EESC 9945

Geodesy with the Global Positioning System

Class 8: *Relative Positioning using Carrier-Beat Phase II*

Rectifying cycle slips

- Last class, we introduced the **Melbourne-Wübbena wide lane**

$$\Delta_{\mathrm{mw}} = \phi_1 - \phi_2 - \frac{1}{c}\frac{(f_1 - f_2)}{(f_1 + f_2)}\left[f_1 R_1 + f_2 R_2\right]$$

- This is used to identify cycle slips

- Why does this work?

Phase observation equations

- Phase observation equations:

$$\phi_k = \frac{\rho}{\lambda_k} + N_k + f_k\delta + \frac{A}{\lambda_k} - \frac{1}{\lambda_k}\left(\frac{\Delta_{\text{ion}}}{f_k^2}\right)$$

- $k = L1, L2$; Phase in cycles; clock δ in units of time; atmospheric delay A and ion delay $\Delta_{\text{ion}}/f_k^2$ in units of length

Phase wide-lane combinations

- Phase observation equations with frequency:

$$\phi_k = \frac{f_k \rho}{c} + N_k + f_k \delta + \frac{f_k A}{c} - \frac{\Delta_{\mathsf{ion}}}{c f_k}$$

- Phase wide-lane combination

$$\Delta\phi_{\mathsf{WL}} = \phi_1 - \phi_2 = \frac{(f_1 - f_2)}{c}\left[\rho + c\delta + A + \frac{\Delta_{\mathsf{ion}}}{f_1 f_2}\right] + N_1 - N_2$$

- Flip in sign of ion: $\frac{1}{f_1} - \frac{1}{f_2} = \frac{f_2 - f_1}{f_1 f_2} = -\frac{f_1 - f_2}{f_1 f_2}$

Pseudorange observation equations

- Pseudorange observation equations:

$$R_k = \rho + c\delta + A + \frac{\Delta_{\text{ion}}}{f_k^2}$$

- Pseudorange in units of length

- Note change in sign of ion delay compared to phase equation

Pseudorange wide-lane combinations

- Pseudorange wide-lane combination

$$f_1 R_1 + f_2 R_2 = (f_1 + f_2)\Big[\rho + c\delta + A + \frac{\Delta_{\text{ion}}}{f_1 f_2}\Big]$$

$$\begin{aligned}
\Delta R_{\text{WL}} &= \frac{1}{c}\frac{(f_1 - f_2)}{(f_1 + f_2)}\left[f_1 R_1 + f_2 R_2\right] \\
&= \frac{(f_1 - f_2)}{c}\Big[\rho + c\delta + A + \frac{\Delta_{\text{ion}}}{f_1 f_2}\Big]
\end{aligned}$$

Melbourne-Wübbena wide-lane combination

- Phase wide-lane combination

$$\Delta\phi_{\mathsf{WL}} = \frac{(f_1 - f_2)}{c}\left[\rho + c\delta + A + \frac{\Delta_{\mathsf{ion}}}{f_1 f_2}\right] + N_1 - N_2$$

