EESC 9945

Geodesy with the Global Positioning System

Class 6: *Point Positioning using Pseuduorange*

GPS Positioning Solutions

- **Point Positioning:** Determination of the coordinates of a point with respect to an implied coordinate system

- **Relative Positioning:** Determination of the location of one point with respect to another

- Pseudorange solutions are point positions

- What is the "implied coordinate system?"

Least-square solutions

1. Data y and measurement errors $\epsilon \rightarrow \Lambda = \langle \epsilon\epsilon^T \rangle$
2. Model $y = f(x; t, \ldots) + \epsilon$
3. Parameters x to be estimated
4. *a priori* parameter estimates $x_\circ$
5. Pre-fit residuals $\Delta y = y - y_\circ = y - f(x_\circ)$
6. Partial derivatives $\partial f(x)/\partial x|_{x=x_\circ} \rightarrow A$
7. Solve $\hat{x} = x_\circ + \Delta\hat{x} = x_\circ + (A^T\Lambda^{-1}A)^{-1}A^T\Lambda^{-1}\Delta y$
8. Iterate if non-linear (use $\hat{x}$ as new a priori and go to step 5)
9. **NOTE: Need matrix multiplication/inversion routines**

RINEX data

- RINEX naming convention `ssssdddf.yyt`
- `ssss`: 4-character site ID
- `ddd`: Day of year (001–366) of first record
- `f`: File sequence number (`f` = 0 for daily file)
- `yy`: two-digit year
- `t`: File type (= “o” for observation file)

RINEX observation header

```
     2.11           OBSERVATION DATA    G (GPS)             RINEX VERSION / TYPE
teqc  2008Feb15     UNAVCO Archive Ops  20080410 14:43:33UTCPGM / RUN BY / DATE
Solaris 5.10|UltraSparc IIIi|cc -xarch=v9 SC5.8|=+|*Sparc   COMMENT
BIT 2 OF LLI FLAGS DATA COLLECTED UNDER A/S CONDITION       COMMENT
XXXX                                                        MARKER NAME
                                                            MARKER NUMBER
Davis, James        EESC 9945                               OBSERVER / AGENCY
3841A24273          TRIMBLE 4000SSI     7.19B               REC # / TYPE / VERS
0220136818          TRM29659.00     SCIS                    ANT # / TYPE
        0.0000        0.0000        0.0000                  APPROX POSITION XYZ
        0.0000        0.0000        0.0000                  ANTENNA: DELTA H/E/N
     1     1                                                WAVELENGTH FACT L1/2
     7    L1    L2    C1    P2    P1    S1    S2            # / TYPES OF OBSERV
    30.0000                                                 INTERVAL
RINEX file created by UNAVCO GPS Archive.                   COMMENT
For more information contact archive@unavco.org             COMMENT
                                                            COMMENT
End of DB comments                                          COMMENT
 SNR is mapped to RINEX snr flag value [1-9]                COMMENT
  L1: 3 -> 1; 8 -> 5; 40 -> 9                               COMMENT
  L2: 1 -> 1; 5 -> 5; 60 -> 9                               COMMENT
  2008     4     9     0     0   30.0000000     GPS         TIME OF FIRST OBS
                                                            END OF HEADER
```

Types of observations

```
     7    L1    L2    C1    P2    P1    S1    S2                  # / TYPES OF OBSERV
```

- Lists all observations in file, in order
- L1 and L2: Carrier-beat phase in cycles
- C1: C/A code pseudorange in meters
- P1 and P2: P-code pseudorange in meters
- S1 and S2: SNR
- Most receivers do not have L1 P-code pseudorange, so to calculate ion-free pseudorange we combine C1 and P2

RINEX observation block

- First line of block: Epoch, flag, and satellite list

```
 08  4  9  1 47  0.0000000  0 11G31G 7G 6G25G16G13G 4G 3G20G23G27
```

- For each satellite, five observation types per line in order defined in the “observation type” header line

- Thus, seven observation types require two lines

- Each observation type has three values (F14.3,I1,I1): Observation, LLI flag, and signal strength flag

```
 -12295400.45442  -9731348.60754  25175212.6024   25175215.5784
        4.7504          4.5004

L1-->           L2-->           C1-->           P2-->           P1-->
S1-->           S2-->
```

Ionosphere-free observation and model

- As per last week, we first form the ionosphere-free ionospheric observable from the C1 and P2 pseudoranges for each satellite s at each epoch t_k

$$R^s_{\mathrm{LC}}(t_k) = \left[1 - \left(\frac{f_2}{f_1}\right)^2\right]^{-1} \left[R^s_1(t_k) - \left(\frac{f_2}{f_1}\right)^2 R^s_2(t_k)\right]$$

- The model we had developed for the pseudorange (in meters) that neglects ionosphere is

$$R^s_{\mathrm{LC}}(t_k) = \tilde{\rho}^s_r(t_k) + \tilde{\delta}_r(t_k) - \tilde{\delta}^s(t_k) + \tau^z m(\epsilon^s(t_k))$$

- $\tilde{\rho} = \rho_\circ / \left[1 + \vec{\beta} \cdot \hat{\rho}_\circ\right]$ is the site-satellite distance corrected for the satellite motion

- Topocentric vector $\vec{\rho}_\circ = \vec{x}^s - \vec{x}_r$

Ionosphere-free observations

- $\tilde{\delta}_r(t_j)$ is the unknown receiver clock error in meters $(= c\delta_r)$

