EESC 9945

Geodesy with the Global Positioning System

Class 8: *Terrestrial Reference Frames in GPS*

Many of the concepts and text in this lecture come from *Global Terrestrial Reference Systems and Frames: Application to the International Terrestrial Reference System/Frame* by Zuheir Altamimi, talk given at the Summer school on Space Geodesy and the Earth System, Shanghai, August 2012

Terrestrial Reference System (TRS)

- Ideal, mathematical, theoretical reference system system

- Tridimensional Euclidian affine space of dimension three

- Defines theoretical origin, axes, length scale

- Really an approach to creating a terrestrial reference frame

Terrestrial Reference Frame (TRF)

- Numerical realization of the TRS to which users have access

- Provides set of coordinates of points located on the Earth's surface

- The TRF is a materialization of the TRS inheriting the mathematical properties of the TRS

- As does the TRS, the TRF has an origin, scale, and orientation

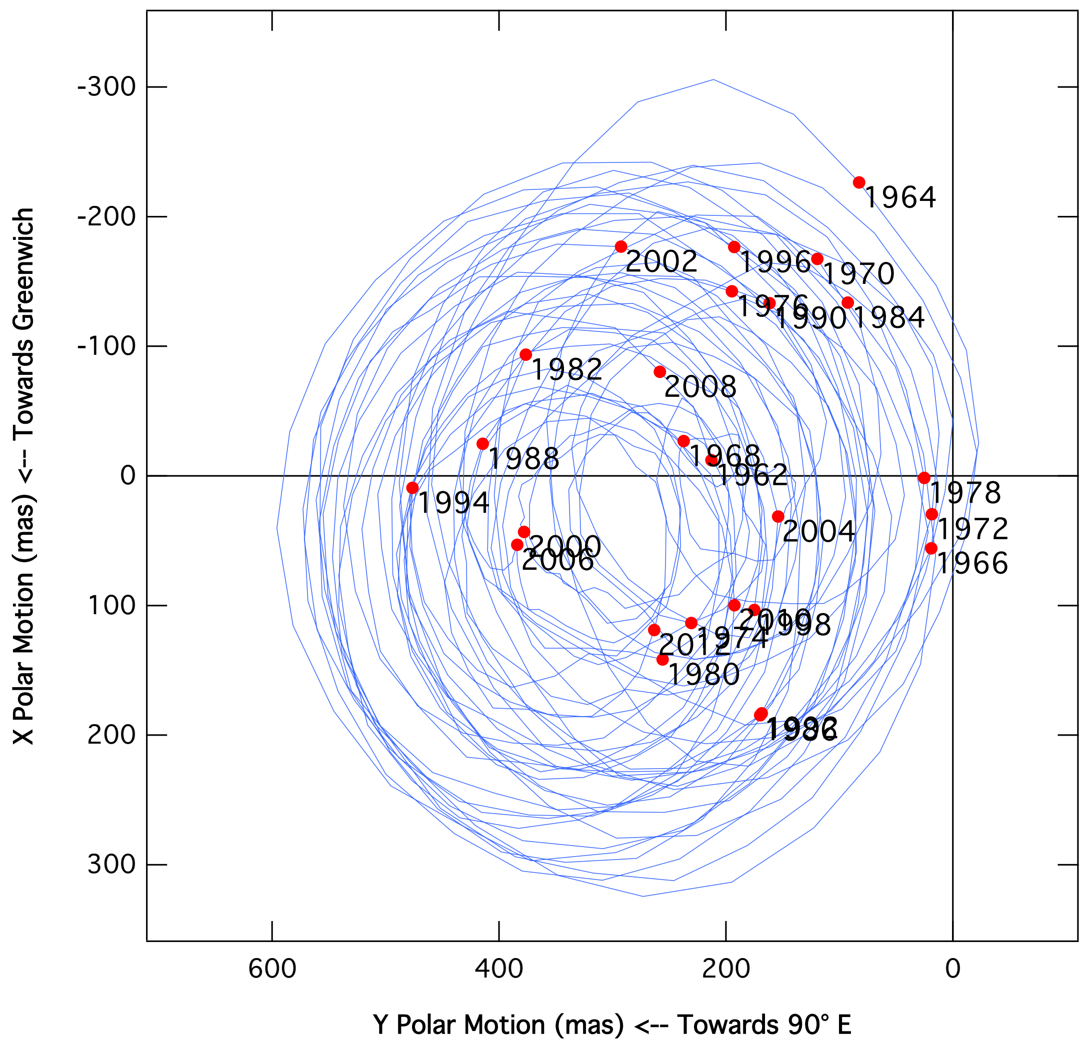
- The TRF is constructed using space geodesy observations