- Pseudorange wide-lane combination

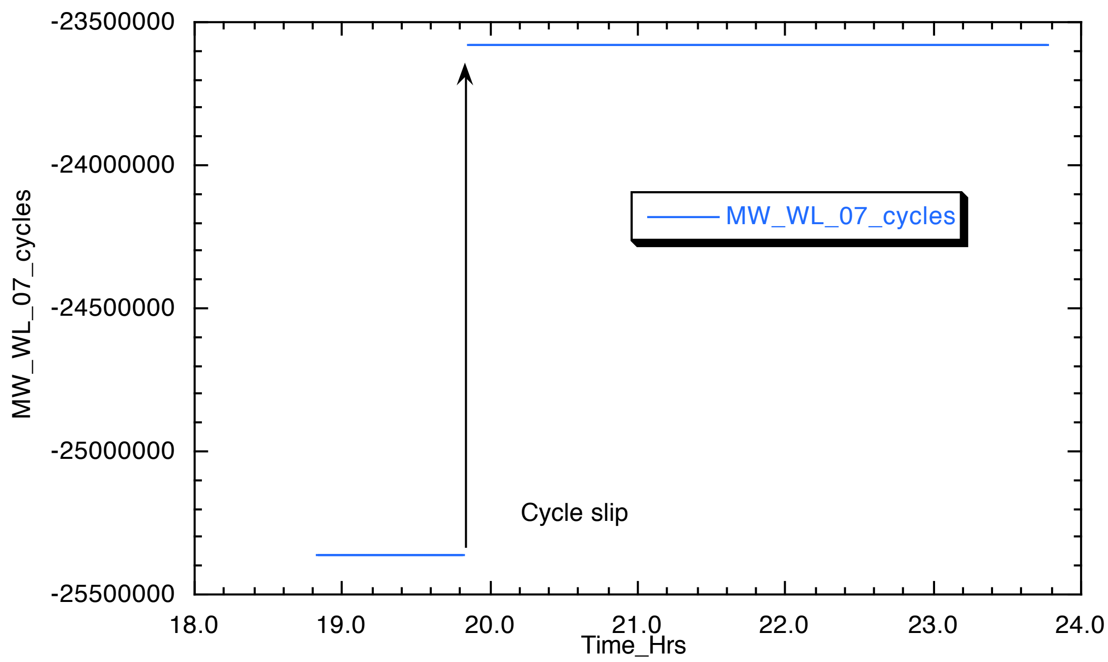
$$\Delta R_{\mathsf{WL}} = \frac{(f_1 - f_2)}{c}\left[\rho + c\delta + A + \frac{\Delta_{\mathsf{ion}}}{f_1 f_2}\right]$$

- Combine phase and pseudorange wide lanes:

$$\Delta_{\mathsf{mw}} = \Delta\phi_{\mathsf{WL}} - \Delta R_{\mathsf{WL}} = N_1 - N_2$$

Melbourne-Wübbena Wide-Lane

Plots of $\Delta_{\mathrm{mw}}(t)$ are plots of $N_1(t) - N_2(t)$:



From Herring, *Principles of the Global Positioning System*

Satellite Orbits

- We developed expressions to relate the GPS broadcast ephemerides to the cartesian positions of the satellites
- For clock solutions, these broadcast ephemerides are probably OK
- But more accurate solutions are available from the International GNSS Service (IGS)
- These more accurate ephemerides come from global GPS solutions in which the orbits have been "improved," i.e., corrections to prior orbits have been estimated in a least-squares sense
- IGS "final" orbits are in "SP3" format: time series of Earth-centered, Earth-fixed cartesian coordinates, every 15 minutes
- Polynomial interpolation is used to obtain the position at any epoch

Satellite Orbits

- We looked at central force problem:

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

- This is force exerted on satellite by point-mass planet (and vice-versa)

- This resulted in Keplerian orbits

- For more accurate orbital modeling, we'll use

$$\vec{a}(\vec{x}) = \nabla V(\vec{x})$$

- $\vec{a}$: Acceleration of satellite at $\vec{x}$

- $V(\vec{x})$: Gravitational potential from "background model" at $\vec{x}$

Gravitational Potential

- Gravitational potential is solution to Laplace's equation

$$\nabla^2 V = 0$$

- Since Earth is nearly spherically symmetric, this is solved in a spherical coordinate system using spherical harmonics

$$V(r,\theta,\phi) = 4\pi G \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (2\ell+1)^{-1} r^{-(\ell+1)} Y_{\ell m}(\theta,\phi) \int_0^a dr' (r')^{\ell+2} \rho_{\ell m}(r')$$

with

$$\rho_{\ell m}(r) = \iint_\Omega d\Omega \; \rho(r,\theta,\phi) Y_{\ell m}(\theta,\phi)$$

where $\rho(r,\theta,\phi)$ is Earth's density and a its radius

Gravitational Potential

- The $\ell = 0$ term is just $V_{00}(r, \theta, \phi) = GMr^{-1}$
- The $\ell = 1$ terms are identically zero if the coordinate origin is the center of mass
- The $\ell = 2$ terms are dominated by Earth's equatorial bulge, which is the $m = 0$ term (i.e., no dependence on ϕ).
- To a good approximation, then, the Earth's gravity field can be written as

$$V = \frac{GM}{r}\left[1 - J_2\left(\frac{a}{r}\right)^2 P_2(\cos\theta)\right]$$

