EESC 9945

# Geodesy with the Global Positioning System

Classes 11: *Stochastic Filters II*

# Review: Kalman Filter equations

Prediction

$$\widehat{x}_{k|k-1} = S_k \, \widehat{x}_{k-1|k-1}$$

$$\Lambda_{k|k-1} = S_k \, \Lambda_{k-1|k-1} \, S_k^T + R_k \, Q_k \, R_k^T$$

Gain

$$K_k = \Lambda_{k|k-1} A_k^T \left( A_k \Lambda_{k|k-1} A_k^T + G_k \right)^{-1}$$

Update

$$\widehat{x}_{k|k} = \widehat{x}_{k|k-1} + K_k \Delta y_k$$

$$\Lambda_{k|k} = (I - K_k A_k) \, \Lambda_{k|k-1}$$

## Example: Random walk

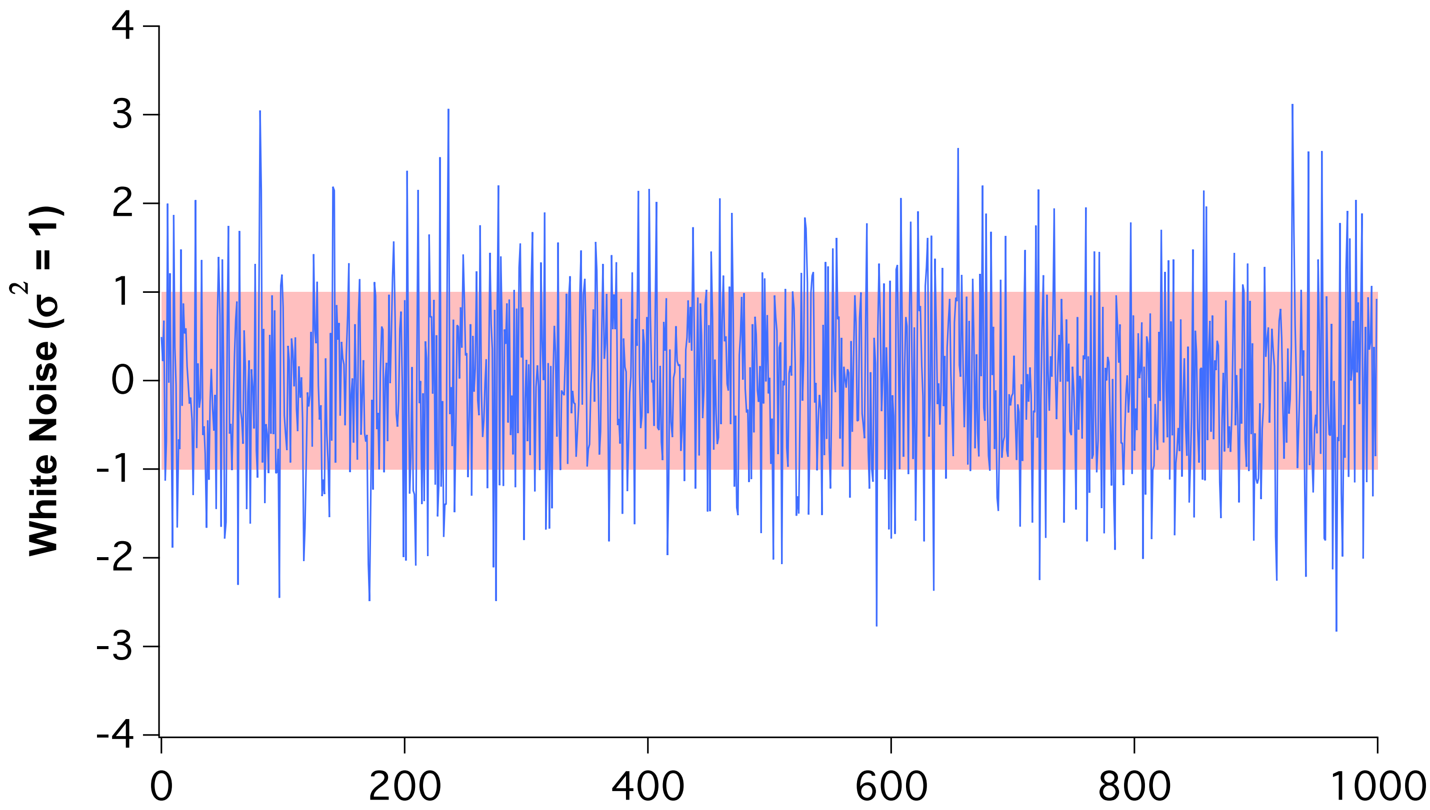
- Let us take the simple 1-D example
$$z_{k+1} = z_k + \xi_k$$
$\xi_k$ a zero-mean white-noise process with variance $\sigma_\xi^2$

- $z_k$ is a **random-walk** process

- Let us also assume an observation
$$y_k = z_k + \epsilon_k$$
with $\epsilon_k$ our observation error: $\langle \epsilon_k^2 \rangle = \sigma_y^2$

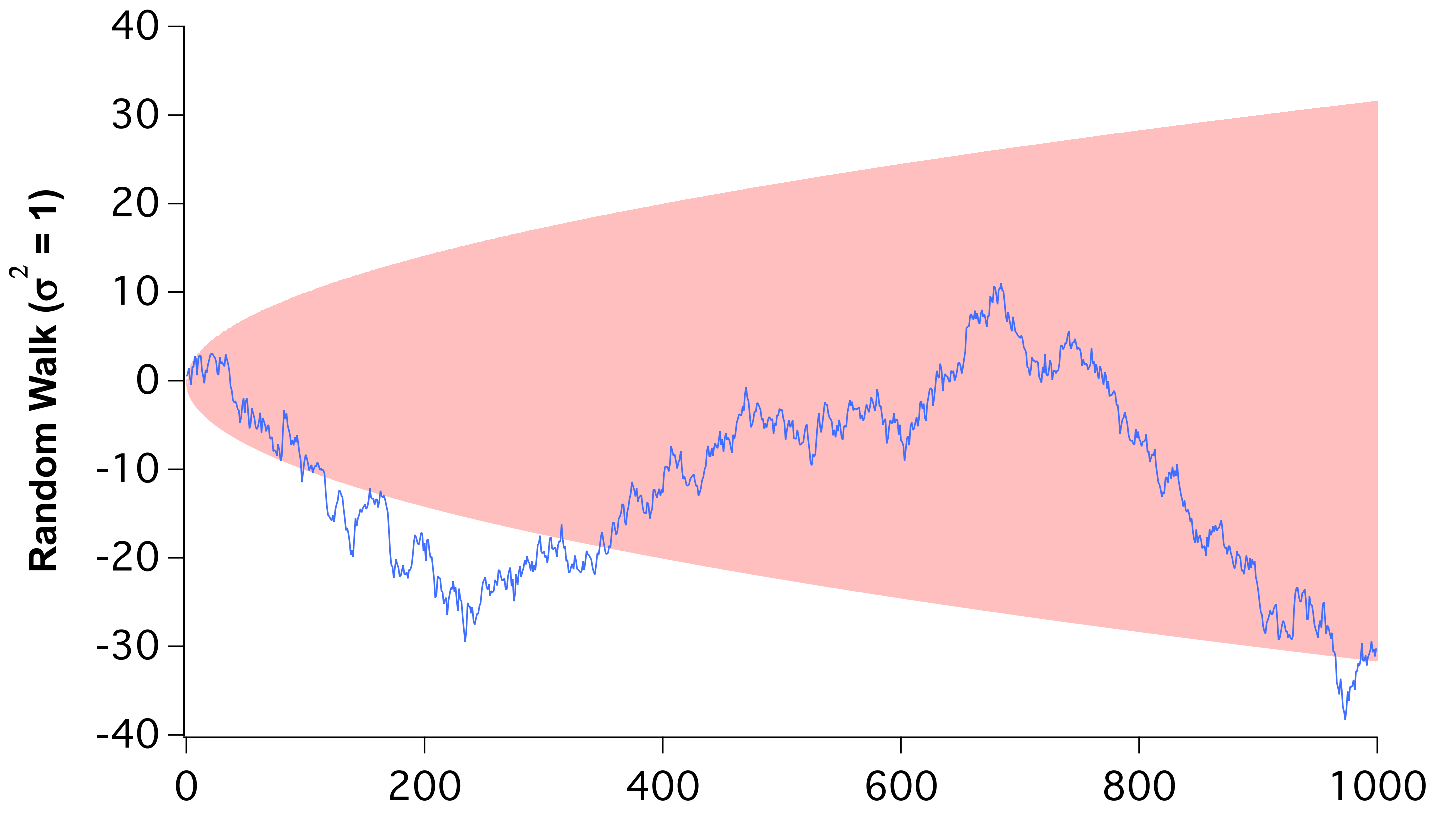
- Then $S_k = R_k = A_k = I_{1\times 1} = 1$