- $\tilde{\delta}^s(t_j)$ is the satellite clock error in meters

- τ^z is the zenith (neutral) atmospheric propagation delay

- $m(\epsilon)$ is the mapping function depending on elevation angle ϵ

Pseudorange solution: Parameters

$\vec{x}_r$	Site vector	Unknown	m	3
$\vec{x}_s$	Satellite vector	Calc from nav file	m	
$\vec{\beta}$	Satellite velocity	Calc from nav file	1	
$\tilde{\delta}^s$	Satellite clock error	Calc from nav file	m	
δ_r	Receiver clock error	Unknown	m	N_{epoch}
τ^z	Zenith delay	Known at $\sim$0.1–0.2 m	m	
$m(\epsilon)$	Mapping function	Formula	1	
ϵ	Elevation angle	Calc from $\vec{x}_r$, $\vec{x}^s$	rad	

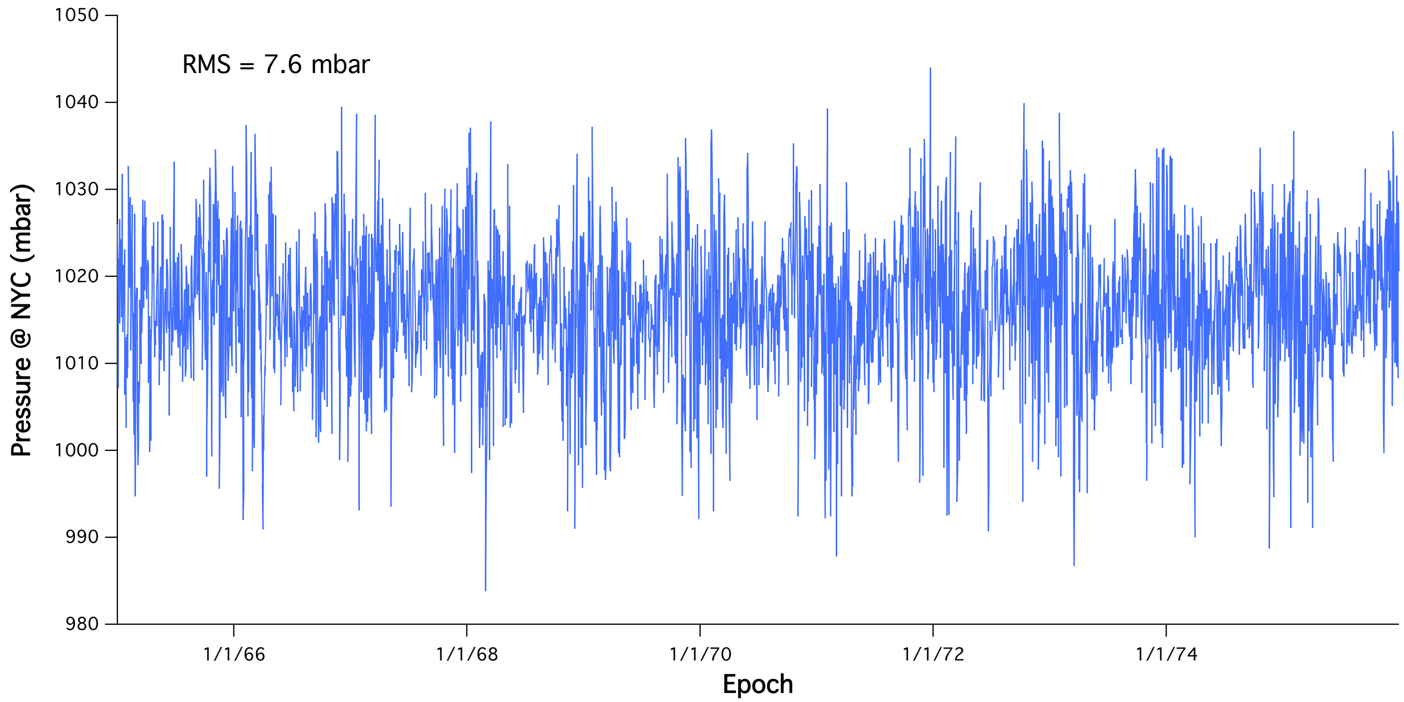
Total number of parameters: $N_{\text{par}} = 3 + N_{\text{epoch}}$

Zenith delay

- The zenith delay is too large to ignore; the data errors (1–10 m) make it hard to estimate accurately

- We know $\tau^z \simeq (2.2 \text{ mm/mbar}) \times P_{\text{surf}} + \tau^z_{\text{wet}}$

Pressure at NYC



7.6 mbar $\rightarrow$ 17 mm hydrostatic delay

Hydrostatic Zenith Delay

- Typical RMS pressure 3–13 mbar, 7–30 mm

- Sometimes seasonal

- **Mean** pressures can vary site-to-site by $>$ 200 mbar (∼0.5 m), depending on site altitude

- Reasonable expression for Earth is $P(H) \simeq P_{\circ} e^{-H/H_{\circ}}$ with $P_{\circ} \simeq$ 1013 mbar and $H_{\circ} \simeq$7.5 km

- Large pressure drops can occur in storms, hurricanes, etc.

