EESC UN3201
Solid Earth Dynamics
Spring 2023

Bill Menke, Instructor

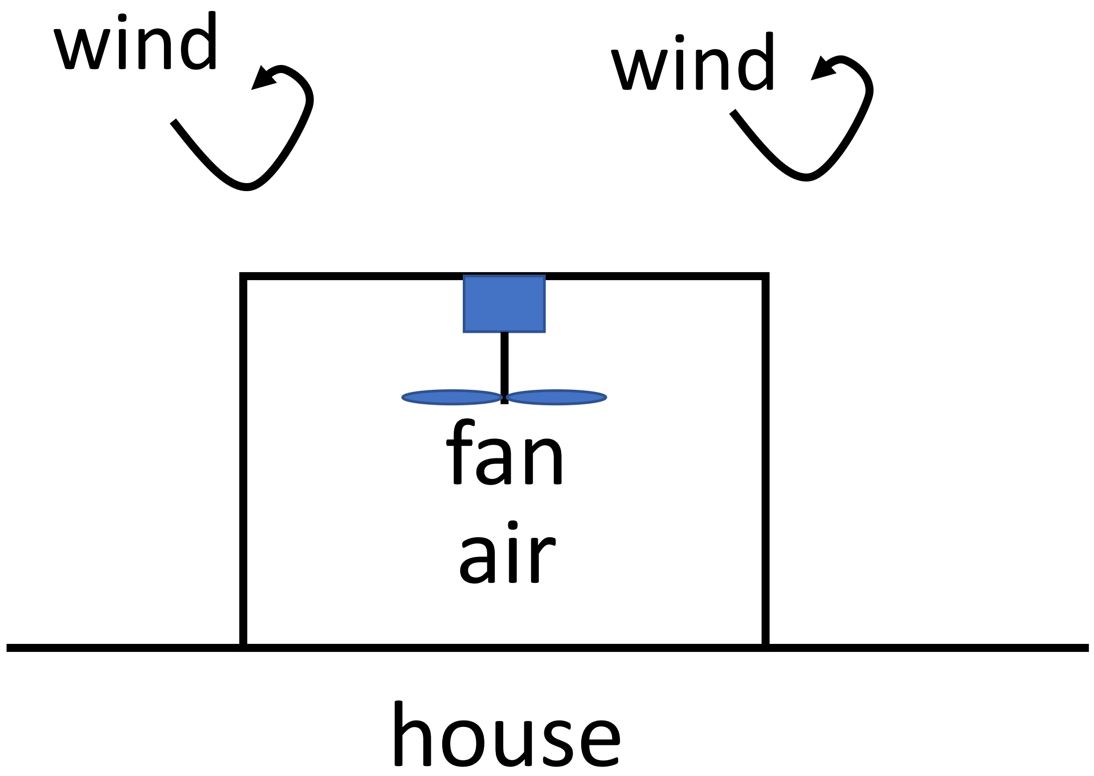
Lecture 3

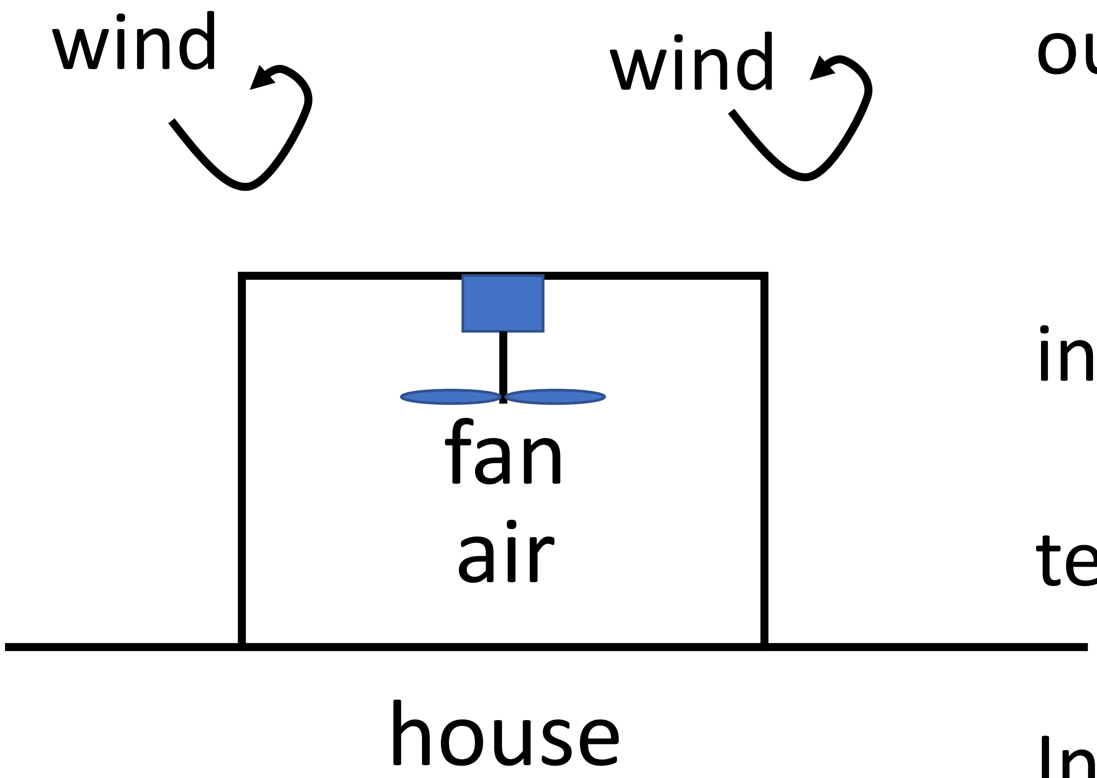
Today:

Heat flow: Cooling

- of a house
- of a dike
- of the ground


Temperature
12/12/22 – 12/19/22
65°F
60°F
55°F
50°F
45°F
40°F
Dec 13
Dec 14
Dec 15
Dec 16
Dec 17
Dec 18
Dec 19