# White Noise Process




$\pm 1$ standard deviation region shown

# Random Walk Process

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# Random walk

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- Then $S_k = R_k = A_k = I_{1 \times 1} = 1$

## Random walk

- The prediction is

$$\widehat{z}_{k|k-1} = \widehat{z}_{k-1|k-1}$$

$$\sigma^2_{k|k-1} = \sigma^2_{k-1|k-1} + \sigma^2_{\xi}$$

- The Kalman gain is

$$K_k = \frac{\sigma^2_{k-1|k-1} + \sigma^2_{\xi}}{\sigma^2_{k-1|k-1} + \sigma^2_{\xi} + \sigma^2_{y}}$$

## Random walk

- The update is

$$\widehat{z}_{k|k} = \widehat{z}_{k-1|k-1} + \left( \frac{\sigma^2_{k-1|k-1} + \sigma^2_{\xi}}{\sigma^2_{k-1|k-1} + \sigma^2_{\xi} + \sigma^2_{y}} \right) \left[ y_k - \widehat{z}_{k-1|k-1} \right]$$

$$\sigma^2_{k|k} = \left( \frac{\sigma^2_{y}}{\sigma^2_{k-1|k-1} + \sigma^2_{\xi} + \sigma^2_{y}} \right) \sigma^2_{k-1|k-1}$$

## Random walk

$$\widehat{z}_{k|k} = \widehat{z}_{k-1|k-1} + \left( \frac{\sigma^2_{k-1|k-1} + \sigma^2_\xi}{\sigma^2_{k-1|k-1} + \sigma^2_\xi + \sigma^2_y} \right) \left[ y_k - \widehat{z}_{k-1|k-1} \right]$$

$$\sigma^2_{k|k} = \left( \frac{\sigma^2_y}{\sigma^2_{k-1|k-1} + \sigma^2_\xi + \sigma^2_y} \right) \sigma^2_{k-1|k-1}$$

- Suppose we have an initial "guess" of $\widehat{z}_{0|0} = Z_\circ$ with a large uncertainty, so $\sigma^2_{0|0} \gg \sigma^2_\xi, \sigma^2_y$

- Then after the first step ($k = 1$) we have

$$\widehat{z}_{1|1} \simeq y_1 \qquad \sigma^2_{1|1} \simeq \sigma^2_y$$

## Random walk

- After $n$ steps we have

$$\sigma^2_{n|n} = \left[\frac{\beta}{(1+\beta)^n - 1}\right]\sigma^2_y, \quad \beta = \sigma^2_\xi/\sigma^2_y$$

- Limits:

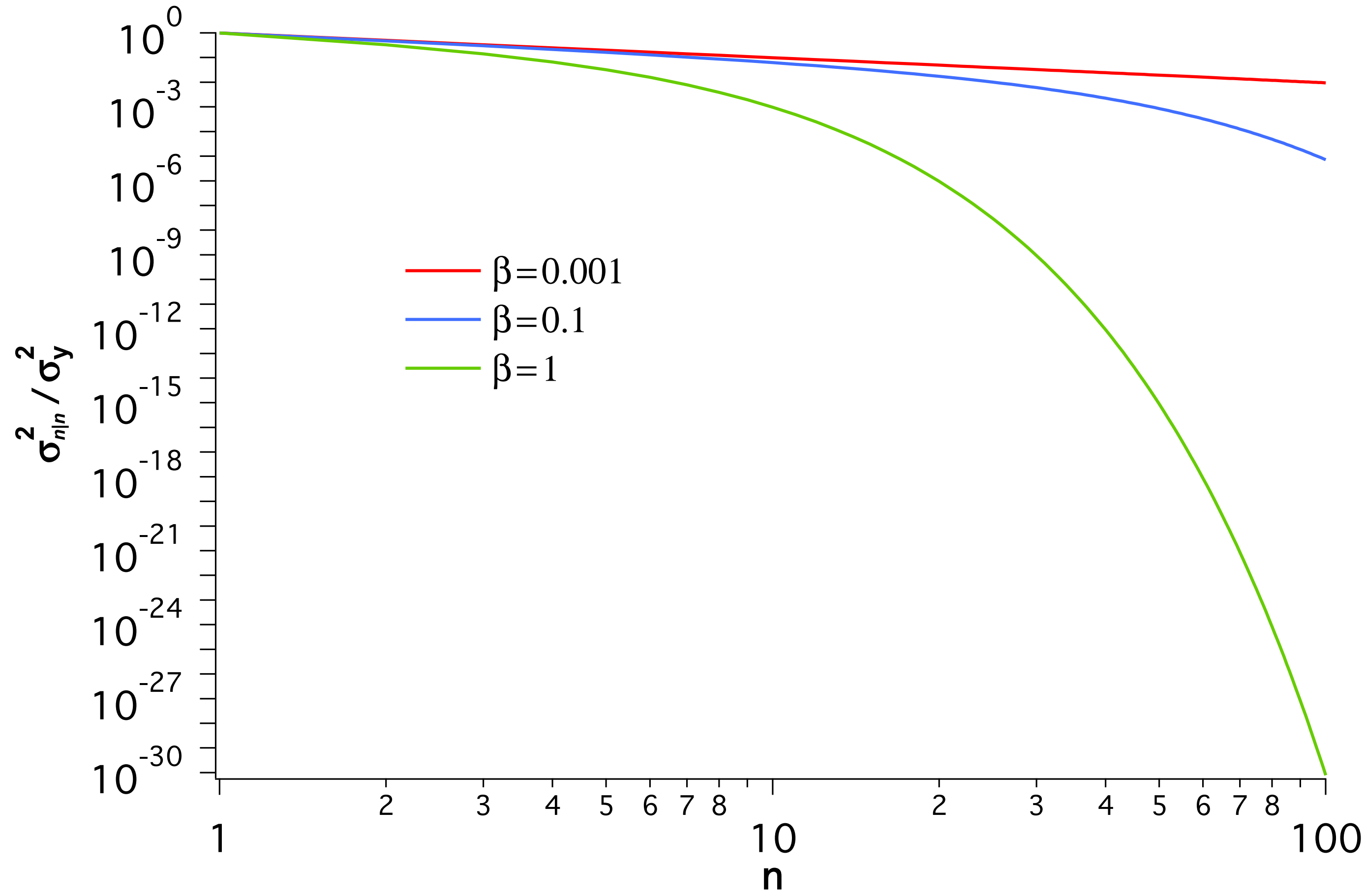
  1. $\beta \to 0$ ($\sigma^2_\xi \to 0$):