- P_2 is the degree-2 Associated Legendre Polynomial and J_2 is

$$J_2 = \frac{C - A}{Ma^2} \simeq 1.08 \times 10^{-3}$$

  where C and A are Earth's moments of inertia about the spin (z) axis and the equatorial axis (x)

- What about the contribution due to Earth's spin?

Impact of J_2 on GPS orbits

- Analytical expressions for the perturbation relative to $J_2 = 0$ have been determined

- J_2 causes secular drifts in:
  - Right ascension of ascending node Ω, causing a precession of the orbit
  - The mean motion n
  - The argument of perigee ϖ

Orbit perturbations for GPS

- Higher order gravity field (up to degree and order 8 or so)
- Tidal
- Third-body
- Atmospheric drag (negligible)
- Direct (important) and indirect (negligible) solar radiation pressure

Some magnitudes of gravitational acceleration for GPS (m/s^2)

- $\ell = 0$: $9.8 \times (6300/26400)^2 \simeq 0.6$

- $\ell = 2$: $0.6 \times (6300/26400)^2 \times (1.08 \times 10^{-3}) \times 1.5 \simeq 5 \times 10^{-5}$

- Other degrees/orders: $\sim 10^{-7}$

- Earth/ocean tides: $\sim 10^{-9}/\sim 10^{-10}$

- Other bodies: $\sim 10^{-6}$

Numerical orbit integration

- Equation of motion

$$\ddot{\vec{r}} = -\frac{GM}{r^3}\vec{r} + \vec{k}$$

- Two first-order differential equations

$$\dot{\vec{v}} = -\frac{GM}{r^3}\dot{\vec{r}} + \vec{k} \qquad \dot{\vec{r}} = \vec{v}$$

- Calculations must be performed in inertial coordinate system, i.e., not rotating with the Earth

- The accuracy of the orbit integration depends on the accuracy of the numerical approach as well as the accuracy of the perturbation model $\vec{k}$

- Need six initial conditions

Estimation of orbit parameters

- How would a GPS network be used to estimate improved orbits, i.e., better than the broadcast ephemerides?

- (Or how are the broadcast ephemerides themselves determined?)

- We know how, in principle, to estimate parameters from observations using least squares

- What we need (besides observations):

  1. A model for the observations

  2. Partial derivatives of the model with respect to the parameters to be estimated

  3. Prior estimates of the parameters

Estimation of orbit parameters: Approach

- Phase model:

$$\phi = \frac{|\vec{x}^s - \vec{x}_r|}{\lambda} + N_k + f\delta + \frac{A}{\lambda} - \frac{1}{\lambda}\left(\frac{\Delta_{\text{ion}}}{f^2}\right)$$

- We've viewed $\vec{x}^s$ as a "given"

- But we can view the satellite position as a parameterized function, e.g., $\vec{x}^s = f(\vec{x}_\circ, \vec{v}_\circ)$

- The orbit integration approach can also be used to calculate sensitivities of orbital position with respect to parameters such as initial conditions