wind
wind
fan
air
house



outside air isothermal due to wind

$$\Delta T = 0$$

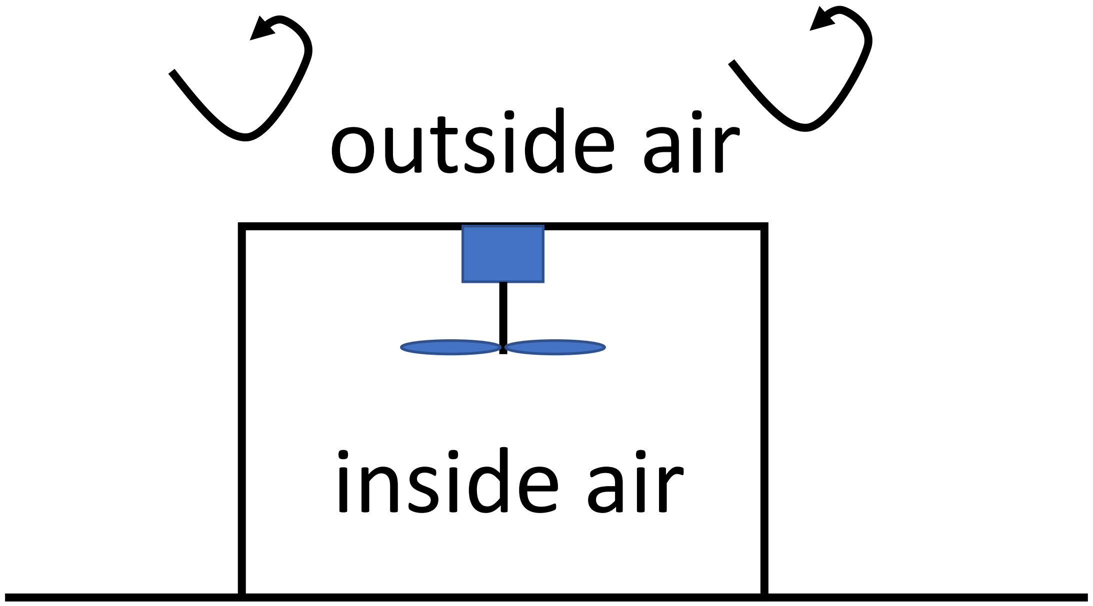
inside air isothermal due to fan

temperature $\Delta T(t)$

Initial temperature of the house

$$\Delta T(t = 0) = \Delta T_0$$

house cools by conduction through walls with total area, A

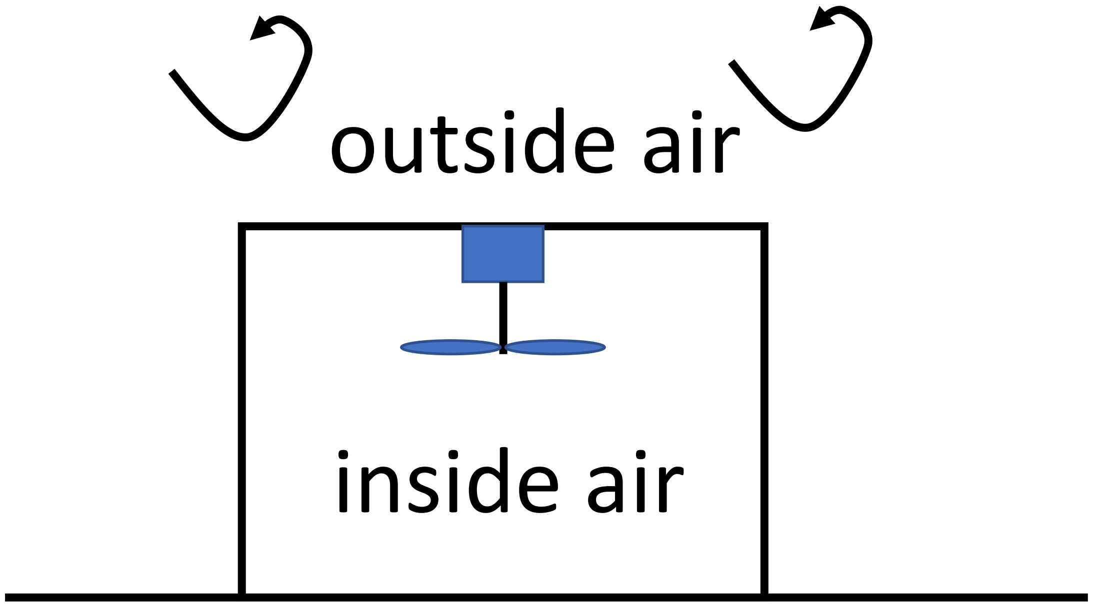


Conservation of energy

change in heat in house with time

=

heat loss thru walls