  $$\sigma^2_{n|n}/\sigma^2_y \to \frac{1}{n}$$

  2. $\beta \to \infty$ ($\sigma^2_y \to 0$): $\sigma^2_{n|n} \to \sigma^2_y/\beta^{n-1}$

  $$\sigma^2_{n|n}/\sigma^2_y \to \frac{1}{\beta^{n-1}}$$

# Random-Walk Example



## Random walk: Interpretation of results

- In this example, the state is a random walk ($z_{k+1} = z_k + \xi_k$), and we directly observe the state ($y_k = z_k + \epsilon_k$)

- Why does result for $\sigma^2_{k|k}$ depend on $\beta = \sigma^2_\xi / \sigma^2_y$?

- Recall that the derivation of the sequential least-squares used the weighted mean of the predicted state and the state derived from the observation

- From the point of view of least-squares, each of these state estimates is a random variable, equivalent except for their standard deviations

## Random walk: Interpretation of results

- A small value for $\beta$ is roughly equivalent to the statement, *The state is walking around, but nearly imperceptibly compared to our ability to observe it*

- Thus, the filter acts to mostly keep the state fixed in time, and to ascribe any variability to observational error

- Hence, $\sigma_{n|n} \sim n^{-1}$ just like a constant mean value is being estimated

## Random walk: Interpretation of results

- A large value for $\beta$ is roughly equivalent to the statement, *The state is walking around, and compared to this variability we can observe it nearly perfectly*

- Thus, the filter ascribes any variability to real variability of the state itself...

- ...and zero uncertainty to error in the estimate of the state

## Random walk: Interpretation of results

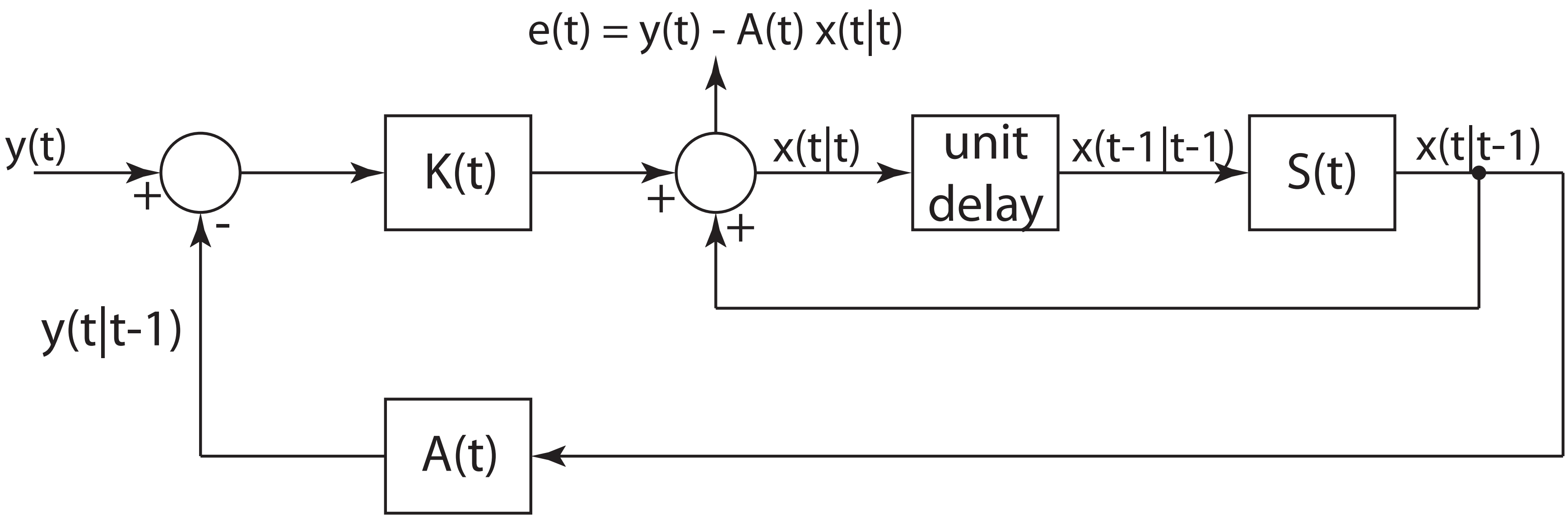
- The preceding interpretation shows why this is called a *filter*

- The filter takes (in this example) or a random input observations), and a stochastic model, and based on the details of the model separates the signals into two components:

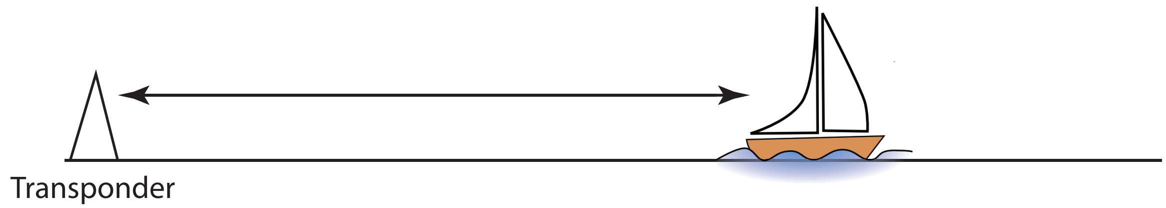
  - The estimate of the state vector $x_k$

  - The post-fit residual $\epsilon_k = y_k - A_k x_k$

# Kalman Filter Schematic



# 1-D position with random-walk speed




- Speed varies as random walk (wind? currents?)

- Temporal variability ($\xi_k$ zero-mean white-noise, $\sigma_\xi^2$):

$$z_k = z_{k-1} + v_{k-1}\Delta t$$

$$v_k = v_{k-1} + \xi_k$$

- Must include both position and speed in state

# 1-D position with random-walk speed

- Dynamic model for state:

$$x_k = \begin{bmatrix} z_k \\ v_k \end{bmatrix} = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} x_{k-1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_k$$

- Observe two-way range delay (obs. sigma $\sigma_y^2$):

$$y_k = \begin{pmatrix} \frac{2}{c} & 0 \end{pmatrix} x_k + \epsilon_k$$

- The partial derivative with respect to speed is zero!

- Can we estimate the speed?