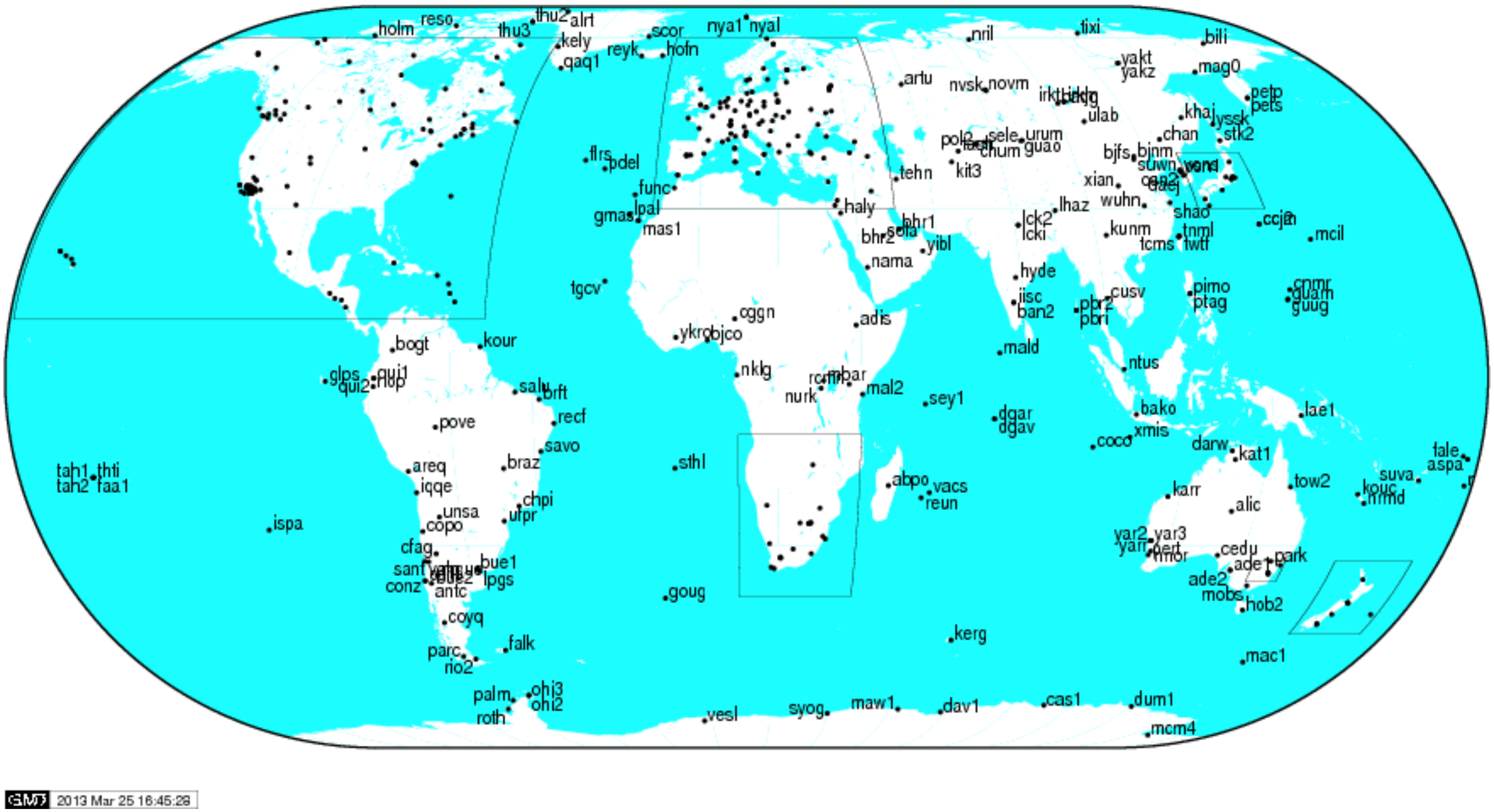
- Then apply chain rule, e.g.,

$$\frac{\partial \phi}{\partial \vec{x}^s} = \frac{\partial \phi}{\partial \rho}\frac{\partial \rho}{\partial \vec{x}^s}$$