Conventional Terrestrial Reference System

- System of axes with origin at center of mass, right-handed cartesian, with axes coincident with axes of principal ellipsoid of inertia

- z-axis defines north pole

- The IERS monitors the motions of this system with respect to the International Celestial Reference Frame (ICRF)
  - Precession and nutation
  - Polar motion
  - Length of day



100 milliarcsecond $\rightarrow \sim$3 m

The International Terrestrial Reference System (ITRS)

- Origin: The center of mass being defined for the whole earth, including oceans and atmosphere

- Scale: The unit of length is the meter (SI)

- Orientation: Initially given by the BIH orientation at 1984.0

- Time evolution: The time evolution of the orientation is ensured by using a no-net-rotation condition with regard to horizontal tectonic motions over the whole earth

- Oversight: International Earth Rotation Service (IERS), formerly International Polar Motion Service (IPMS) and the earth-rotation section of the Bureau International de l'Heure (BIH)

IUGG(1991) Resolution on the Conventional Terrestrial Reference System

The International Union of Geodesy and Geophysics,

considering

the need to define a Conventional Terrestrial Reference System (CTRS) which would be unambiguous at the millimeter level at the Earth's surface and that this level of accuracy must take account of relativity and of Earth deformation, and

noting

the resolutions on Reference Systems adopted by the XXIst General Assembly of the International Astronomical Union (IAU) at Buenos Aires, 1991,

endorses

the Reference System as defined by IAU at the XXIst General Assembly at Buenos Aires, 1991 and

recommends

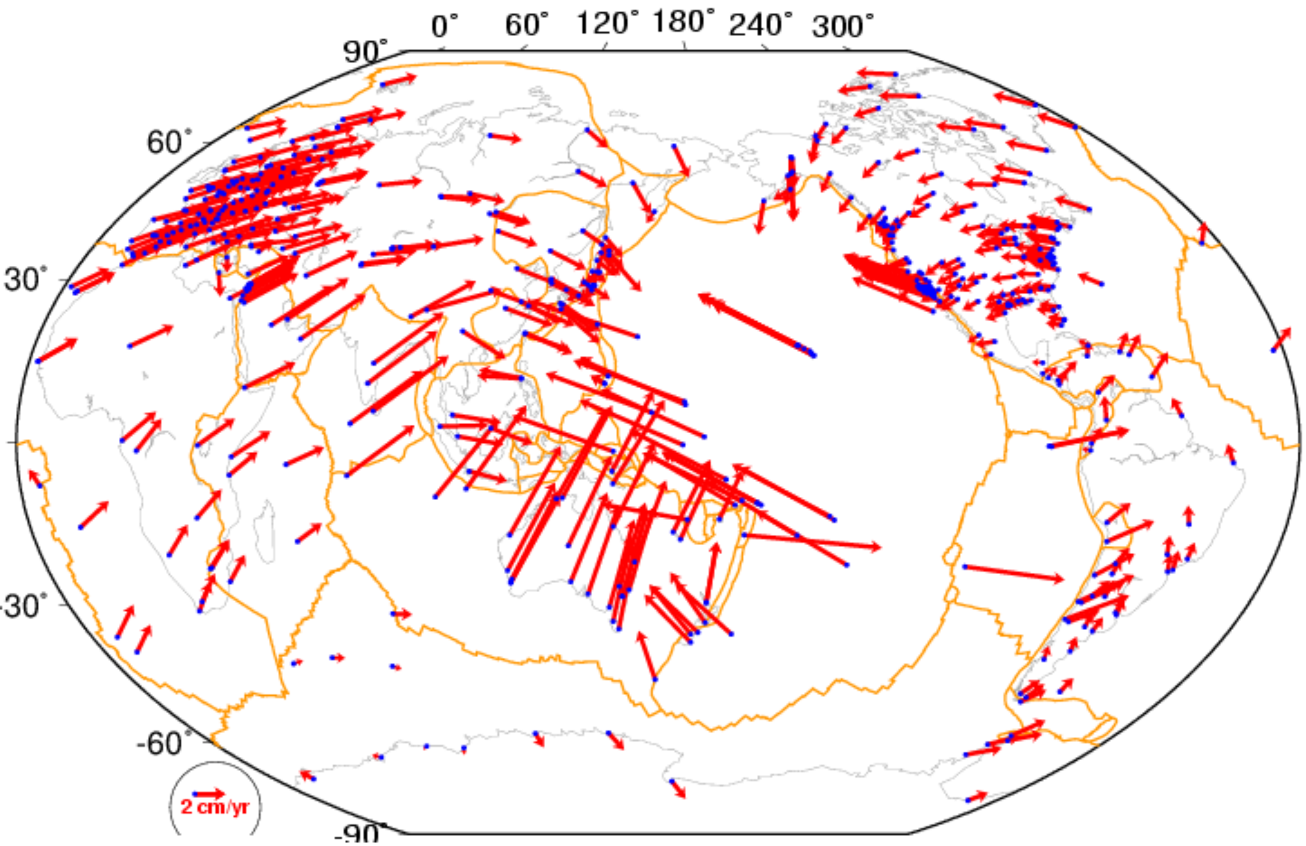
the following definitions of the CTRS:

1. **CTRS to be defined from a geocentric non-rotating system by a spatial rotation leading to a quasi-Cartesian system,**
2. **the geocentric non-rotating system to be identical to the Geocentric Reference System (GRS) as defined in the IAU resolutions,**
3. **the coordinate-time of the CTRS as well as the GRS to be the Geocentric Coordinate Time (TCG),**
4. **the origin of the system to be the geocenter of the Earth's masses including oceans and atmosphere, and,**
5. **the system to have no global residual rotation with respect to horizontal motions at the earth's surface.**

The International Terrestrial Reference Frame (ITRF)

- Realization of ITRF
- Multiple geodetic techniques combined to establish ITRF:
  - GNSS (GPS)
  - Very Long Baseline Interferometry (VLBI)
  - Satellite Laser Ranging (SLR)
  - Doppler Orbitography and Radiopositioning Integrated by Satellite (DORIS)
- Update fairly frequently: ITRF92, ITRF93, ITRF94, ITRF96, ITRF97, ITRF2000, ITRF2005, ITRF2008
- List of site coordinates and velocities

ITRF 2008 Velocity Field ($T > 3$ years)



Zuheir Altamimi

Transformation between reference systems

- A 7-parameter similarity transformation is generally used:

$$\vec{x}_2 = \vec{T} + \lambda \cdot R \cdot \vec{x}_1$$

- $\vec{T}$ is a translation vector (3 components)

- λ is a scale factor (1 scalar value)

- R is a rotation matrix (3 angles)

Rotation matrix for similarity transformation

$$R = R_x R_y R_z$$

with

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \qquad R_y = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformation between reference systems

- We usually use a linearized version of the transformation:

$$\vec{x}_2 = \vec{x}_1 + \vec{T} + s\vec{x}_1 + \tilde{R}\vec{x}_1$$

  where $\lambda = 1 + s$ and $R = I + \tilde{R}$

- Typical is $s, \theta \lesssim 10^{-5}$

- Terms of order $10^{-10} \simeq 0.6$ mm are neglected

- Velocity transformation:

$$\dot{\vec{x}}_2 \simeq \dot{\vec{x}}_1 + \dot{\vec{T}} + \dot{s}\vec{x}_1 + \dot{\tilde{R}}\vec{x}_1$$

- Neglected $s\dot{\vec{x}}_1, \theta\dot{\vec{x}}_1 \lesssim 10^{-3}$ mm/yr

Reference frame for a fixed-orbit solution

- GPS observables all have

$$\rho = |\vec{x}^s - \vec{x}^r|$$

- As we have discussed, one mode of processing GPS data is to assume the orbits, taken from a particular source, are "perfect" and hence held fixed (not estimated) in a GPS solution

- Suppose $\{\vec{x}^s_\circ\}$ are a set of satellite positions we use in a GPS least-squares solution, which results in site position estimates $\{\vec{x}^r_\circ\}$

