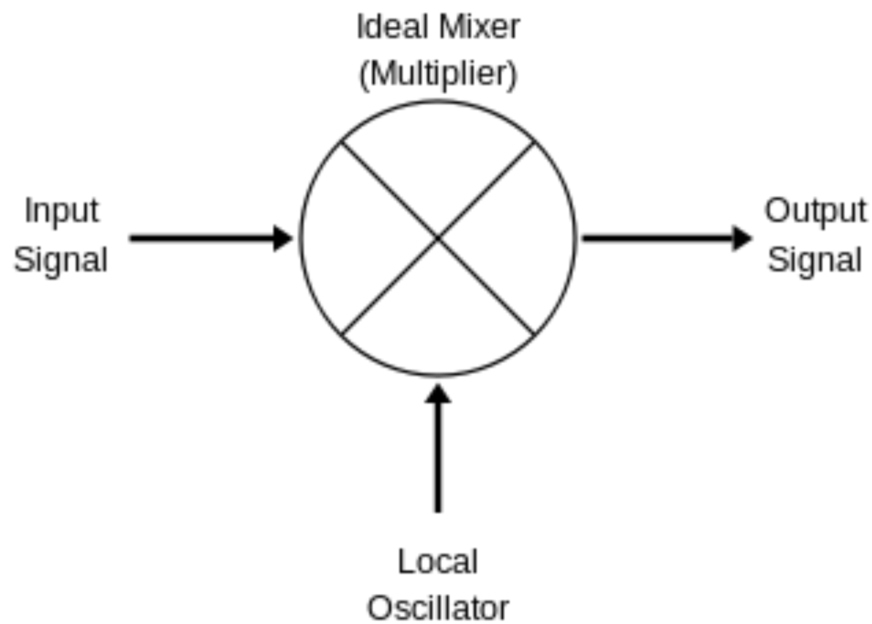
- Note that $\sigma_n^2 + \sigma_e^2 + \sigma_u^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$

Carrier beat phase observable

- Code (pseudrange) solutions are intended for instantaneous positioning with 1–100 m accuracy

- In geodesy, we need a much more accurate observable. Therefore geodetic receivers use the **carrier beat phase**.

- The idea of “beating” comes from the concept of “mixing” (or “heterodyning”) two signals by multiplying them


Ideal Mixer
(Multiplier)
Input
Signal
Output
Signal
Local
Oscillator

Signal mixing

- Mix signals $S_1(t) = A_1 \cos 2\pi f_1 t$ and $S_2(t) = A_2 \cos 2\pi f_2 t$:

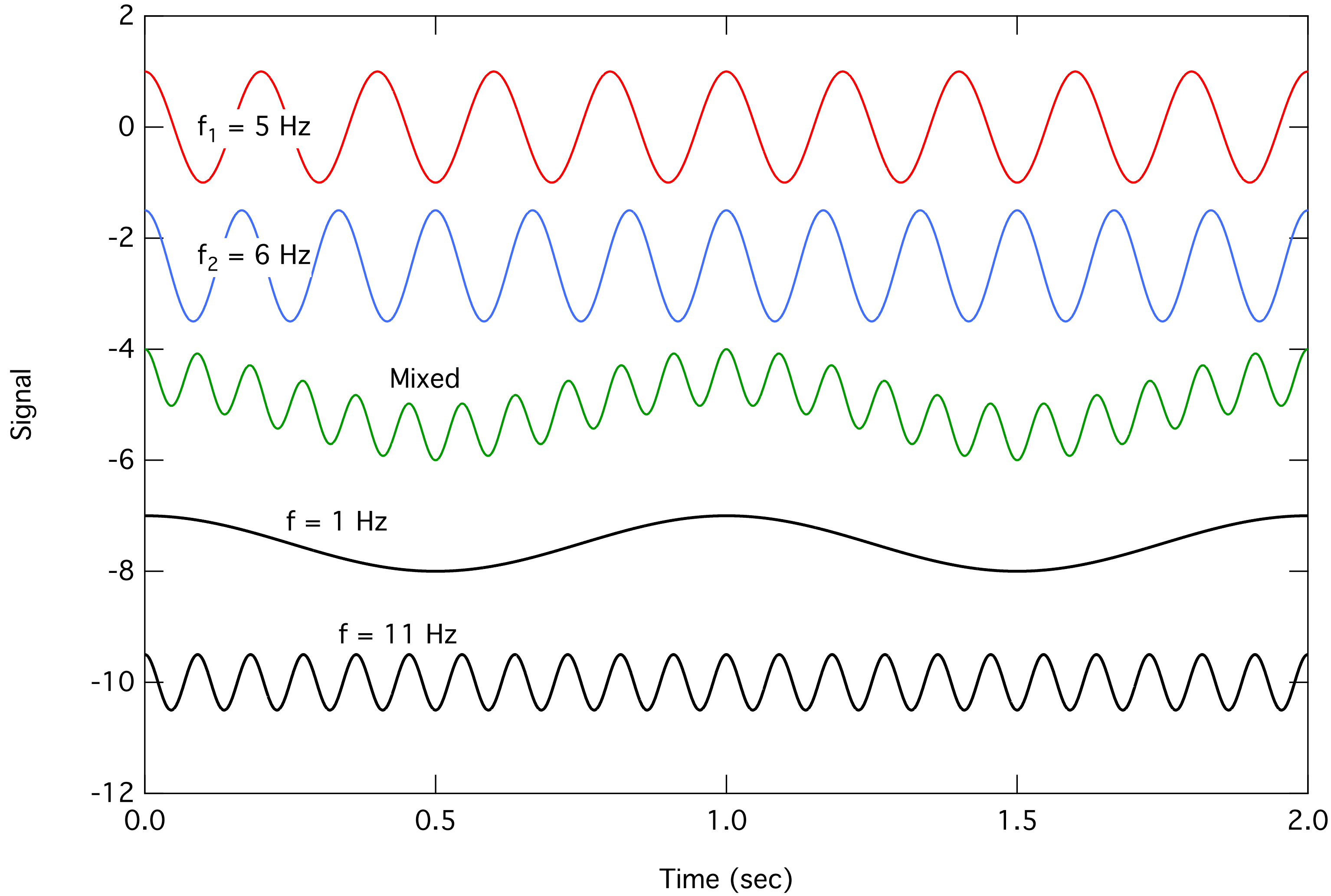
  $M(t) = A_1 A_2 \cos 2\pi f_1 t \;\cos 2\pi f_2 t$

- Trig identity $2 \cos\alpha \cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$

  $M(t) = \frac{A_1 A_2}{2} \cos 2\pi (f_1 + f_2) t + \frac{A_1 A_2}{2} \cos 2\pi (f_1 - f_2) t$

- Mixed signal has two frequency components: $f_1 + f_2$ and $f_1 - f_2$

- Band-pass filtering can be used to select one of the components

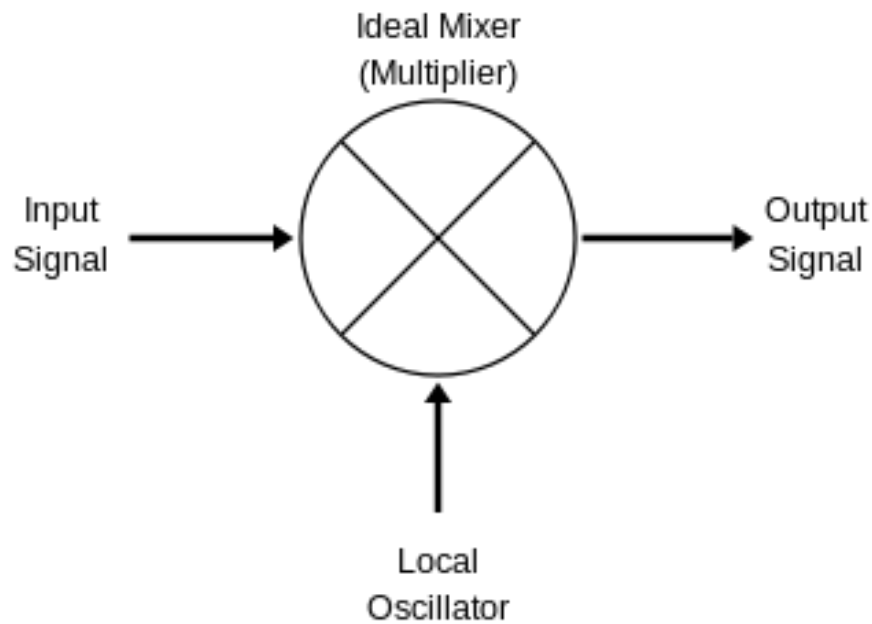

f1 = 5 Hz
f2 = 6 Hz
Mixed
f = 1 Hz
f = 11 Hz
Signal
Time (sec)
2
0
-2
-4
-6
-8
-10
-12
0.0
0.5
1.0
1.5
2.0