Wet Zenith Delay

- Mean varies $\sim$5–30 cm
- RMS $\sim$4–10 cm
- Highly dependent on climate
- Strong seasonal variability
- Scale height of $\sim$2 km

Zenith Delay for Solutions

- For pseudorange solution, we'd like $\sigma_{\tau^z} \lesssim$ 0.1–0.3 m

- Could use model and no estimation

- Simple model: $\tau^z \simeq (2.31\ \text{m})\, e^{-H/H_h} + (0.15\ \text{m})\, e^{-H/H_w}$ with $H_h = 7.5$ km and $H_w = 2$ km

- Probably $\sigma_{\tau^z} \lesssim$ 20 cm; OK for pseudorange solutions

- For phase solutions, we'd like $\sigma_{\tau^z} \lesssim$ 5 mm

Mapping Function

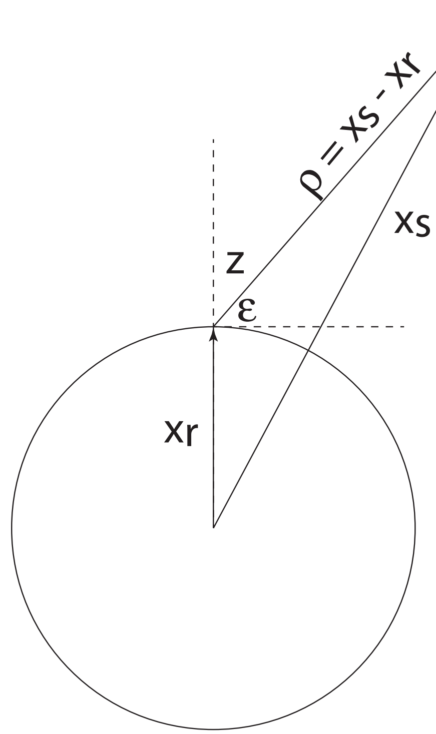
- Want something slightly better than cosecant law

- One of simplest to use is *Davis et al.* [1985]:

$$m(\epsilon) = \cfrac{1}{\sin\epsilon + \cfrac{a}{\tan\epsilon + \cfrac{b}{\sin\epsilon + c}}} \qquad (1)$$

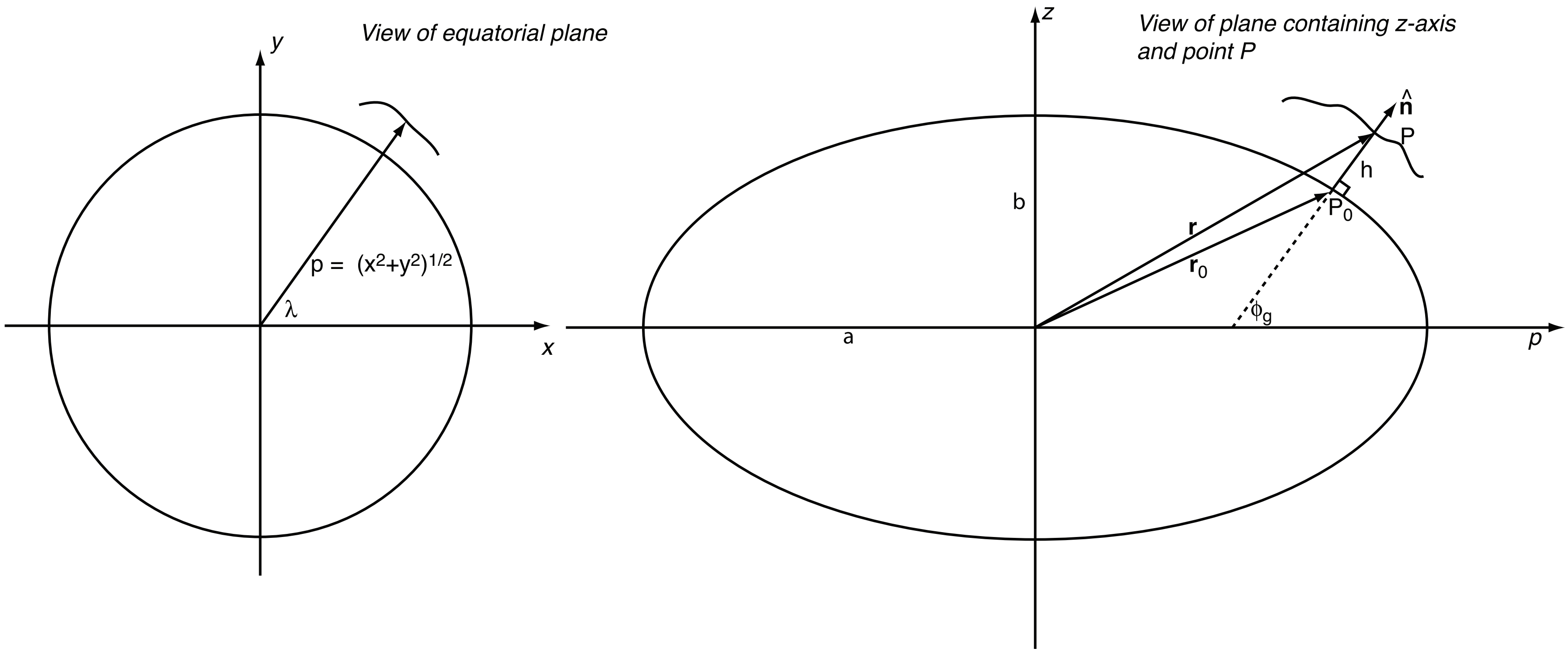
- Nominal constants: $a = 0.001185$; $b = 0.001144$; $c = -0.0090$