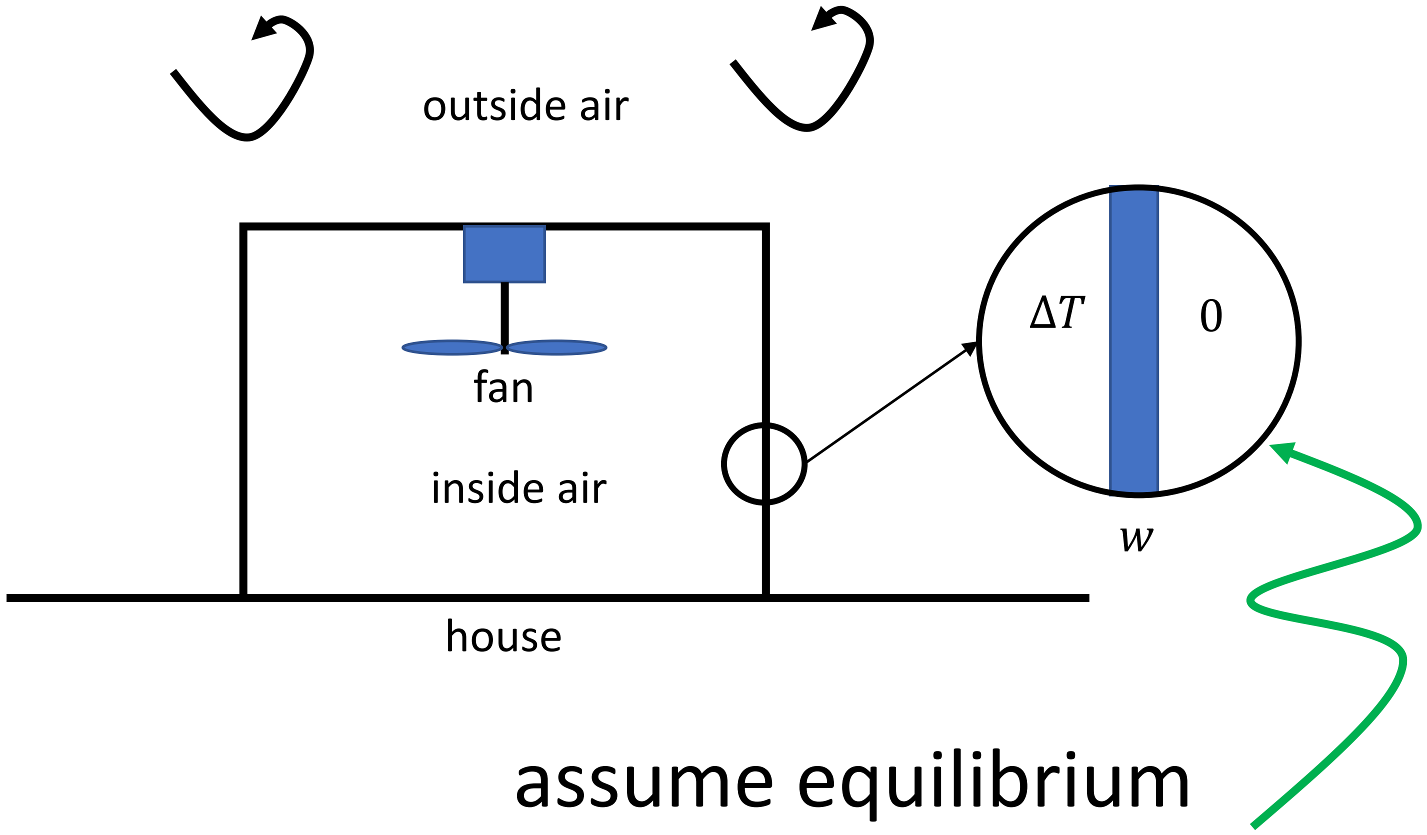
Conservation of energy

$$\rho c_p V \frac{d\Delta T}{dt}$$

$$=$$

$$-Aq$$

Conservation of energy

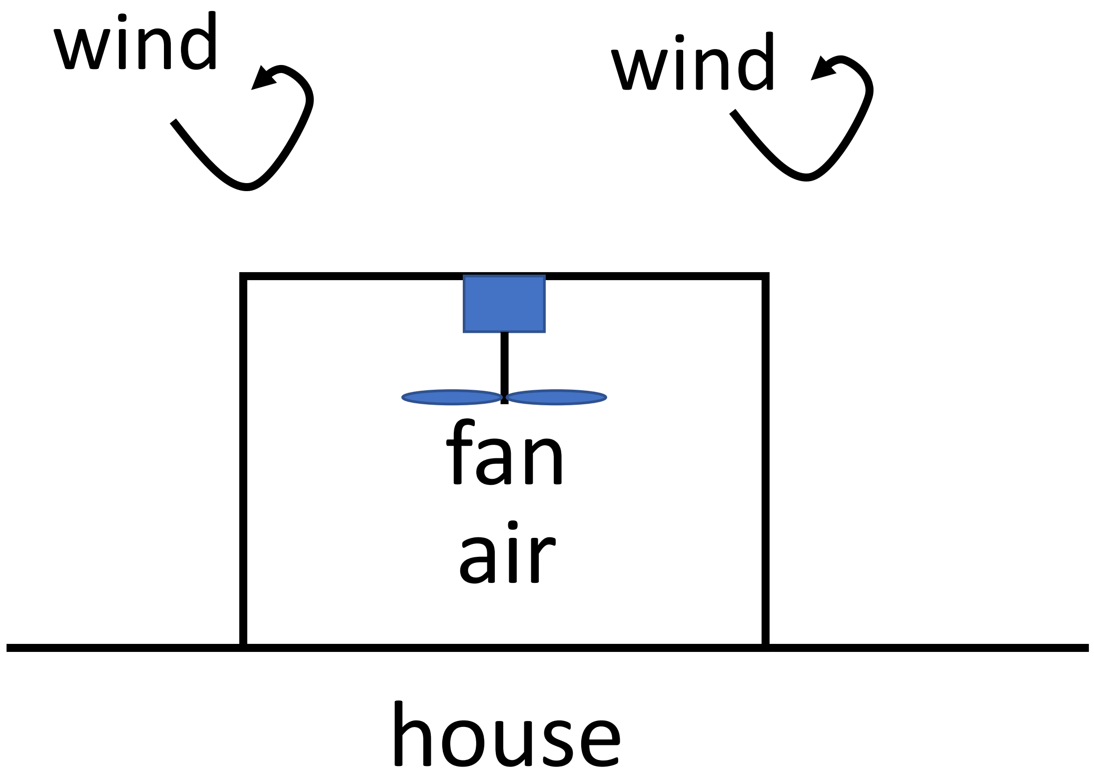


assume equilibrium

ΔT linear in wall

$$\frac{d\Delta T}{dx} = \frac{\Delta T - 0}{w} = \frac{\Delta T}{w}$$

$$\rho c_p V \frac{d\Delta T}{dt} = -Aq = -\frac{Ak}{w}\Delta T$$



Ever hear of an R-Value in connection with home insulation?