## 1-D position with random-walk speed

- Kalman gain

$$K_k = \Lambda_{k|k-1} A_k^T \left( A_k \Lambda_{k|k-1} A_k^T + G_k \right)^{-1}$$

- Since $\left( A_k \Lambda_{k|k-1} A_k^T + G_k \right)^{-1}$ is $1 \times 1$:

$$K_k \sim \Lambda_{k|k-1} A_k^T$$

- For $\Lambda_{k|k-1} = \begin{pmatrix} \sigma_z^2 & \sigma_{zv} \\ \sigma_{zv} & \sigma_v^2 \end{pmatrix}$:

$$K_k \sim \begin{pmatrix} \sigma_z^2 \\ \sigma_{zv} \end{pmatrix}$$

## 1-D position with random-walk speed

- Thus, there is information to estimate *both* parameters even though observation = distance

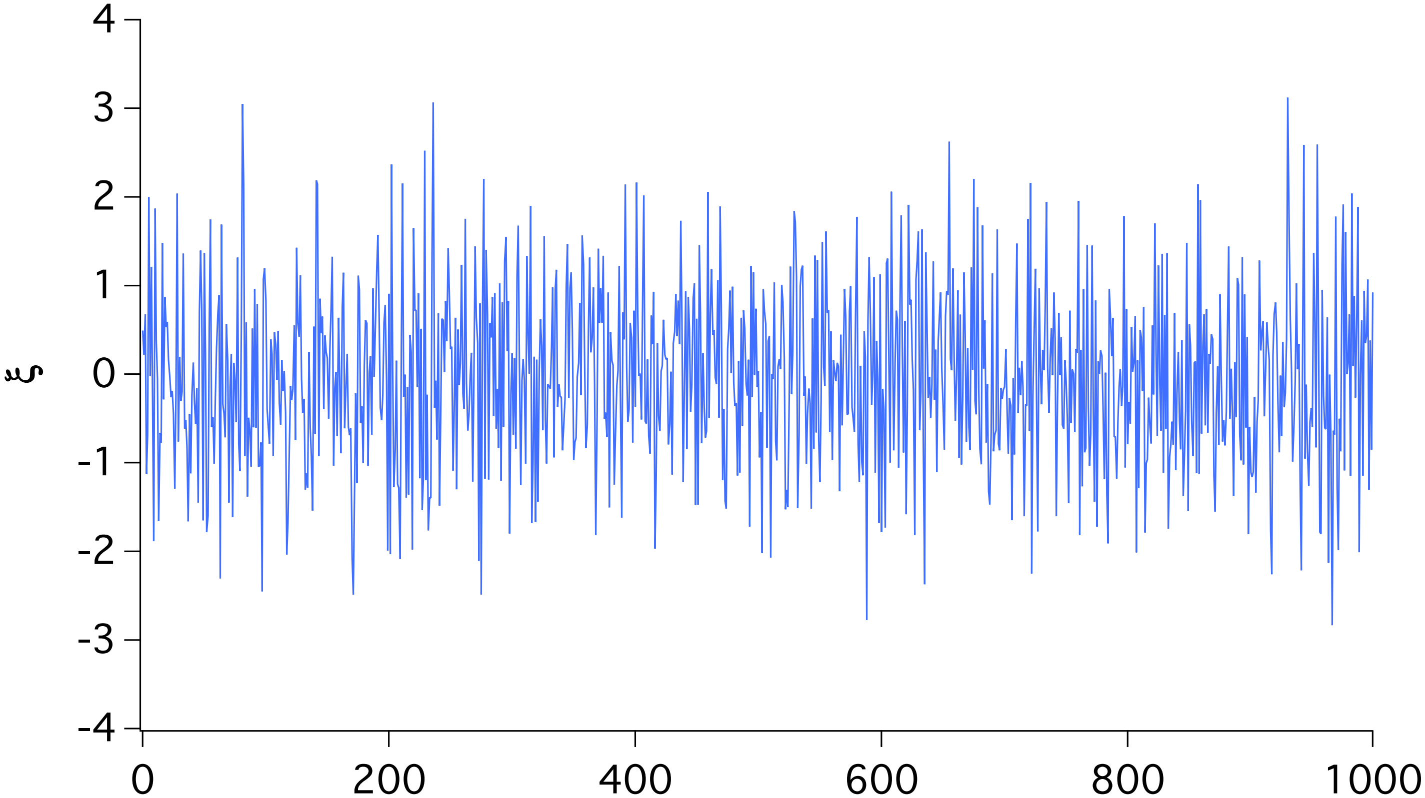
- Even on *first* epoch, if $\Lambda_{0|0} = \begin{pmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}$, then

$$\Lambda_{1|0} = \begin{pmatrix} \sigma_z^2 + (\Delta t)^2 \sigma_v^2 & \sigma_v^2 \Delta t \\ \sigma_v^2 \Delta t & \sigma_v^2 + \sigma_\xi^2 \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}_z^2 & \sigma_v^2 \Delta t \\ \sigma_v^2 \Delta t & \sigma_v^2 + \sigma_\xi^2 \end{pmatrix}$$