Calculation of Elevation Angle

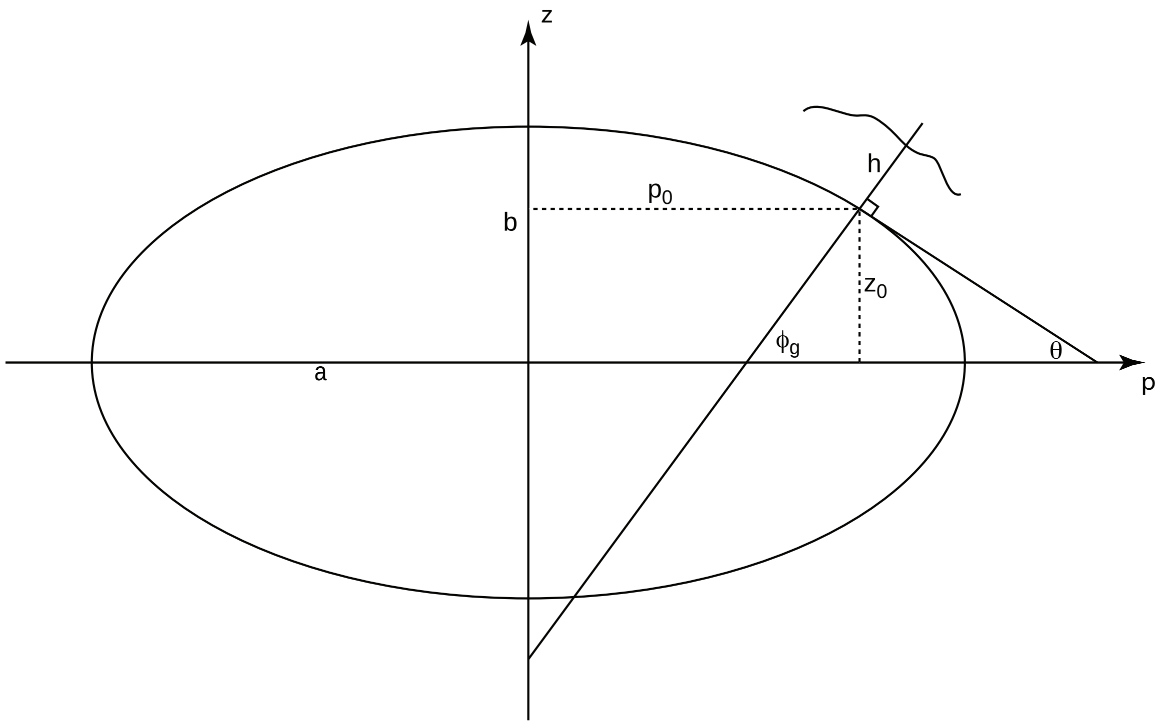


Spherical Earth approximation: $\hat{x}_r \cdot \hat{\rho} = \cos z = \sin \epsilon$

Calculations on a Reference Ellipsoid



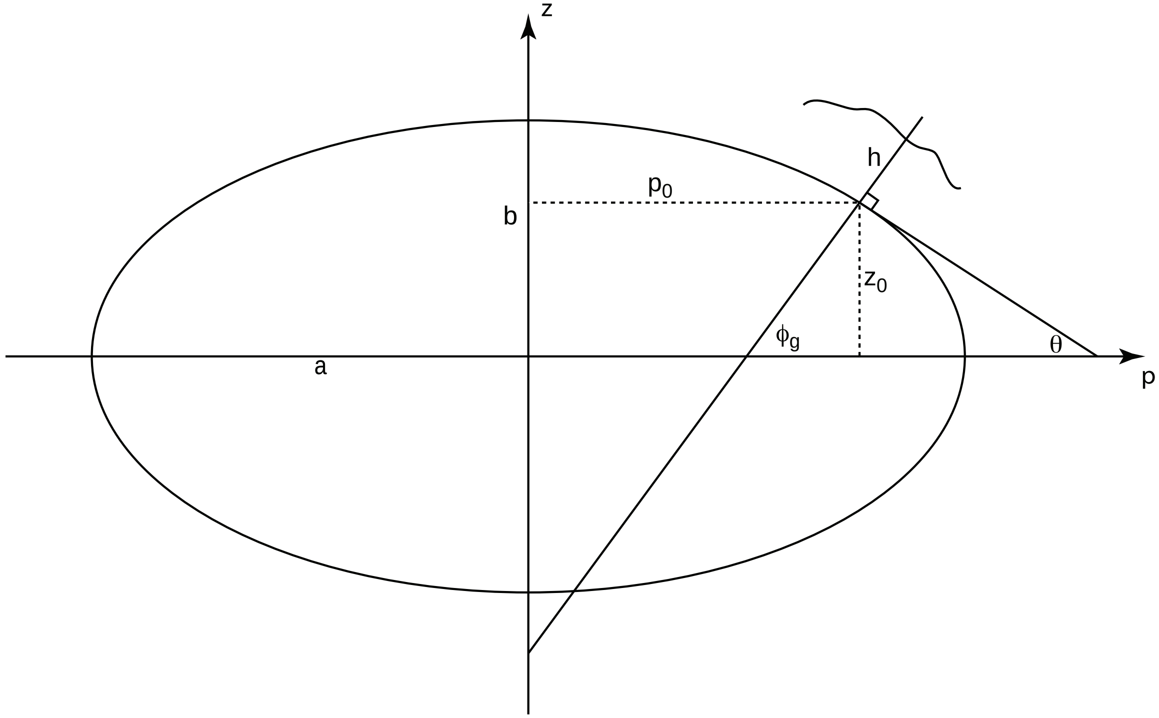
Coordinates are longitude λ, geodetic latitude ϕ_g, and ellipsoidal height h.