Conservation of energy

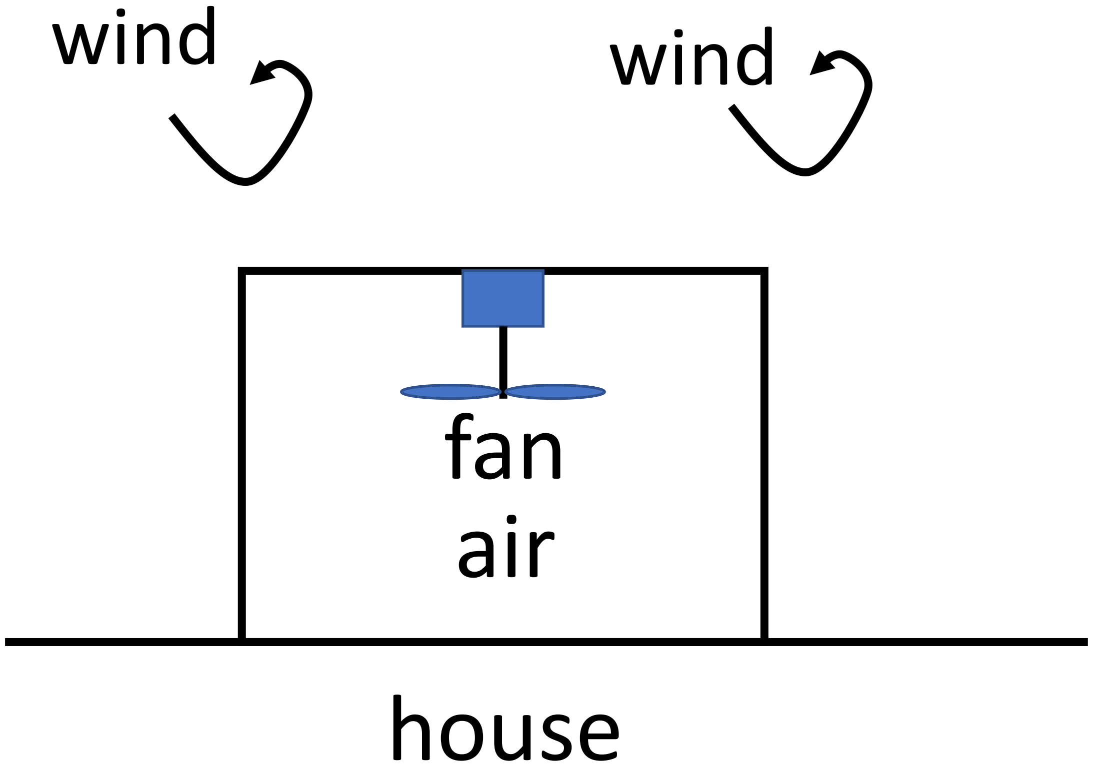
$$\rho c_p V \frac{d\Delta T}{dt}$$

$$=$$

$$-Aq$$

$$= -\frac{A}{R}\Delta T$$

with R=w/k

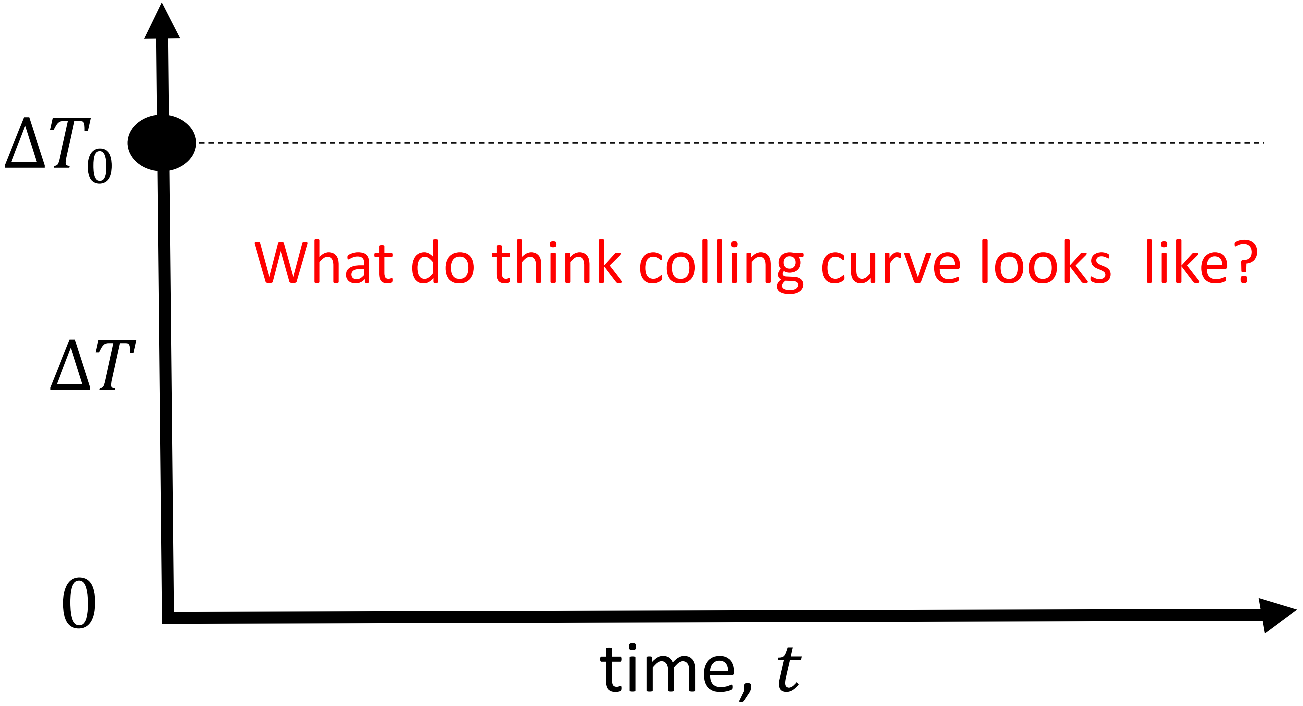


Conservation of energy

$$\frac{d\Delta T}{dt} = -c\Delta T$$

with

$$c = \frac{Ak}{w\rho c_p V}$$

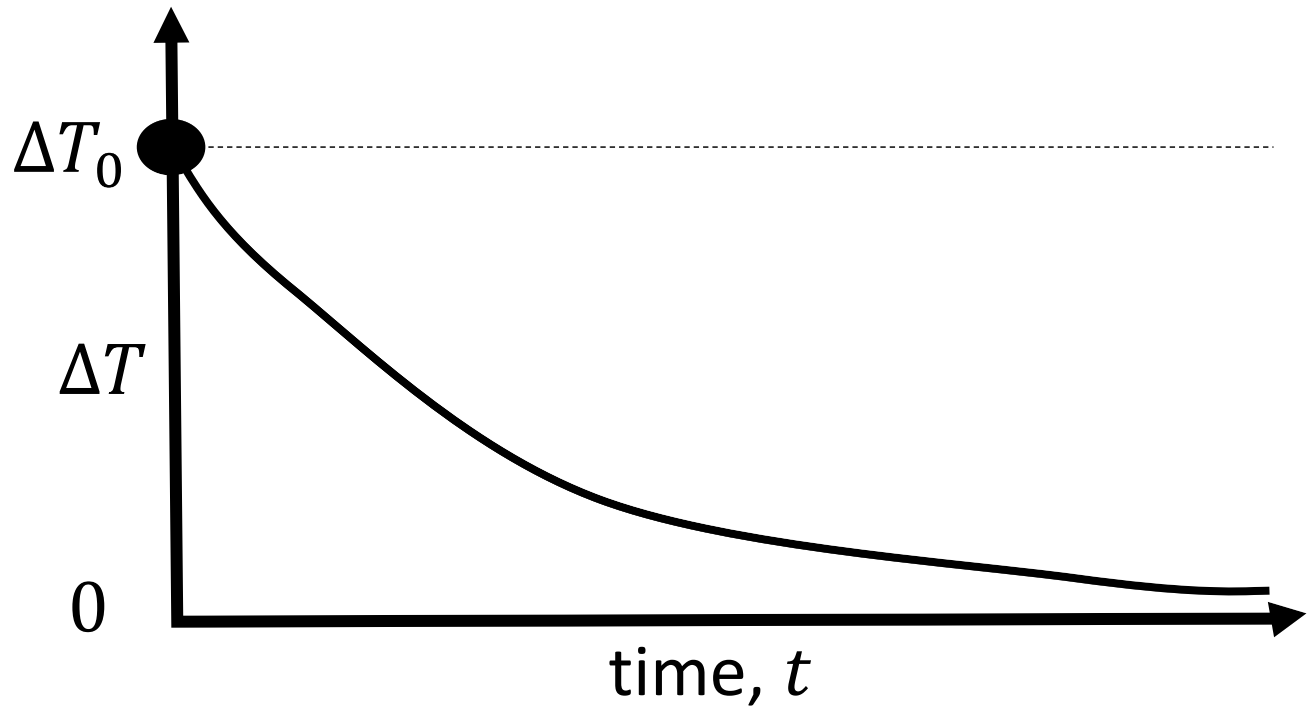


Conservation of energy

$$\frac{d\Delta T}{dt} = -c\Delta T$$

with

$$c = \frac{Ak}{w\rho c_p V}$$