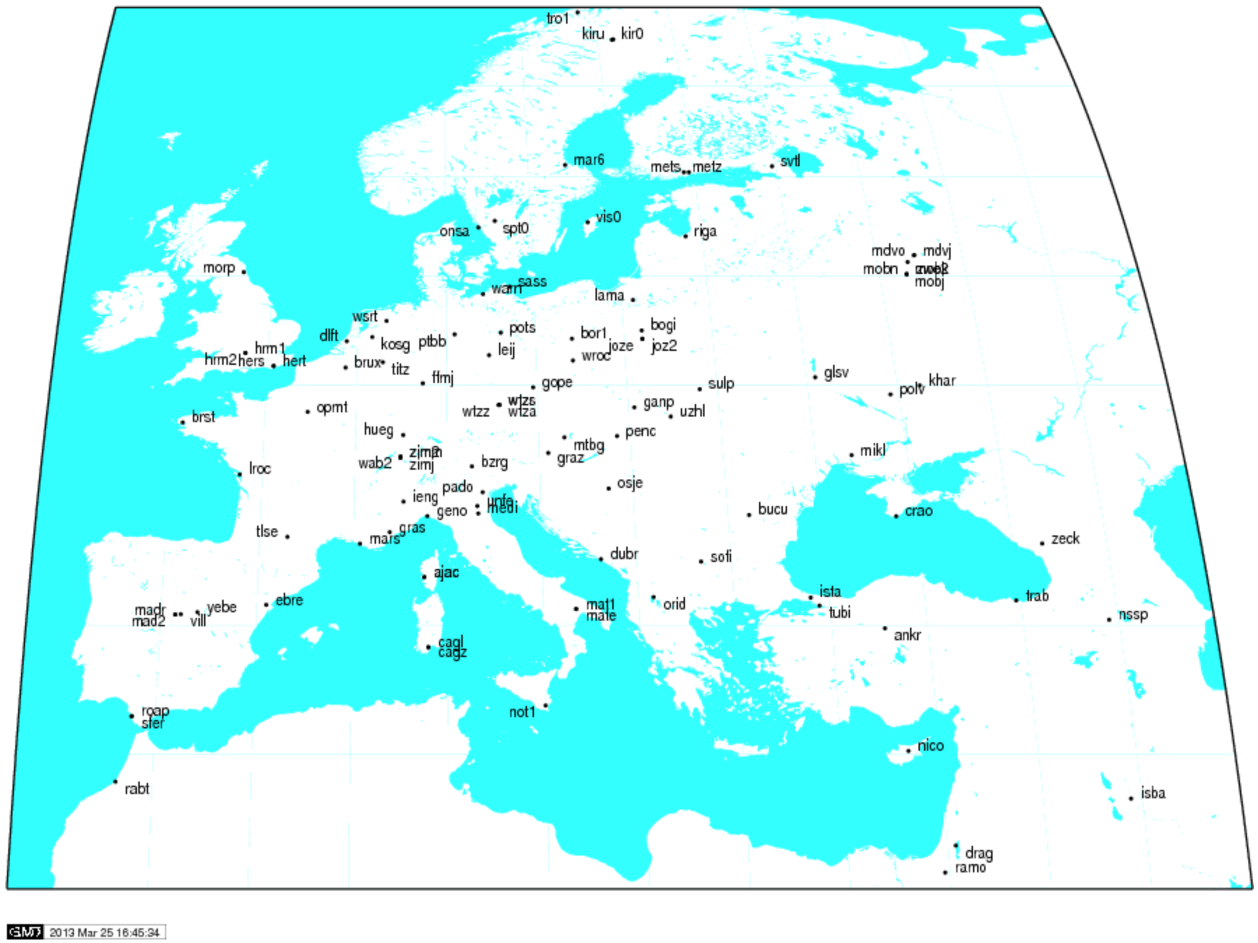
Estimation of orbit parameters: IGS Approach

- Use global network of GPS sites with long time history, good quality control, and good documentation

IGS Tracking Network



IGS Tracking Network



IGS Tracking Network

Estimation of orbit parameters: IGS Approach

- Use global network of GPS sites with long time history, good quality control, and good documentation
- Number of different analysis centers download raw phase and pseudorange data and perform solutions, estimating orbital (and other) parameters
- Different ACs may use different software
- Data sets are organized by UT day
- An analysis center coordinator combines orbit results
- SP3 file contains time series of cartesian coordinates