The projection along the local normal onto the ellipsoid is $p = p_\circ$ and $z = z_\circ$

$p_\circ$ and $z_\circ$ are related by

$$\frac{p_\circ^2}{a^2} + \frac{z_\circ^2}{b^2} = 1$$

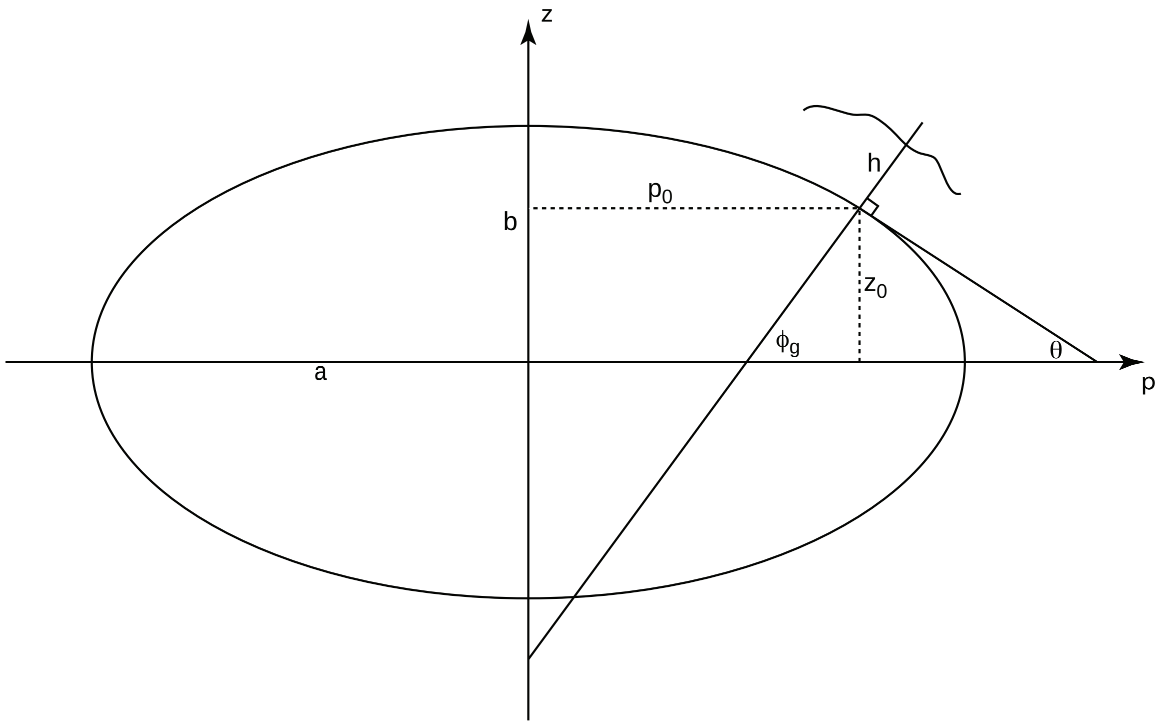


The relation to the geodetic latitude (angle of orthogonal projection of point onto ellipsoid with z-axis) is

$$-\frac{dz_\circ}{dp_\circ} = \tan\theta = \tan\left(\frac{\pi}{2} - \phi_g\right) = \cot\phi_g$$