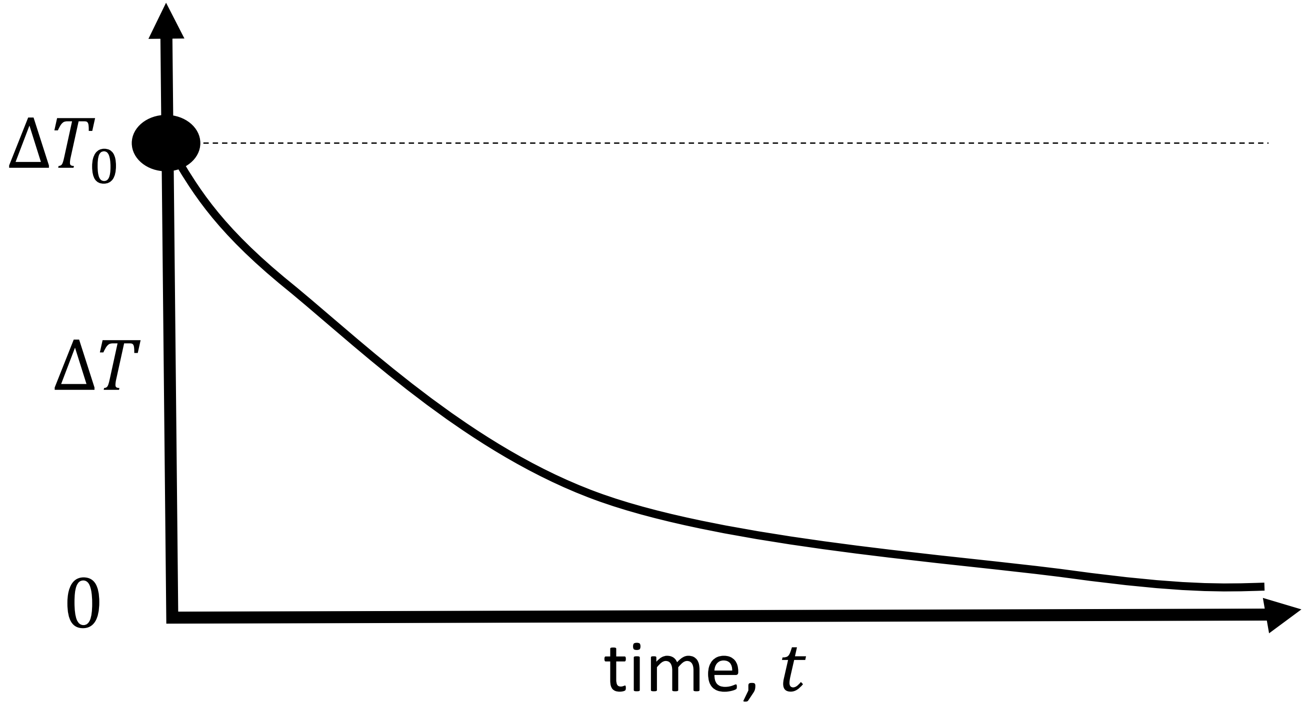


Conservation of energy

$$\frac{d\Delta T}{dt} = -c\Delta T$$

with

$$c = \frac{Ak}{w\rho c_p V}$$



Conservation of energy

$$\frac{d\Delta T}{dt} = -c\Delta T$$

with

$$c = \frac{Ak}{w\rho c_p V}$$

$$\Delta T = \Delta T_0 \quad e^{-at}$$

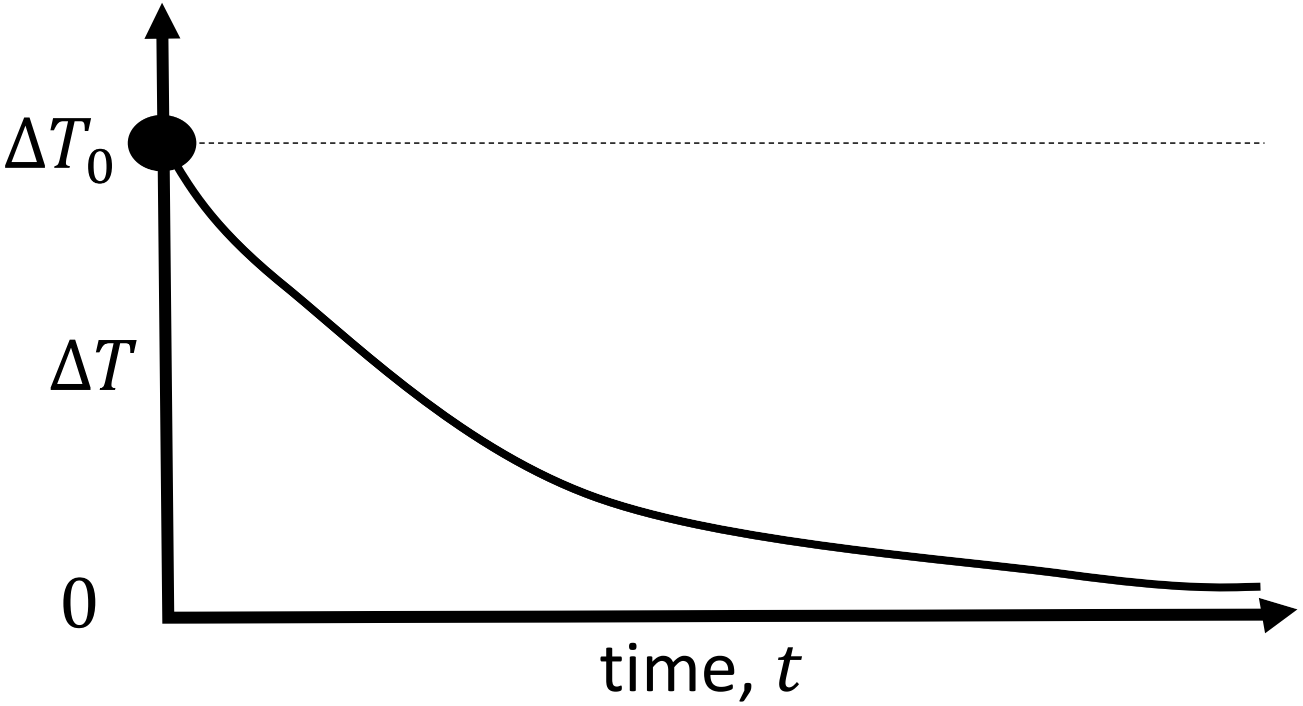
unknown constant

“Exponential” function

$$e^t \quad = (2.71\cdots)^t = \exp(t)$$