IGS Orbit Files

```
/* FINAL ORBIT COMBINATION FROM WEIGHTED AVERAGE OF:
/* cod emr esa gfz jpl mit ngs sio
/* REFERENCED TO IGS TIME (IGST) AND TO WEIGHTED MEAN POLE:
/* PCV:IGS05_1421 OL/AL:FES2004  NONE     Y  ORB:CMB CLK:CMB
*  2007  7 30  0  0  0.00000000
PG01  17933.606176  10456.200236 -16398.511349    142.896385 11  7  8 163
PG02 -17671.173068   2787.655640 -19737.224725    125.242146 10 12 12 183
PG03  18903.630496   1936.324295  18469.216582     93.835488 11 10 10 179
PG04 -12618.686115  -9268.588836 -21500.975032      4.406072 10  9  8 182
PG05 -11025.833402  15177.271670 -18993.727115    235.518415 12 10  9 188
PG06   3943.736939  22887.449896 -12664.637763    241.293399  8  7  9 148
PG07   6880.211181  23239.243558 -10456.876665    177.738265  7  7  9 167
PG08  -4620.646504 -17357.389821  19291.717095   -116.858671  8 10  9 153
PG09 -14640.227549  21214.393770   5324.665981     80.908863  8  4  9 156
PG10 -26700.900925    715.428896   1030.746462    105.458130 10 10 12 189
PG11  12096.316700 -22881.463167   5116.113240     21.359245 10 10 10 172
PG12 -17476.694796  11668.484594 -16101.421004    -50.980199 11  9 10 186
PG13  -3540.437830 -23143.673559 -12675.027708    194.056941 13 10 12 181
PG14  15873.890583  20739.994885  -5003.564925     32.638879 10  8 11 179
PG16  26120.976079   2783.894276  -4391.596445    165.997064 11  9 11 204
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Using IGS Orbits

- IGS orbits are generally used instead of broadcast orbits for all geodetic positioning

- If rapid (i.e., near-real-time) solutions are required, other accurate IGS orbit products can be used (as we've seen)

- The orbits are often used assuming the IGS orbits are perfect

- What if the user has baselines $\sim R_e$, so that IGS orbit uncertainties can't be ignored?

Constrained least squares

- What if some subset of parameters in a least squares solution are known at some level?

- That is, what if for some subset of parameters we have prior estimates including a covariance matrix that describes the level of uncertainty associated with the estimates?

Combining parameter estimates

- Suppose we have two independent estimates $\hat{x}_1$ and $\hat{x}_2$ of parameter vector x

- We also have covariance matrices Λ_1 and Λ_2

- Each has multivariate normal pdf

$$P(\hat{x}_k) = \frac{1}{\sqrt{2\pi|\Lambda_k|}} \exp\left[-\frac{1}{2}(\hat{x}_k - x)^T \Lambda_k^{-1} (\hat{x}_k - x)\right]$$

- Independent means the joint pdf is

$$\begin{aligned} P(\hat{x}_1, \hat{x}_2) &= \frac{1}{2\pi\sqrt{|\Lambda_1||\Lambda_2|}} \exp\left[-\frac{1}{2}(\hat{x}_1 - x)^T \Lambda_1^{-1} (\hat{x}_1 - x)\right] \\ &\quad \times \exp\left[-\frac{1}{2}(\hat{x}_2 - x)^T \Lambda_2^{-1} (\hat{x}_2 - x)\right] \end{aligned}$$

Combining parameter estimates

- Given this joint pdf, what is the best (least-squares) estimate of x?

- We can use a maximum likelihood approach. The Likelihood function $L(x) = -\log P$ is

$$L(x) = C + \frac{1}{2}\left[[(\widehat{x}_1 - x)^T \Lambda_1^{-1} (\widehat{x}_1 - x) + (\widehat{x}_2 - x)^T \Lambda_2^{-1} (\widehat{x}_2 - x) \right]$$

- Maximizing with respect to x at the combined estimate $\widehat{x}_c$ yields

$$\left.\frac{\partial L(x)}{\partial x}\right|_{x=\widehat{x}_c} = 0 = \Lambda_1^{-1} (\widehat{x}_1 - \widehat{x}_c) + \Lambda_2^{-1} (\widehat{x}_2 - \widehat{x}_c)$$