or

$$\frac{dp_\circ}{dz_\circ} = -\tan\phi_g$$

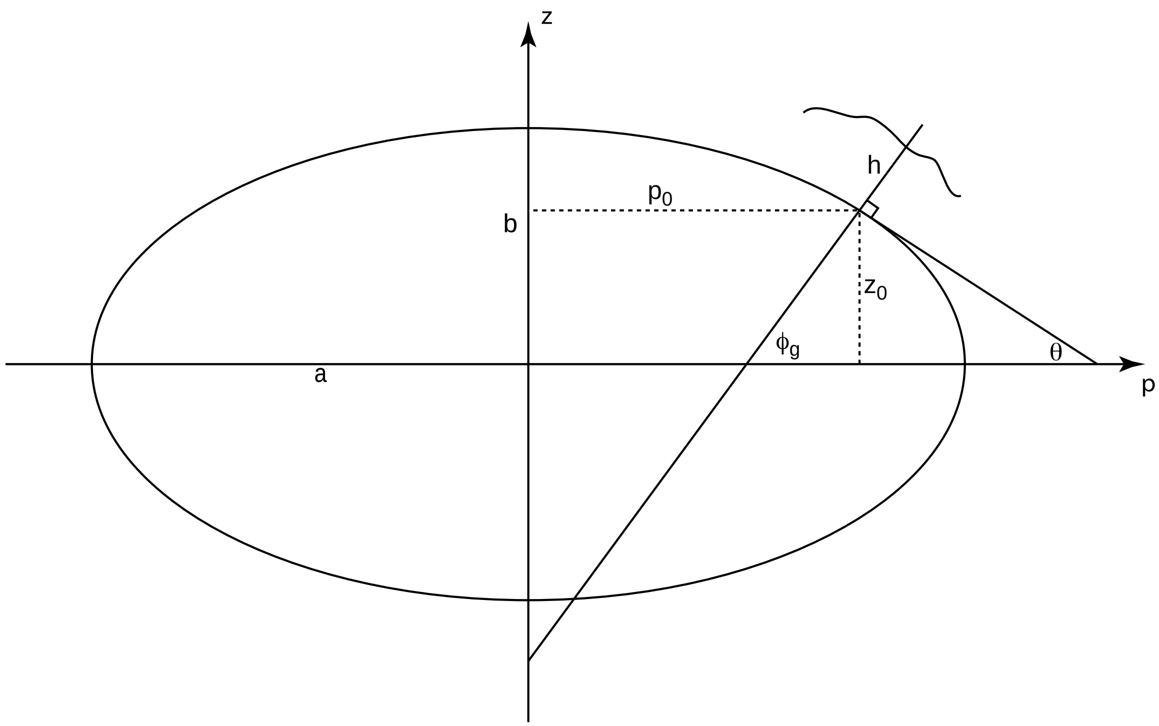


Differentiating eq. of ellipsoid ($\frac{p_\circ^2}{a^2} + \frac{z_\circ^2}{b^2} = 1$) w.r.t. $z_\circ$, we get