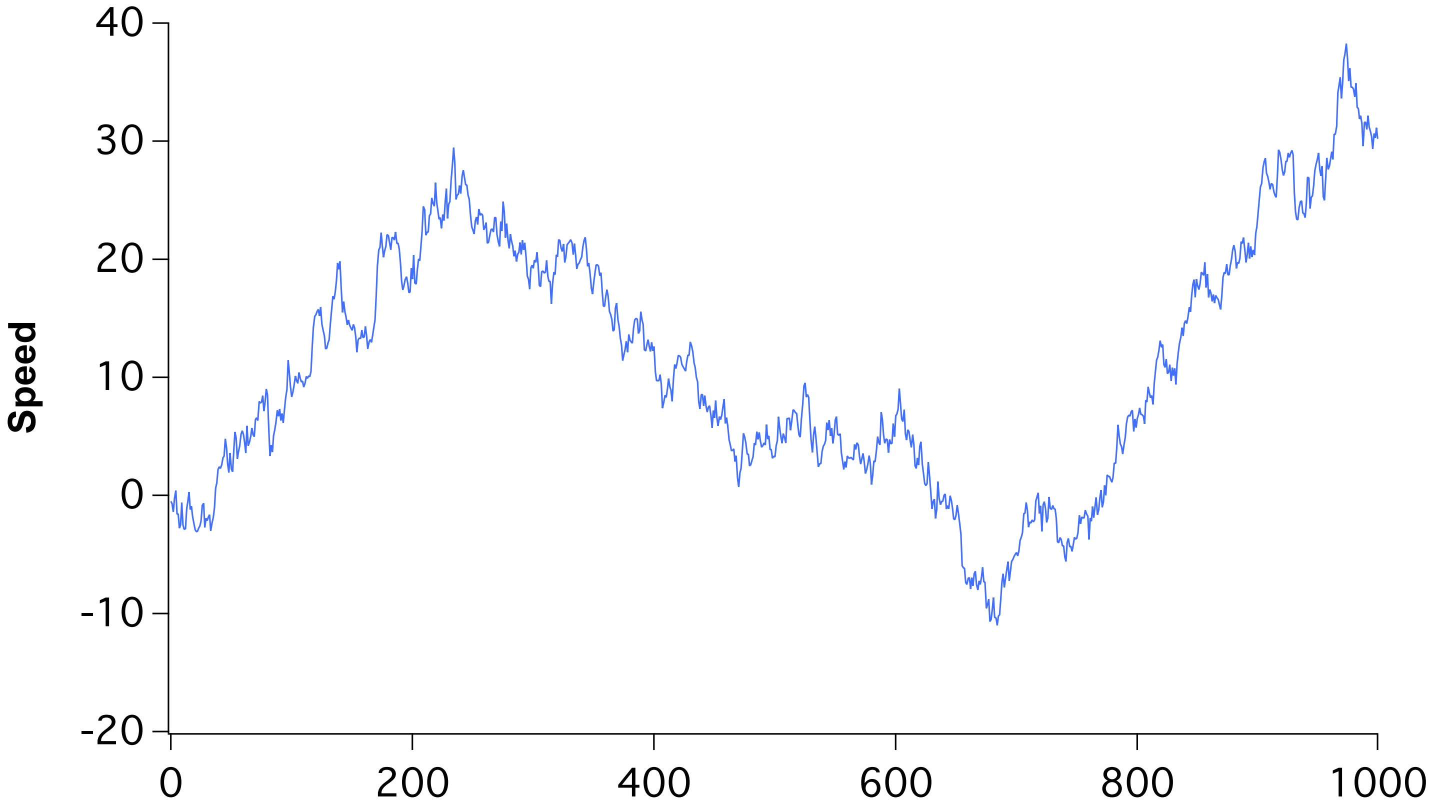
- And Kalman gain

$$K_1 = \left(\frac{c}{2}\right) \left[\tilde{\sigma}_z^2 + \left(\frac{c^2}{4}\right) \sigma_y^2\right]^{-1} \begin{pmatrix} \tilde{\sigma}_z^2 \\ \sigma_v^2 \Delta t \end{pmatrix}$$

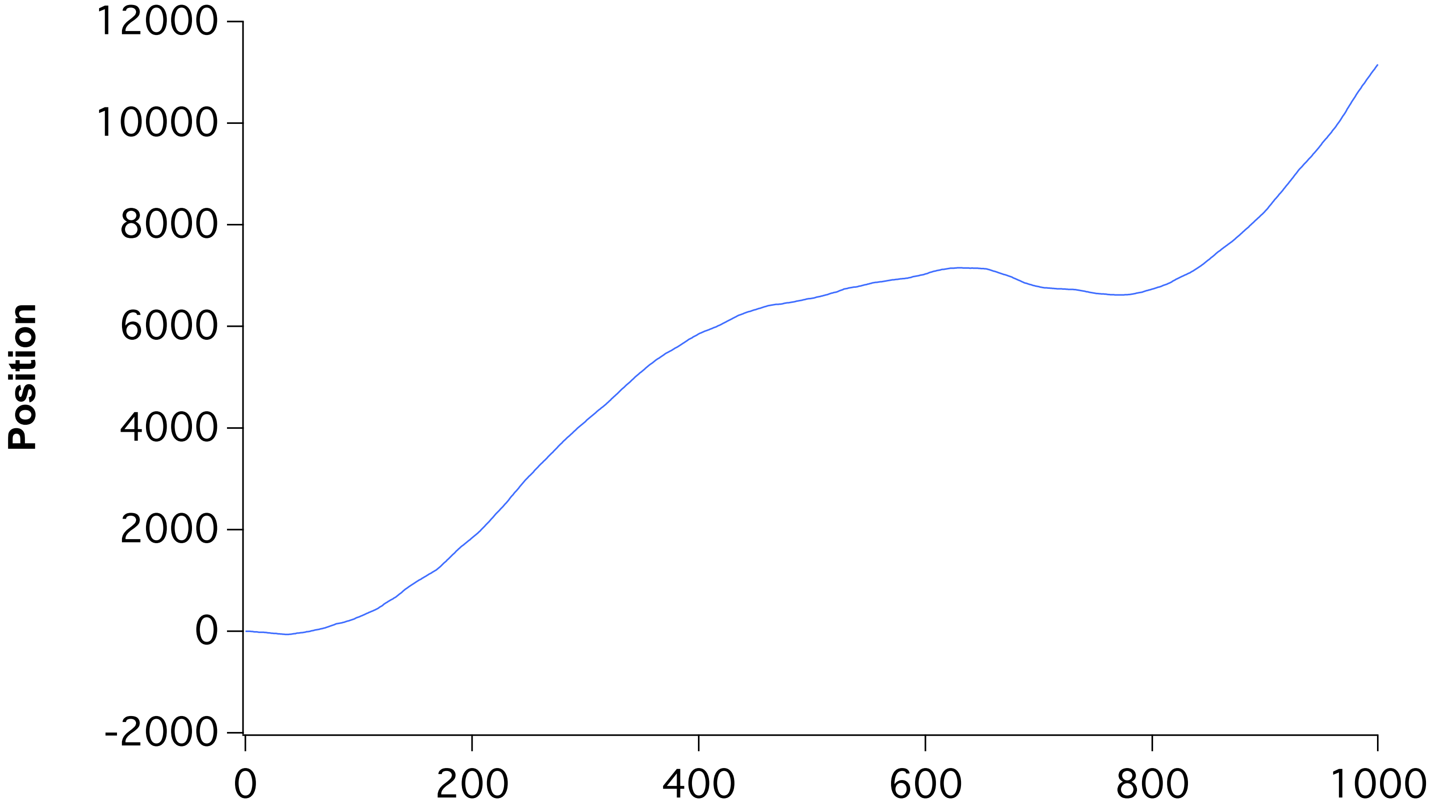
# $\xi$ (White Noise)



# Speed (Random Walk)



# Position (Integrated Random Walk)



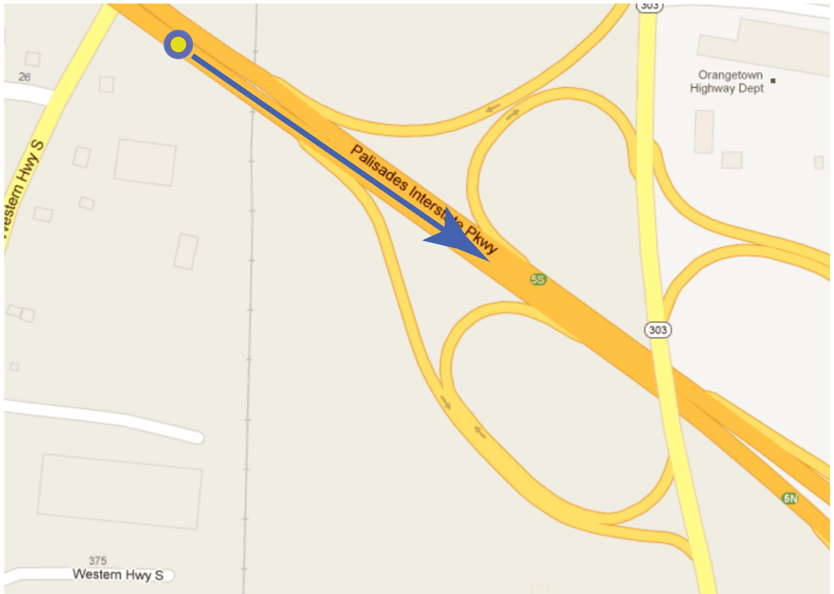
# Application to GPS positioning

- Stochastic parametrization is used by various software packages for
  - Clock errors
  - Atmospheric (wet) zenith delay
  - Zenith-delay gradient parameters
  - Time-dependent positions
  - Earth orientation/rotation
  - Orbit parameters