Carrier beat phase

- Once we have tracked the codes on L1 and L2, we can reconstruct the carrier signals

- The receiver can then mix the reconstructed carrier signal with a sinusoidal signal generated by the local oscillator

- If we LPF, the carrier beat phase is the difference between the phase of the received GPS signal and the internal signal


Ideal Mixer
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Oscillator

Carrier beat phase

- The carrier beat phase is the difference between the phase of the GPS signal and the phase of the LO

- The phase of the LO (cycles) is

  $\phi_{LO}(t) = f_{LO}(t - t_\circ) + \phi^\circ_{LO}$

- The phase of the GPS signal is the phase of the transmitted signal, delayed by the time it took to propagate from the GPS SV to the antenna:

  $\phi_{rec}(t) = \phi_{SV}(t - \tau) = f_{SV}(t - \tau - t_\circ) + \phi^\circ_{SV}$

Carrier beat phase

- The carrier beat phase (in cycles) is the difference

$$
\begin{aligned}
\Delta\phi &= \phi_{rec}(t) - \phi_{LO}(t) \\
&= f_{SV}(t - \tau - t_\circ) + \phi^\circ_{SV} - f_{LO}(t - t_\circ) - \phi^\circ_{LO}
\end{aligned}
$$

- Initially phase (in radians) is measured modulo 2π (Once lock achieved, phase is tracked continuously)

- Unknown integer number of cycles N ("ambiguity")

- Sign is arbitrary (i.e., could define $\Delta\phi = \phi_{LO} - \phi_{rec}$)

Carrier beat phase

- Ideally, the frequency used in the SV and in the receiver are the GPS L1 or L2 frequencies ad therefore the same as each other

- In practice, there is a small variation in the frequencies used in the satellite and receiver oscillators:

$$f_{SV} = f_\circ + \delta f_{SV}(t) \quad f_{LO} = f_\circ + \delta f_{LO}(t)$$

Carrier beat phase

- So we find that

$$
\begin{aligned}
\Delta\phi \;=\; & -f_{\circ}\tau + \Delta\phi^{\circ} + N \\
& + \delta f_{SV}(t)(t - t_{\circ}) - \delta f_{LO}(t)(t - t_{\circ}) \\
& - \delta f_{SV}(t)\tau
\end{aligned}
$$

- We defined $\Delta\phi^{\circ} = \phi^{\circ}_{SV} - \phi^{\circ}_{LO}$

- The last three terms are clock errors and arise for the same reason that the clock errors arise in the pseudorange, since the onboard frequency standard is the SV's clock

Carrier beat phase

$$
\begin{aligned}
\Delta\phi \;=\; & -f_\circ\tau + \Delta\phi^\circ + N \\
& + \delta f_{SV}(t)(t - t_\circ) - \delta f_{LO}(t)(t - t_\circ) \\
& - \delta f_{SV}(t)\tau
\end{aligned}
$$

- Two terms depend on τ
- For $\delta f_{SV}/f_\circ \simeq 10^{-12}$ and $\tau \simeq 0.09$ sec, last term is less than $\sim 10^{-4}$ cycles or 0.02 mm and will be neglected

Carrier beat phase

- Writing clock terms as for the pseudorange, we have

$$\Delta\phi_r^s(t) = -f_\circ\tau_r^s(t) + \Delta\phi_r^{s,\circ} + N_r^s + f_\circ[\delta^s(t) - \delta_r(t)]$$

1. Added explicit notation for receiver r and satellite s
2. Added explicit time dependence
3. Omitted observational error $\epsilon_r^s(t)$

Carrier beat phase

- If we write phase in units of distance, we have

$$\begin{aligned}\Delta\phi_r^s(t) &= \lambda\Big(-f_\circ\tau_r^s(t) + \Delta\phi_r^{s,\circ} + N_r^s + f_\circ[\delta^s(t) - \delta_r(t)]\Big)\\ &= -c\tau_r^s(t) + c[\delta^s(t) - \delta_r(t)] + \lambda\Delta\phi_r^{s,\circ} + \lambda N_r^s\end{aligned}$$

- Note similarity to pseudorange equation

$R_r^s(t) = c\tau_r^s(t) + c\,[\delta_r(t) - \delta^s(t)]$