$$2\frac{p_\circ}{a^2}(-\tan\phi_g) + 2\frac{z_\circ}{b^2} = 0$$

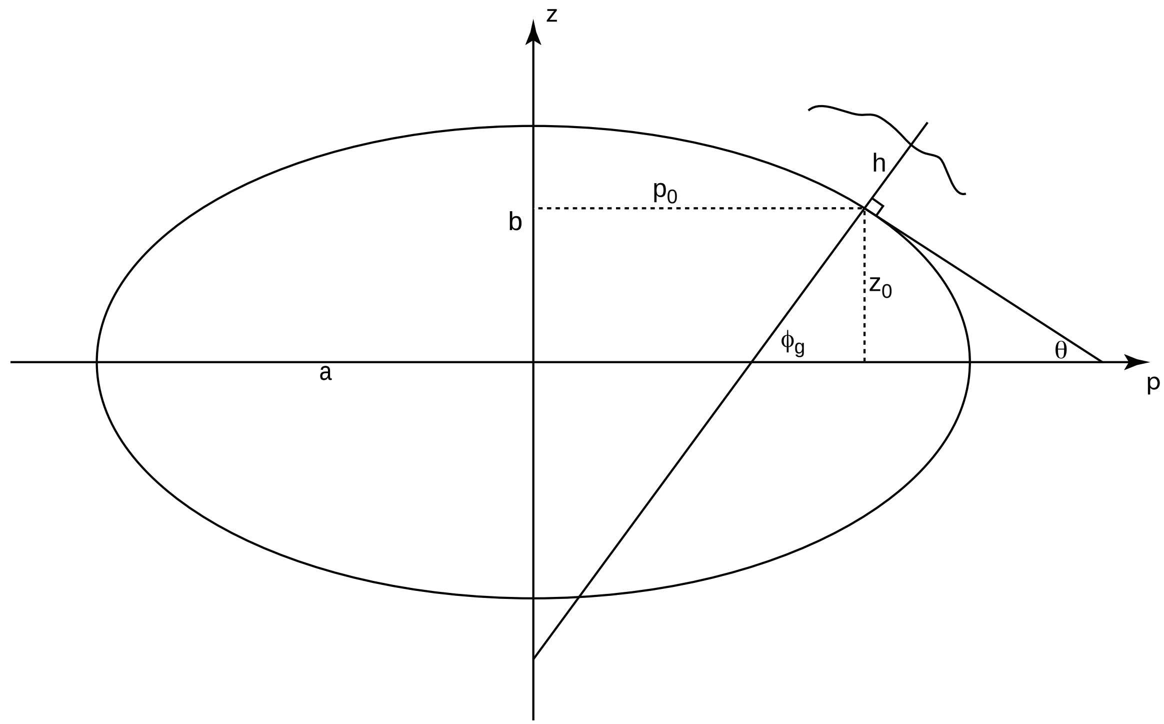
or

$$z_\circ = \left(\frac{b}{a}\right)^2 p_\circ \tan\phi_g$$



Substituting back into eq. and solving for $p_\circ$ yields

$$p_\circ = \frac{a^2 \cos\phi_g}{\sqrt{a^2 \cos^2\phi_g + b^2 \sin^2\phi_g}}$$

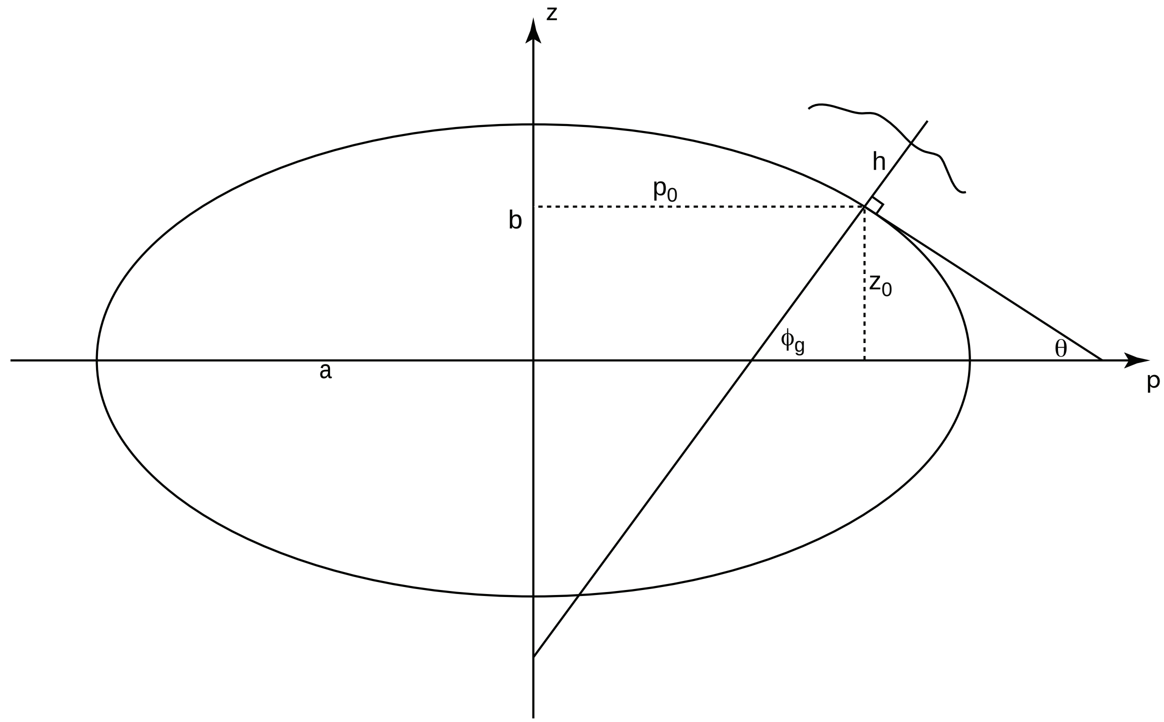


Denoting

$$N_\circ = \frac{a^2}{\sqrt{a^2 \cos^2 \phi_g + b^2 \sin^2 \phi_g}}$$

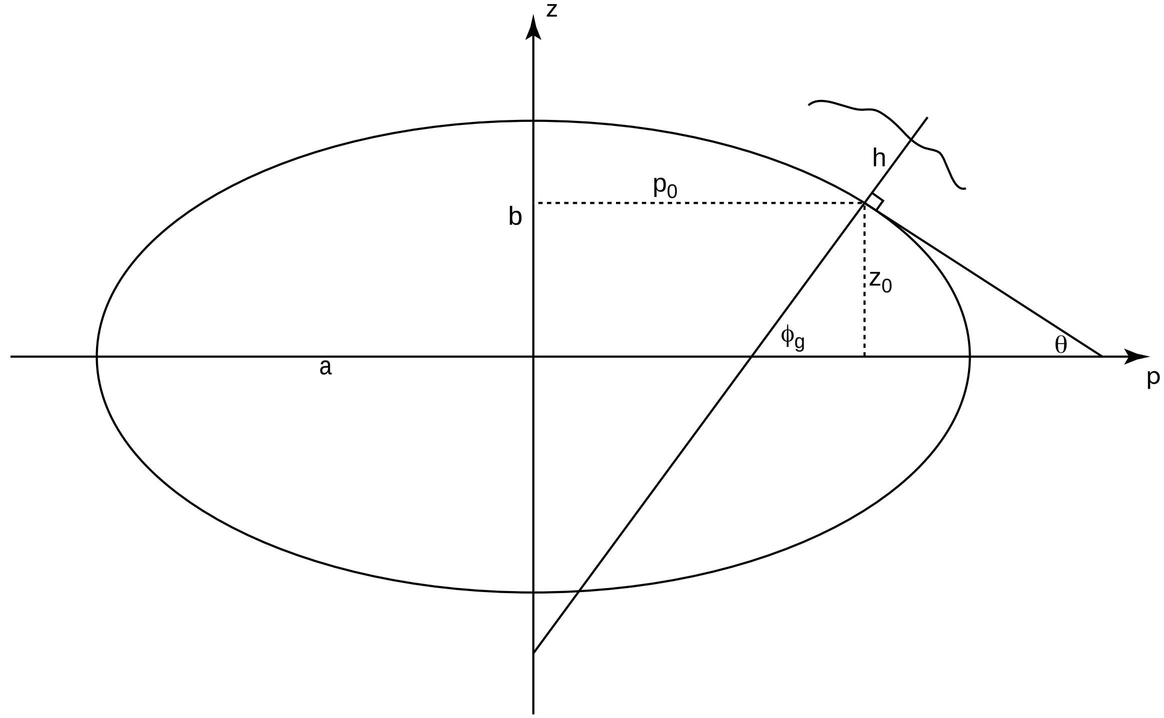
(the prime radius of curvature), then

$$p_\circ = N_\circ \cos \phi_g \qquad \text{and} \qquad z_\circ = \left(\frac{b}{a}\right)^2 N_\circ \sin \phi_g$$