Reference frame for a fixed-orbit solution

- Now suppose that we are given a new set of satellite positions $\{\vec{x}_1^s\}$ that differs from $\{\vec{x}_\circ^s\}$ by a constant offset $\Delta\vec{x}$

- Since the GPS observables depend only on $\vec{x}^s - \vec{x}^r$, a minimum χ^2 solution using $\{\vec{x}_1^s\}$ is found for site positions $\{\vec{x}_1^r\}$, where each

$$\vec{x}_1^r = \vec{x}_\circ^r + \Delta\vec{x}$$

- Thus, a translation in the satellite frame causes a translation of the terrestrial frame

Reference frame for a fixed-orbit solution

- Suppose we have $\{\vec{x}^s_2\}$ that differs from $\{\vec{x}^s_\circ\}$ by a rotation so that $\vec{x}^s_2 = R\vec{x}^s_\circ$

- The site positions also rotate so that $\vec{x}^r_2 = R\vec{x}^r_\circ$:

$$\begin{aligned} \rho_2 &= |\vec{x}^s_2 - \vec{x}^r_2| = |R\vec{x}^s_\circ - R\vec{x}^r_\circ| \\ &= |R\,(\vec{x}^s_\circ - \vec{x}^r_\circ)| = \{[R\,(\vec{x}^s_\circ - \vec{x}^r_\circ)] \cdot [R\,(\vec{x}^s_\circ - \vec{x}^r_\circ)]\}^{1/2} \end{aligned}$$

- **Proof**: The rotation R is a second-order tensor with $R^T R = RR^T = I$. Using the tensor identity $(A\vec{v}) \cdot \vec{u} = \vec{v} \cdot (A^T\vec{u})$ we have

$$\rho_2 = \left[(\vec{x}^s_\circ - \vec{x}^r_\circ)\,R^T R\,(\vec{x}^s_\circ - \vec{x}^r_\circ)\right]^{1/2} = [(\vec{x}^s_\circ - \vec{x}^r_\circ) \cdot (\vec{x}^s_\circ - \vec{x}^r_\circ)]^{1/2} = \rho$$

- A rotation of the satellite reference frame causes a rotation of the terrestrial reference frame

Reference frame when orbits are estimated

- Suppose we are estimating site positions *and* orbital parameters (as discussed last time

- Can we do this using standard least squares?

- Previous slides show that $\exists$ a serious **rank deficiency** in the normal equations if we attempt this

- Normal equations (parameter vector x):

$$Nx = B$$

with $N = A^T \Lambda_y^{-1} A$ (A is design matrix)

Rank deficiencies in least squares

- The **rank** of N is a measure of the linear independence of the system of linear equations represented by $y = Ax$

- Trivial (but relevant) example: $y_k = a - b$, where a and b are unknown parameters to be estimated, and $k = 1, \ldots, n$

- The design matrix is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{pmatrix}$$

Rank deficiencies in least squares

- The normal equations matrix (using $\Lambda_y = \sigma^2 I$) is

$$N = A^T \Lambda_y^{-1} A = \sigma^{-2} \begin{pmatrix} n & -n \\ -n & n \end{pmatrix}$$

- N is rank deficient since the second row is -1 times the first row

- The determinant of N is zero, so the matrix is singular and we cannot invert this matrix to solve for the parameters

Overcoming rank deficiencies

- Suppose instead we had the matrix
$$N = A^T \Lambda_y^{-1} A + \frac{1}{\kappa^2} I = n\sigma^{-2} \begin{pmatrix} 1+\beta^2 & -1 \\ -1 & 1+\beta^2 \end{pmatrix}$$
with $\beta^2 = \sigma^2/n\kappa^2$ and κ^2 large such that $\beta^2 \ll 1$

- N is not singular, but nearly so. Its determinant is
$$n^2\sigma^{-4}\left[\left(1+\beta^2\right)^2 - 1\right] = n\kappa^{-2}\sigma^{-2}\left(1+2\beta^2\right)$$

- The inverse of N is
$$N^{-1} = \frac{\kappa^2}{1+2\beta^2} \begin{pmatrix} 1+\beta^2 & 1 \\ 1 & 1+\beta^2 \end{pmatrix}$$