$\frac{d}{dt}e^t = e^t$ derivative is itself

$$\frac{d}{dt}e^{-at} = -ae^t$$



Conservation of energy

$$\frac{d\Delta T}{dt} = -c\Delta T$$

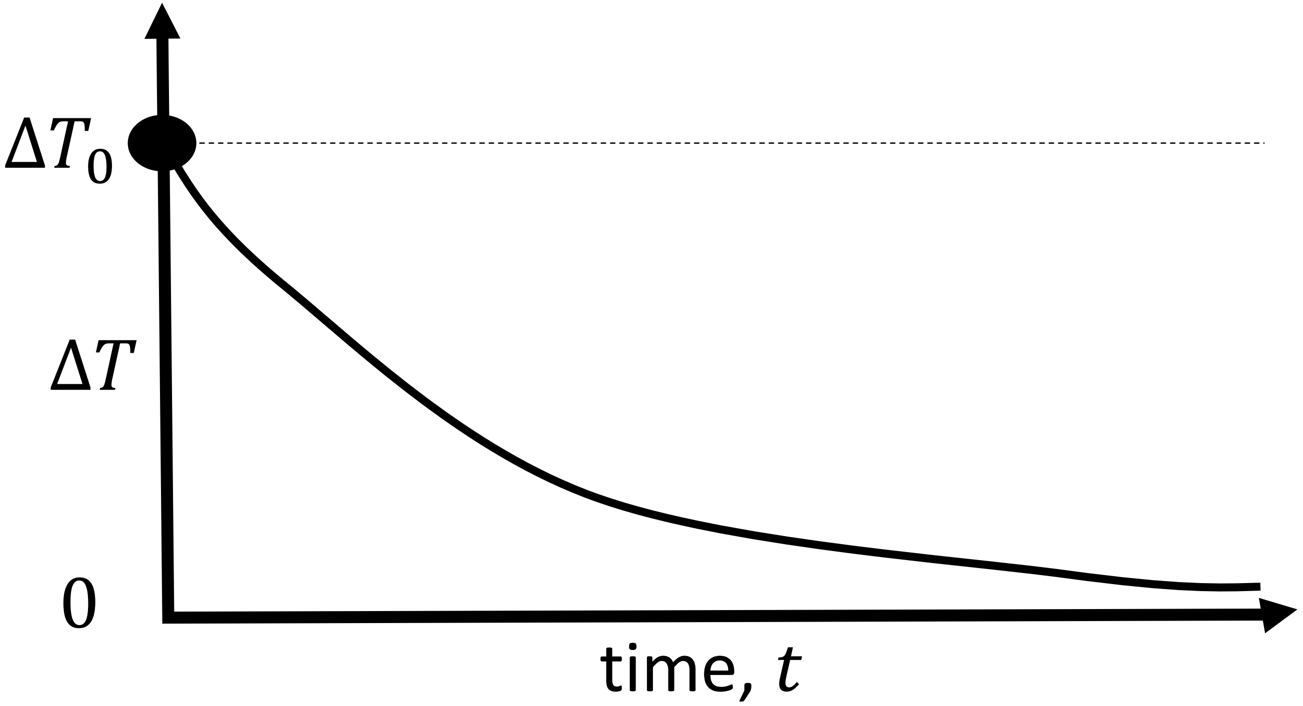
$$c = \frac{Ak}{w\rho c_p V}$$

$$\Delta T = \Delta T_0 \quad e^{-at}$$

$$\Delta T = \Delta T_0 \quad e^{-at}$$

compatible with equation?

$$\frac{d\Delta T}{dt} = -a\Delta T_0 \quad e^{-at}$$



$$\Delta T = \Delta T_0 \quad e^{-at}$$

compatible with equation?