## Stochastic Models in GPS

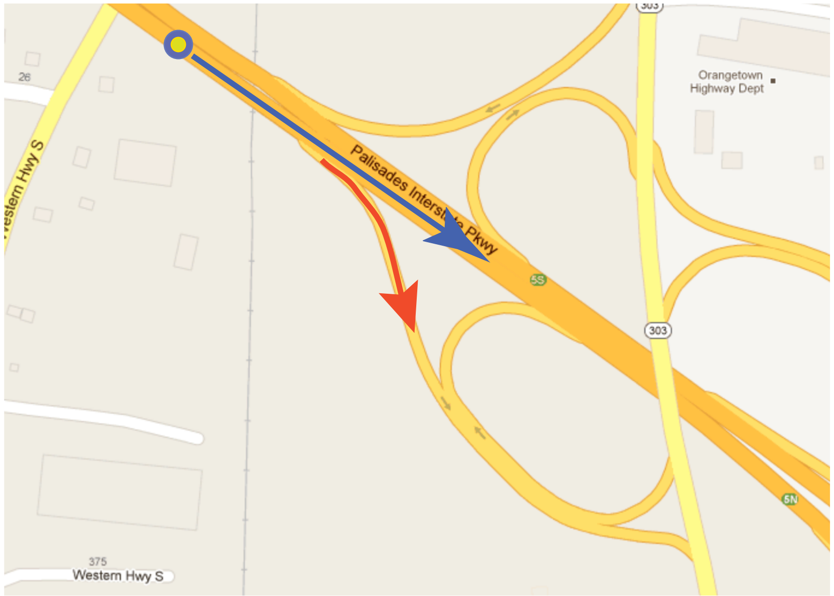
- White noise, random walk, and integrated random walk are most used
  - They are easy to implement in a Kalman filter
  - Nature of variations can be tuned to understanding of physical process
- Stochastic variances have to be "guesstimated," and sensitivity studies done

# GPS Car Navigation Example



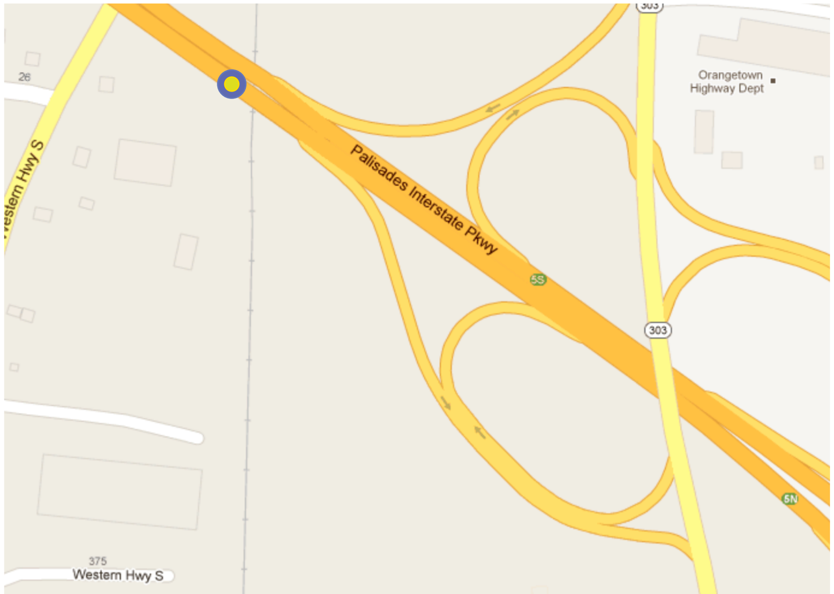

Position of your vehicle and the navigation system's route

# GPS Car Navigation Example



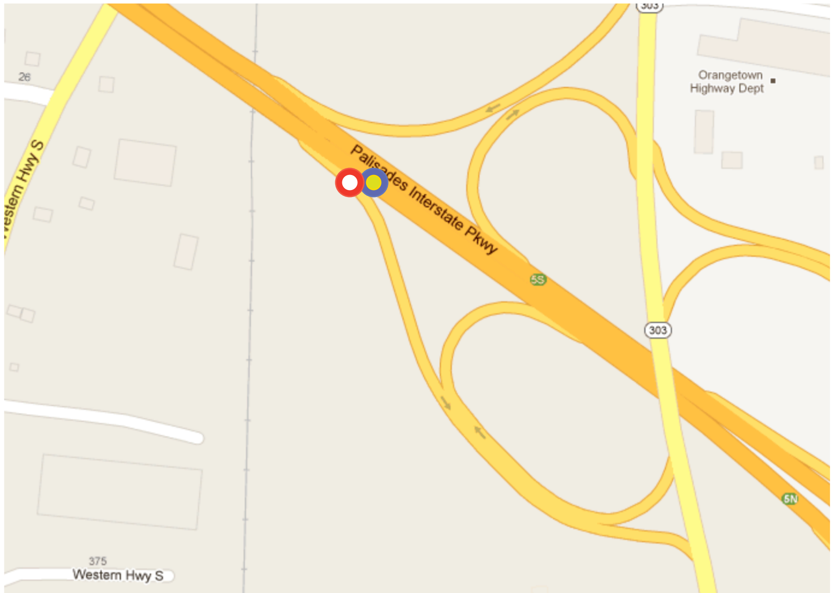

Route you want to take

# GPS Car Navigation Example




Estimated position (blue/yellow) vs. true position (red/white)

# GPS Car Navigation Example

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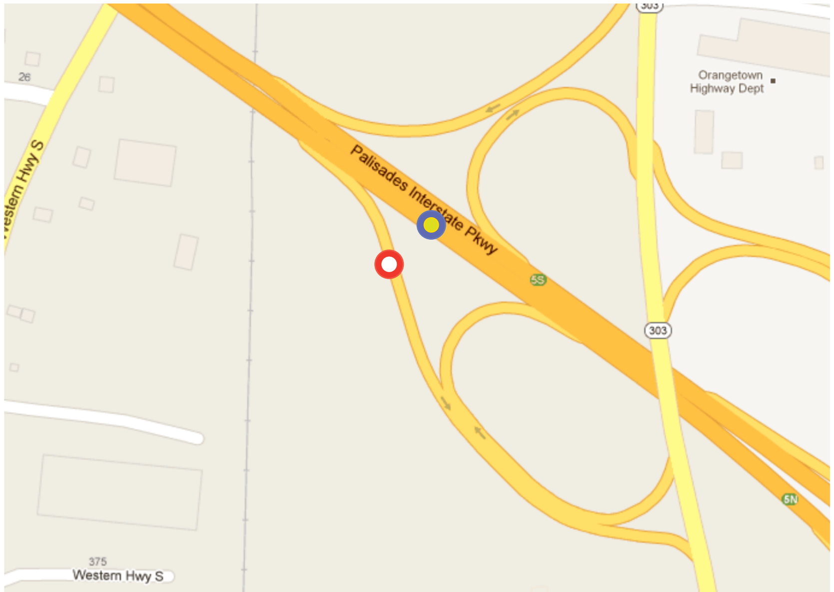