where we have used the matrix calculus identity

$$\frac{\partial}{\partial y}\left(y^T A y\right) = 2Ay$$

for A symmetric

Combining parameter estimates

- The solution for the combined estimate $\widehat{x}_c$ of x is

$$\widehat{x}_c = \left(\Lambda_1^{-1} + \Lambda_2^{-1}\right)^{-1} \left(\Lambda_1^{-1}\widehat{x}_1 + \Lambda_2^{-1}\widehat{x}_2\right)$$

- This is the weighted mean of $\widehat{x}_1$ and $\widehat{x}_2$.

- (For the one-dimensional case, $\Lambda_k \to \sigma_k^2$)

- It can easily be shown that $\widehat{x}_c$ has the covariance matrix

$$\Lambda_c = \left(\Lambda_1^{-1} + \Lambda_2^{-1}\right)^{-1}$$

Constrained least squares

- Suppose we have an estimate $\widehat{x}_1$ of x with covariance Λ_1
- Now we acquire observations. How should we handle this?
- Let's start by performing a least-squares solution using the data:

$$\Delta\widehat{x}_2 = \left(A^T \Lambda_y^{-1} A\right)^{-1} A^T \Lambda_y^{-1} \Delta y = \Lambda_2 A^T \Lambda_y^{-1} \Delta y$$

  with $\Delta\widehat{x}_2 = \widehat{x}_2 - x_{\text{prior}}$ ("adjustments")

- Now let $\Delta\widehat{x}_1 = \widehat{x}_1 - x_{\text{prior}}$
- Then the combined estimate $\Delta\widehat{x}_c$ given $\Delta\widehat{x}_1$ and $\Delta\widehat{x}_2$ is just

$$\Delta\widehat{x}_c = \left(\Lambda_1^{-1} + \Lambda_2^{-1}\right)^{-1} \left(\Lambda_1^{-1}\Delta\widehat{x}_1 + \Lambda_2^{-1}\Delta\widehat{x}_2\right)$$

Constrained least squares

- Now let $x_{\text{prior}} = \widehat{x}_1$. Then $\Delta\widehat{x}_1 = 0$ and

$$\Delta\widehat{x}_c = \left(\Lambda_1^{-1} + \Lambda_2^{-1}\right)^{-1} \Lambda_2^{-1} \Delta\widehat{x}_2$$

- Using $\Lambda_2 = \left(A^T \Lambda_y^{-1} A\right)^{-1}$ and $\Delta\widehat{x}_2 = \Lambda_2 A^T \Lambda_y^{-1} \Delta y$ gives

$$\Delta\widehat{x}_c = \left(A^T \Lambda_y^{-1} A + \Lambda_1^{-1}\right)^{-1} A^T \Lambda_y^{-1} \Delta y$$

and

$$\widehat{x}_c = \widehat{x}_1 + \left(A^T \Lambda_y^{-1} A + \Lambda_1^{-1}\right)^{-1} A^T \Lambda_y^{-1} \Delta y$$

Constrained least squares

$$\widehat{x}_c = \widehat{x}_1 + \left(A^T \Lambda_y^{-1} A + \Lambda_1^{-1}\right)^{-1} A^T \Lambda_y^{-1} \Delta y$$

- This solution is known as **constrained least squares** or **least squares with prior constraints**
- It yields the least-squares solution for x using the observations y given the prior information $\widehat{x}_1$ and Λ_1
- Two limits of interest:
  - Prior information dominates: $\widehat{x}_c \to \widehat{x}_1$ if $\Lambda_1 \ll \Lambda_2$
  - New information dominates: $\widehat{x}_c \to \widehat{x}_2$ if $\Lambda_1 \gg \Lambda_2$

Constraining orbital parameters

- In the user's GPS solution, the IGS orbits can be considered prior information if orbital parameters are estimated
- The user can adjust the prior covariance matrix to reflect the type of solution required
- Very "tight" constraints can be used for regional solutions where the orbits can be taken nearly as a given but the user wishes uncertainty in the orbits to be reflected in the site position uncertainties
- "Looser" constraints can be used for continental-to-global scale solutions where the IGS solutions should have great weight (given their extensive global network) but where an orbit adjustment can account for systematic errors