Conservation of energy

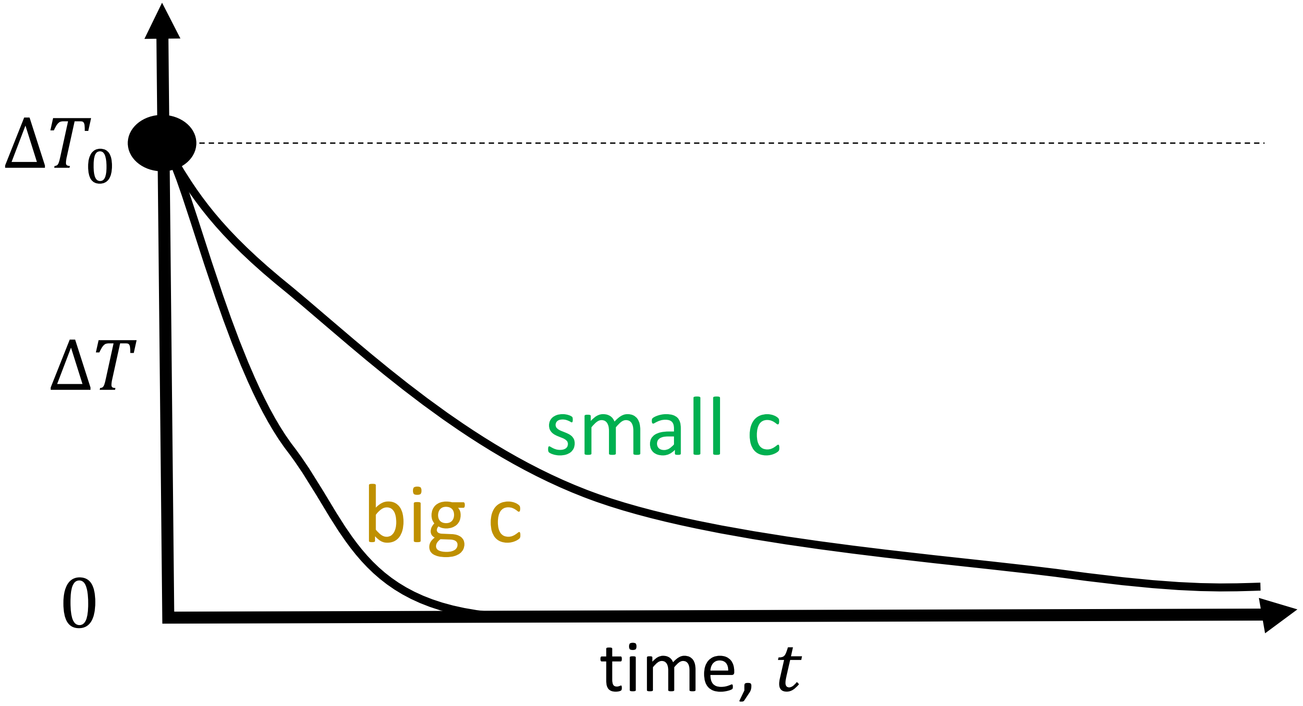
$$\frac{d\Delta T}{dt} = -c\Delta T$$

$$c = \frac{Ak}{w\rho c_p V}$$

$$\Delta T = \Delta T_0 \quad e^{-at}$$

$$\frac{d\Delta T}{dt} = -a\Delta T_0 \quad e^{-at}$$

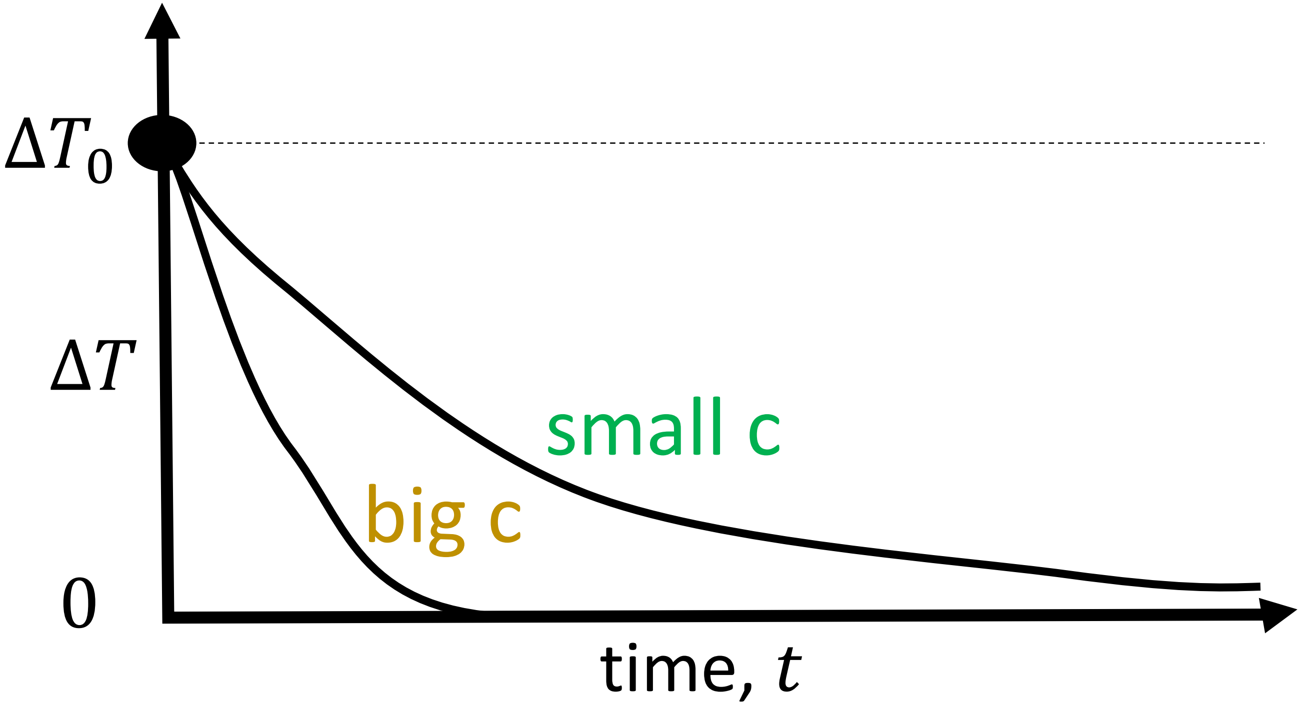
yes if $a = c$



$$\Delta T = \Delta T_0 \quad e^{-ct}$$

$$c = \frac{Ak}{w\rho c_p V}$$

small c	big c
small, A	big, A
small, k	big, k
big w	small w
big V	small V



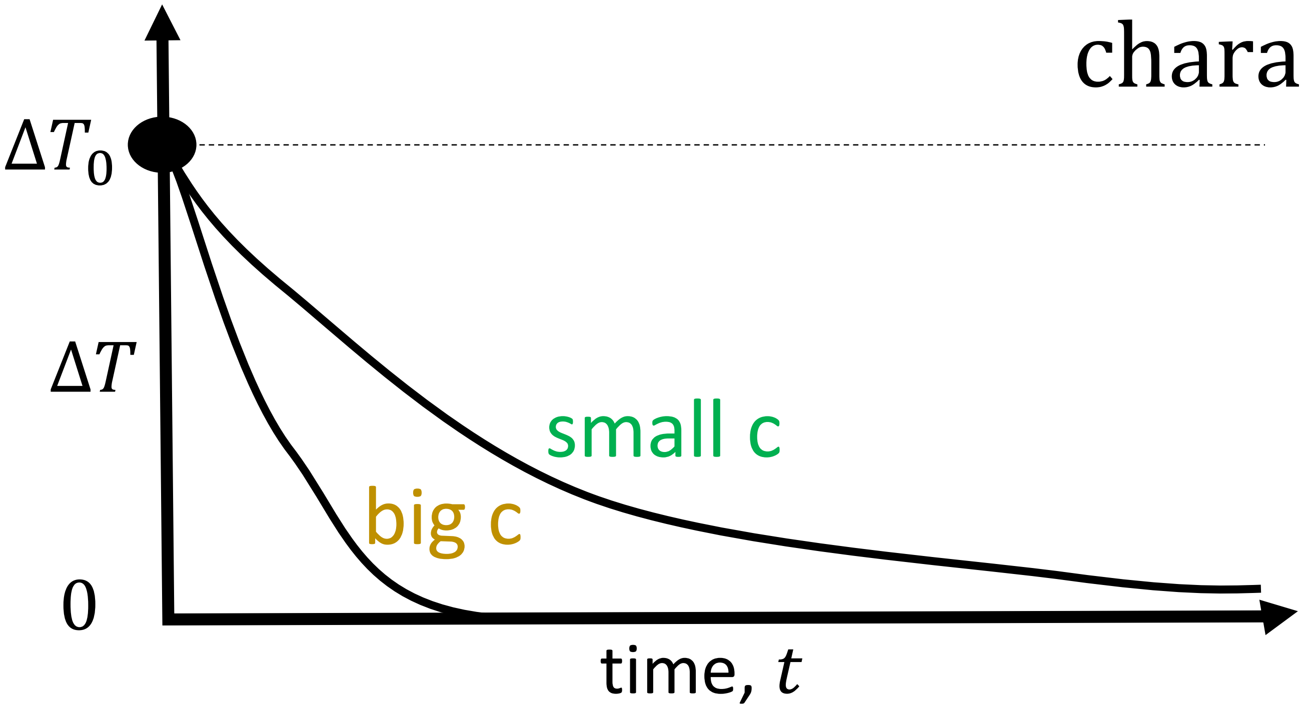
$$\Delta T = \Delta T_0 \quad e^{-ct}$$