Overcoming rank deficiencies

- The variances of the parameters are

$$\sigma_a^2 = \sigma_b^2 = \kappa^2 \frac{1+\beta^2}{1+2\beta^2} \simeq \kappa^2 \left(1-\beta^2\right)$$

- The correlation between the errors of the parameter estimates $\widehat{a}$ and $\widehat{b}$ is

$$r_{ab} = \sigma_b^2 = \frac{1}{1+2\beta^2} \simeq 1-2\beta^2$$

- The uncertainties are large but the correlation is nearly 1

Overcoming rank deficiencies

- We started by adding something to the original normal equations matrix and showed that this removed the rank deficiency

$$N = A^T \Lambda_y^{-1} A + \frac{1}{\kappa^2} I = n\sigma^{-2} \begin{pmatrix} 1+\beta^2 & -1 \\ -1 & 1+\beta^2 \end{pmatrix}$$

- What would justify us doing this?

- Compare to the least-squares solution with prior constraints:

$$\Delta\widehat{x} = \left(A^T \Lambda_y^{-1} A + \Lambda_x^{-1}\right)^{-1} A^T \Lambda_y^{-1} \Delta y$$

Overcoming rank deficiencies

- If $\Lambda_x = \kappa^2 I$ then our procedure to remove the rank deficiency is equivalent to having a prior constraint on the adjustment to the parameters a and b with standard deviation κ

- We said κ was large (wrt to information from the unconstrained solution) so that this is a "loose" constraint

Estimating orbits and site positions

- If we have fairly good estimate of the prior orbital parameters and site positions that we can loosely constrain, we can estimate orbits and site positions simultaneously

- Typical loose constraints might be $\kappa \simeq 1$ m for sites and equivalent for orbits

- Such a solution will yield parameter estimates with standard deviations just slightly under 1 m but with correlations near 1

- This is not useful, in itself

A posteriori conditions

- Suppose we have performed a loosely constrained least-squares solution, resulting in parameters estimates $\widehat{x}$, Λ_x

- Suppose also that we wish to impose a condition on the parameters

- For example, suppose we are looking for a solution that minimizes the adjustments to some subset of the parameters x_1, where

$$x = \left[\begin{array}{c} x_1 \\ \hline x_2 \end{array} \right]$$

A posteriori conditions

- By minimizing the adjustments, we are saying that we wish $\Delta x_s = 0$, more or less

- This is not an observation equation, but we can make it look like an observation equation if we let

$$\Delta y_s = A_s \Delta x = \left[\begin{array}{c|c} I & 0 \end{array}\right] \Delta x$$

  with the dimension of I being the same as x_1

- This creates a “pseudo-observation” if the “observed” value for Δy_s is zero and we assign a covariance matrix Λ_s that expresses the “more or less” above

A posteriori conditions

- Now we can treat the solution as a prior constraint and the condition as a new observation

- The solution is

$$\Delta\widehat{x}_c = \Lambda_c \Lambda_x^{-1} \Delta\widehat{x}$$

  where

$$\Lambda_c^{-1} = \left(\begin{array}{c|c} \Lambda_s^{-1} & 0 \\ \hline 0 & 0 \end{array}\right) + \Lambda_x^{-1}$$

Stacking of TRF time series

1. Perform a series of loosely constrained daily solutions

2. These have large standard deviations, but the covariance matrix is a mathematical expression that these coordinates can be rotated or translated while maintaining relative positions

3. The basic combination model that is imposed is:

   $$x_s^i(t_s) = x_c^i(t_\circ) + v_c^i \times (t_s - t_\circ) + T_s + D_s x_c^i(t_s) + R_s x_c^i(t_s)$$

   Least-squares is used to estimate the station positions at $t_\circ$ and velocities

4. Other nonlinear motions must be accounted for

5. A posteriori conditions are imposed (e.g., no net rotation)

Combining TRFs from different techniques

The ITRF solutions use data products from SLR, VLBI, DORIS, and GPS




Combining TRFs from different techniques

- Inherent sensitivities are different
- For example, SLR is very sensitive to the gravitational field and hence to the geocenter
- VLBI establishes a connection to the celestial reference frame
- Combination has the advantage of being less sensitive to particular sources of systematic error
- Scale is a particularly sensitive parameter

Scale parameter wrt ITRF2008 (ppb)