Estimated position (blue/yellow) vs. true position (red/white)

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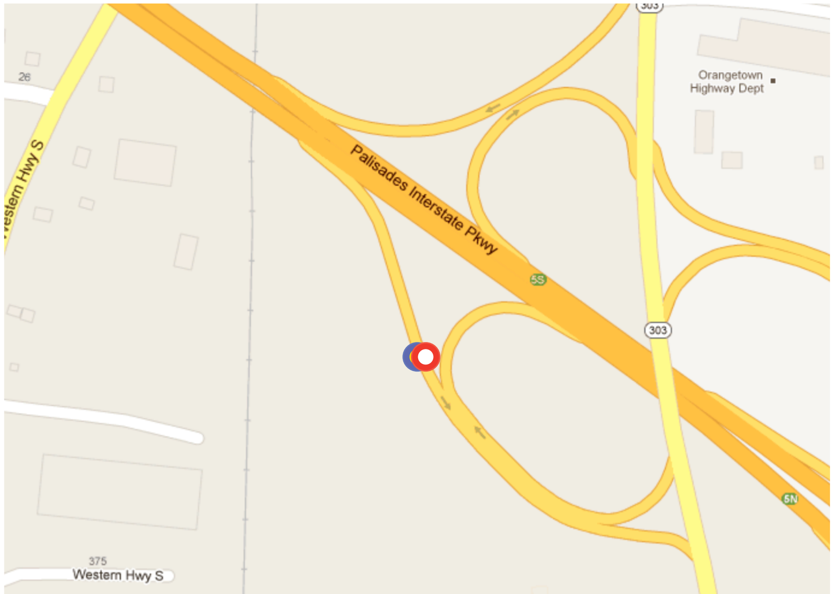



Estimated position (blue/yellow) vs. true position (red/white)

## What's happening?

- Kalman filter for position and velocity (?)

- Vehicle is assumed to be on road

- Identified of road enables prediction with small $\sigma$'s

- As prefit residual pseudoranges $(y_k - A_k\widehat{x}_{k|k-1})$ become large, estimated position changes, but *not enough to be identified as different road*

- Prefit residuals at some point become large enough that a true position update is allowed

## Combining high-accuracy GPS and accelerometer data for strong-motion displacements

- Recently, Yehuda Bock (UCSD) and colleagues have experimented with combining GPS and accelerometer estimates using a Kalman filter

- There are two ways of doing this

- One is to devise a state vector including position and acceleration, for example (for each component):

$$x_k = \begin{bmatrix} z_k \\ v_k \\ a_k \end{bmatrix}$$

## Combining GPS and accelerometer data

- The transition matrix would then be

$$S_k = \begin{pmatrix} 1 & \Delta t & \frac{1}{2}(\Delta t)^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{pmatrix}$$

- The observation vector would include both position $z$ (from GPS) and acceleration $a$ (from the accelerometer)

- This approach would require understanding the statistics of the acceleration changes which would be modeled as noise

## Combining GPS and accelerometer data

- The approach taken by Bock et al. is to use a modified dynamic equation

$$x_k = S_k x_{k-1} + B_k u_k + R_k \xi_k$$

- The state vector is

$$x_k = \begin{bmatrix} z_k \\ v_k \end{bmatrix}$$

- The transition matrix is

$$S_k = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}$$

## Combining GPS and accelerometer data

$$x_k = S_k x_{k-1} + B_k u_k + R_k \xi_k$$

- The input $u_k$ is the accelerometer reading, so that

$$B_k = \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix}$$

- This modified dynamic equation can be implemented by modifying the prediction equation:

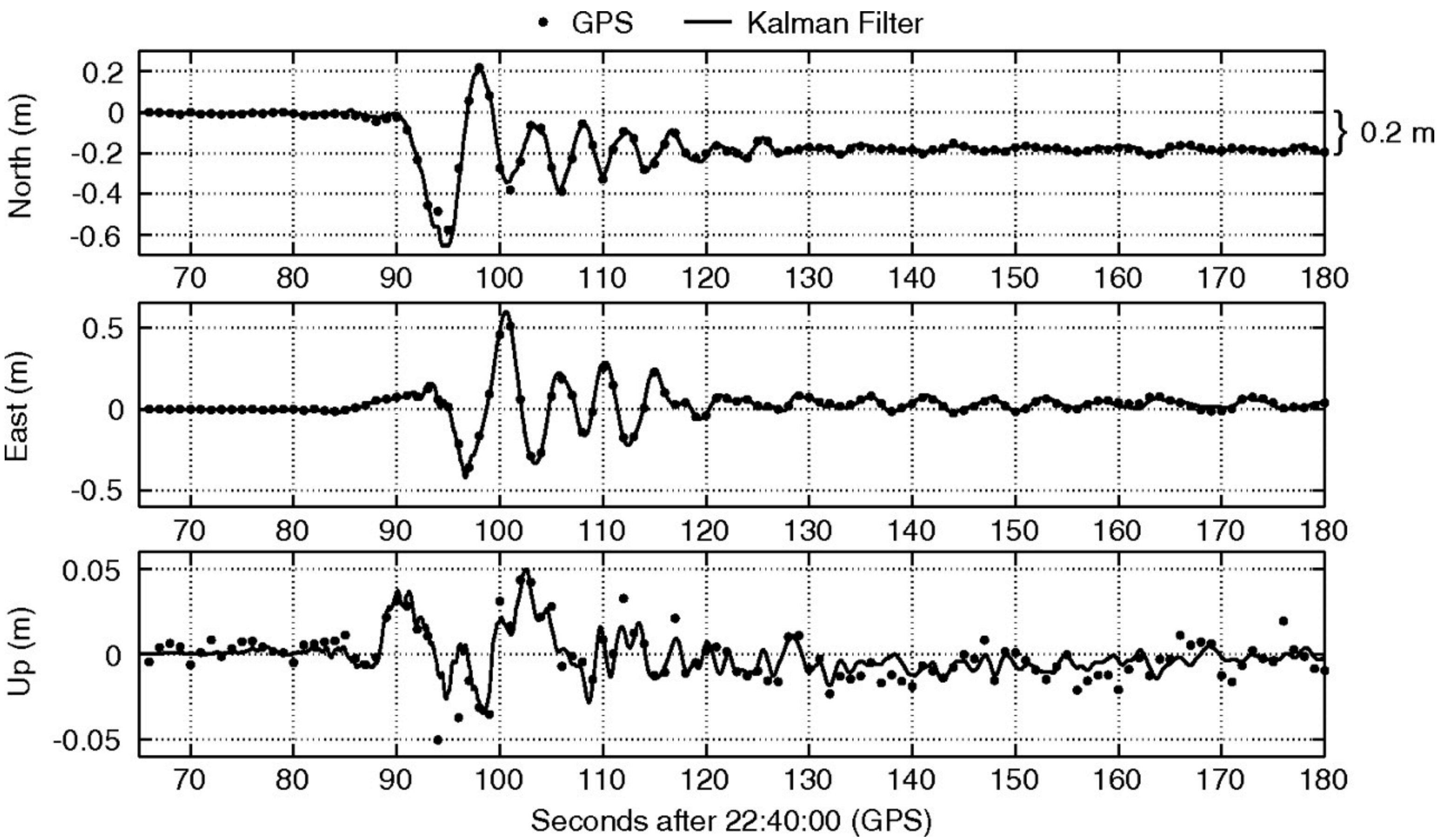
$$\widehat{x}_{k|k-1} = S_k x_{k-1|k-1} + B_k u_k$$