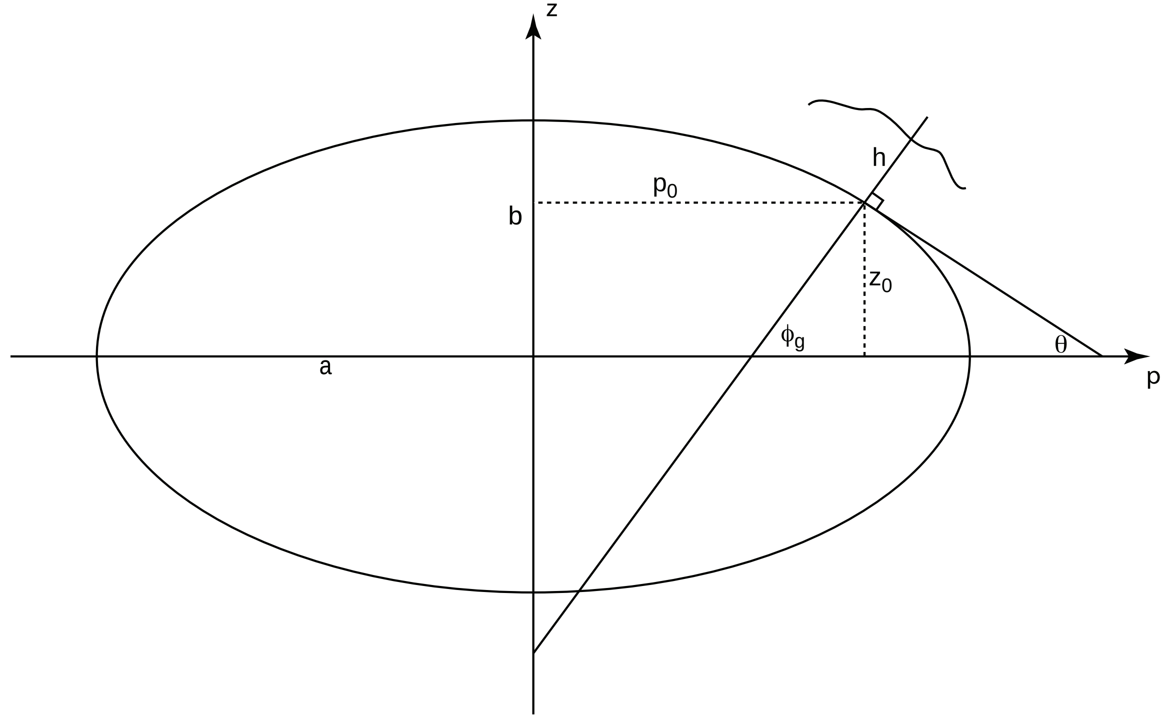
Now we must add the piece between the ellipsoid and the point on the terrain.

The unit normal to the ellipsoid is $\hat{n} = \hat{p}\cos\phi_g + \hat{z}\sin\phi_g$,



The terrain point is thus

$$\vec{x} = p_\circ \hat{p} + z_\circ \hat{z} + h\hat{n} = \hat{p}(N_\circ + h)\cos\phi_g + \hat{z}\left(\frac{b^2}{a^2}N_\circ + h\right)\sin\phi_g$$



So the 3-D position is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (N_\circ + h)\cos\phi_g \cos\lambda \\ (N_\circ + h)\cos\phi_g \sin\lambda \\ \left[\frac{b^2}{a^2} N_\circ + h\right] \sin\phi_g \end{pmatrix}$$

Calculation of Ellipsoidal Coordinates

- Previous gives $(\lambda, \phi_g, h) \rightarrow (x, y, z)$
- No closed-form expression for ϕ_g unless $h = 0$
- To perform the reverse use iterative approach for ϕ_g and h:
  1. $h^{(0)} = 0$; $\tan \phi_g^{(0)} = z/\sqrt{x^2 + y^2}$; calculate $N_\circ^{(0)}$ from $\phi_g^{(0)}$
  2. $\tan \phi_g^{(k+1)} = \left(N_\circ^{(k)} + h^{(k)}\right) z / \left(\frac{b^2}{a^2} N_\circ^{(k)} + h^{(k)}\right) \sqrt{x^2 + y^2}$
  3. $N_\circ^{(k+1)}$ from $\phi_g^{(k+1)}$
  4. $h^{(k+1)} = \sqrt{x^2 + y^2} / \cos \phi_g^{(k+1)} - N_\circ^{(k+1)}$
  5. Go to Step 2 until converged
- Solution for λ is exact

WGS 84 Ellipsoid

- World Geodetic System: Conventional Terrestrial Reference System
- Adopted by GPS
- Definition of coordinate axes:
  - Origin: Earths center of mass
  - Z-Axis: Direction of IERS Reference Pole
  - X-Axis: Intersection of IERS Reference Meridian and plane passing through origin and normal to Z-axis.
  - Y-Axis: Completes right-handed, Earth-centered Earth-fixed orthogonal coordinate system
- Uses $a = 6378137.0$ m
- Flattening $1/f = 298.257223563 = a/(a-b)$
- Therefore $b = 6356752.3142$ m

A priori parameter estimates

- Needed for:
  1. Calculation of $\Delta y = y - y_\circ$
     (a) Topocentric vector
     (b) Elevation angle ϵ for mapping function
     (c) Station height for zenith delay
  2. Calculation of design matrix A (partial derivatives w.r.t. site coordinates)
- Need: A priori site coordinates (x-y-z) and a priori site clock errors (N_{epoch} values)
- What should we use?