## Combining GPS and accelerometer data

- The noise matrix $Q_k$ has a rather complex form to reflect the sampling (with error) of a continuous quantity (acceleration)

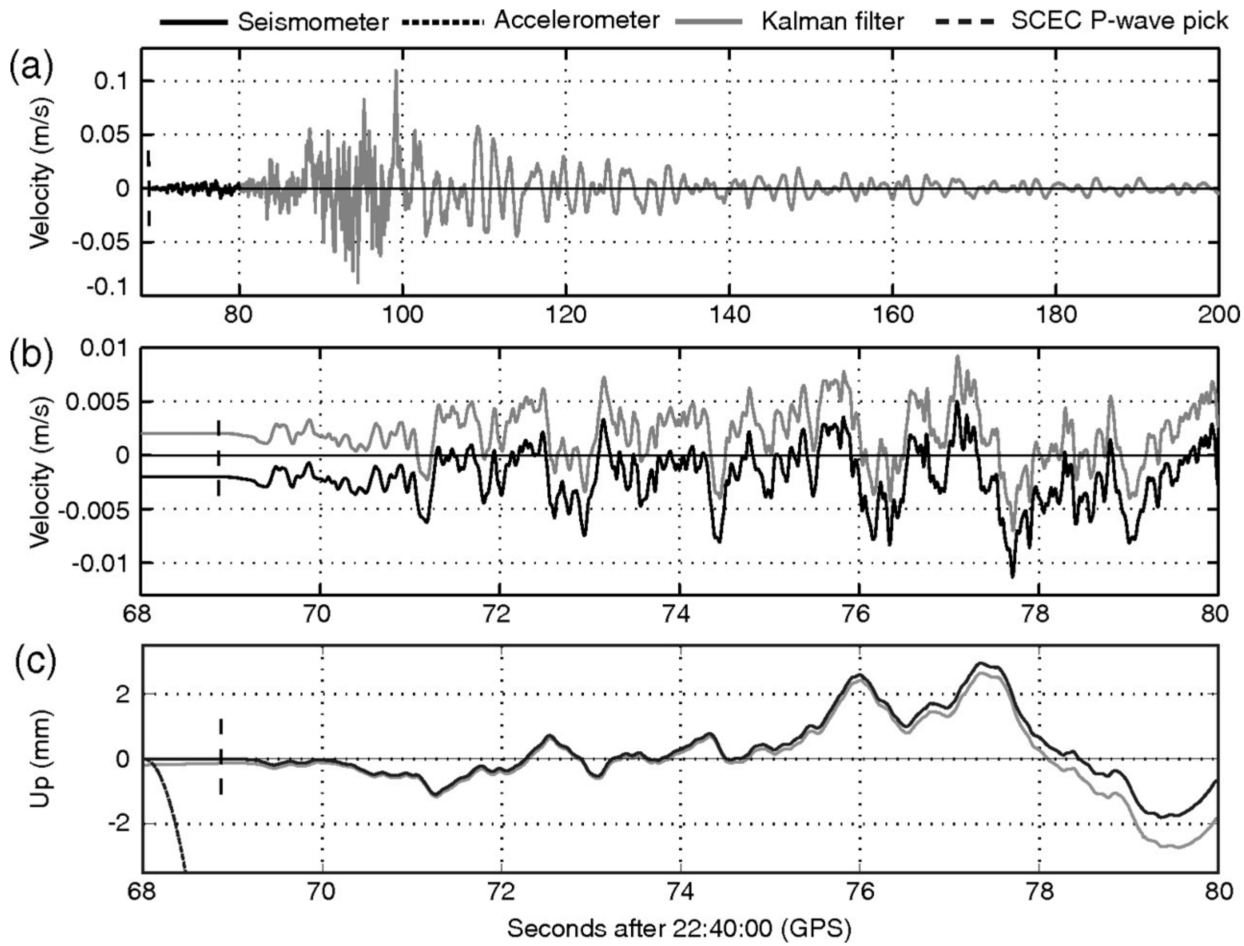
- The GPS receiver and accelerometer have different sampling rates
  - GPS: 10 Hz
  - Accelerometer: 100 Hz

# GPS vs. KF (El Mayor-Cucapah 4/4/2010)



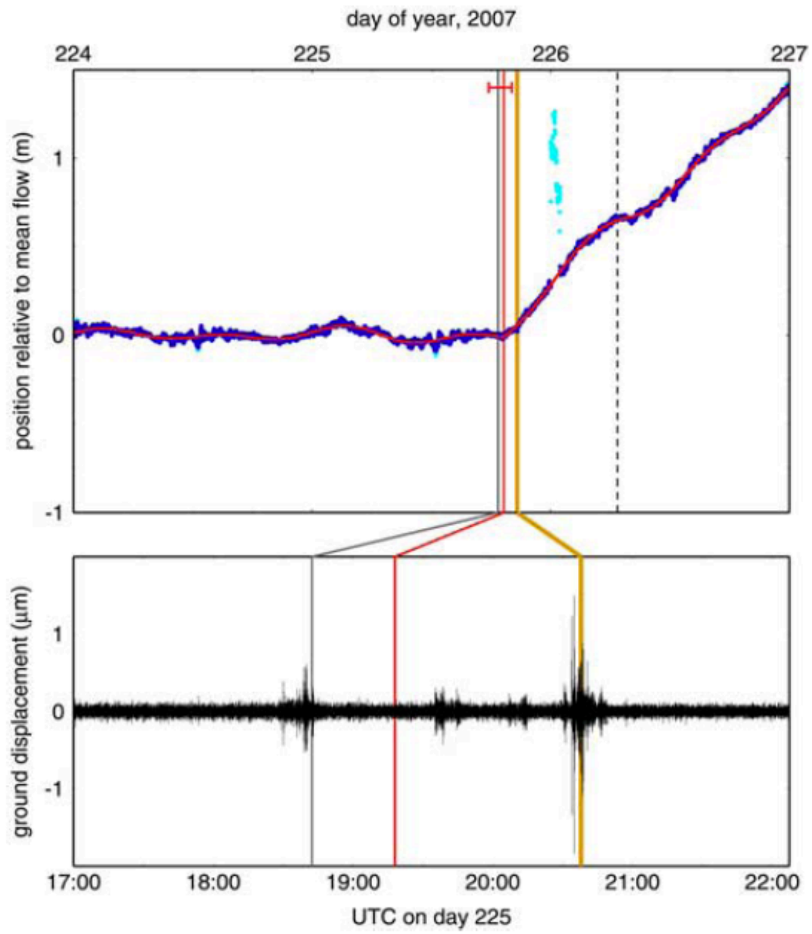

Bock et al. (2011), *BSSA*, *101*, 2904–2925

# Seismometer vs. KF (El Mayor-Cucapah)




Bock et al. (2011), *BSSA*, *101*, 2904–2925

# Position estimates of glacier GPS sites




Nettles et al. (2008), *GRL*, *35*, L24503