$$c = \frac{Ak}{w\rho c_p V}$$

small c	big c
small, A	big, A
small, k	big, k
big w	small w

$$\text{characteristic time, } t_0 = \frac{1}{c} = \frac{w\rho c_p V}{Ak}$$

$$\text{characteristic time, } t_0 = \frac{w \rho c_p V}{Ak}$$



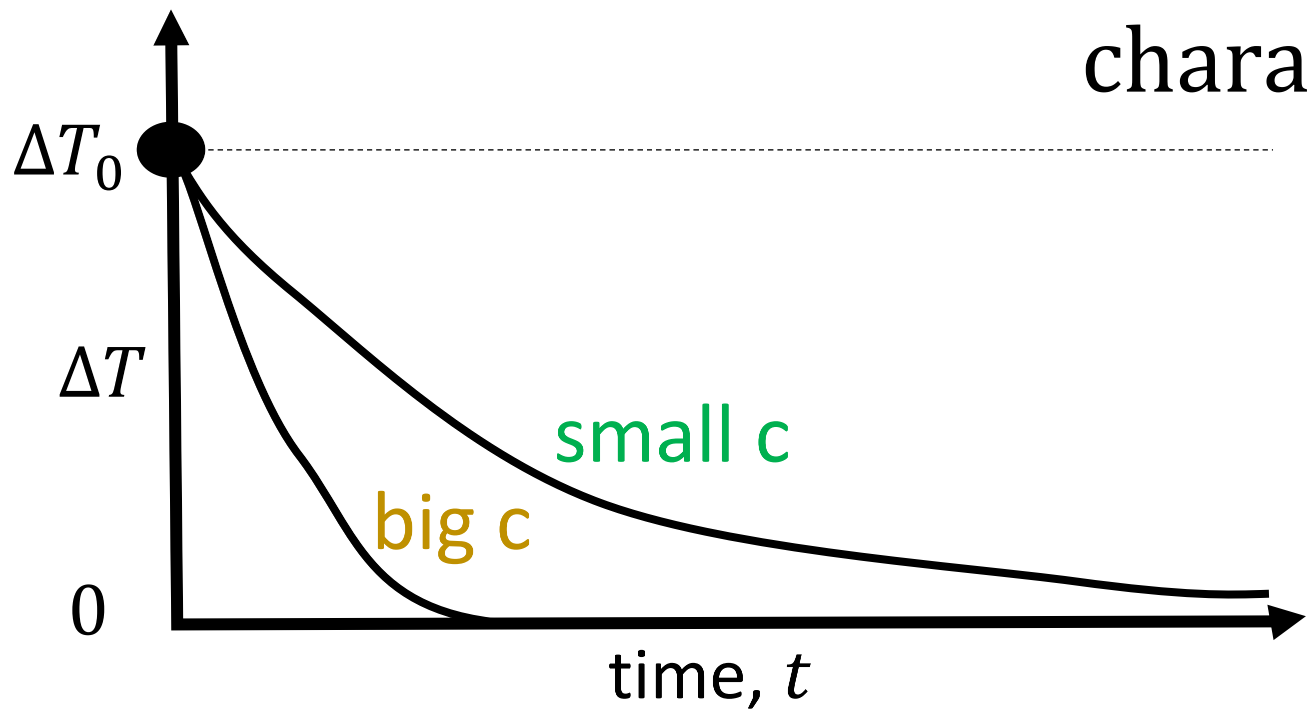
$$A = 5 \times 3 \times 10 = 150\ m^2$$

$$V = 3 \times 10 \times 10 = 300\ m^3$$

$$w = 0.2\ \ m$$

$$k = 0.2\ \ \frac{J}{sm°C}$$ (wood)

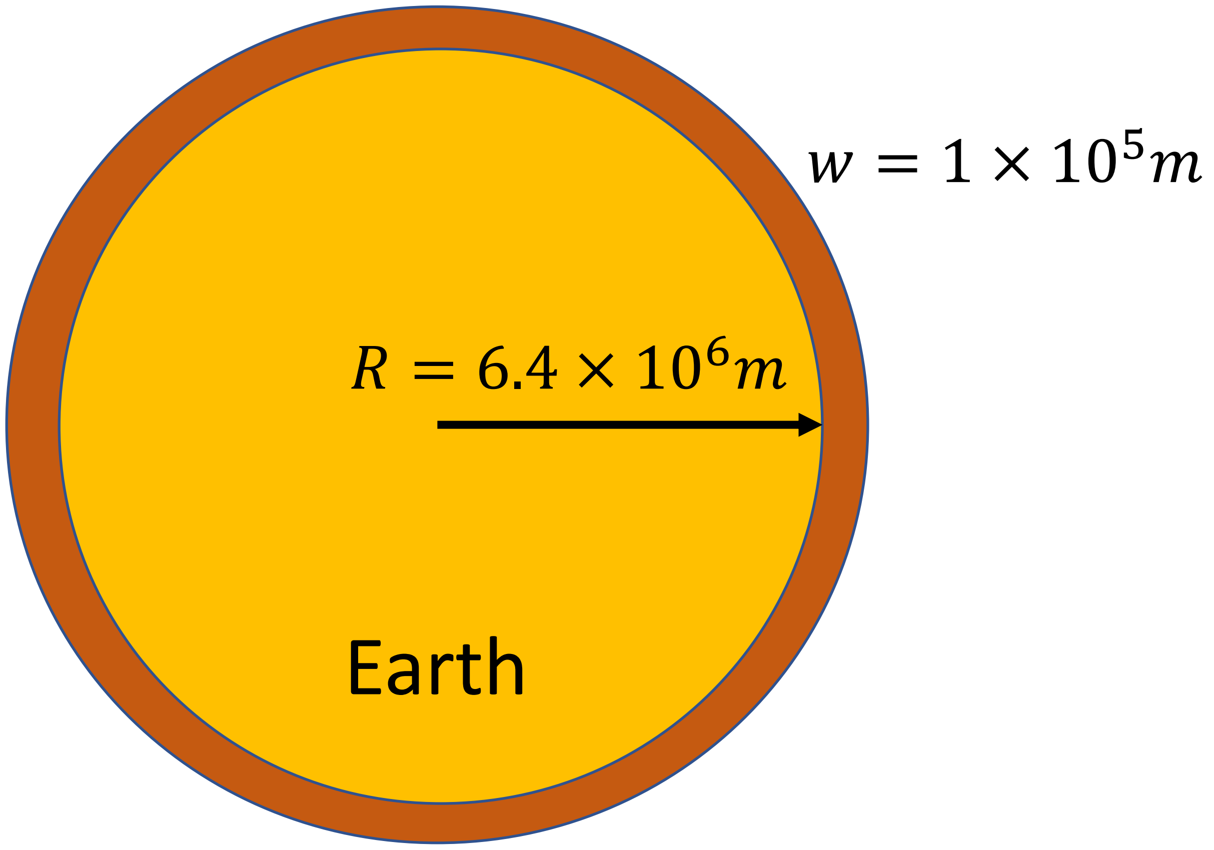
$$c_p = 700 \frac{J}{kg°C} \quad \rho = 1.3 \frac{kg}{m^3}$$



$$\text{characteristic time}, t_0 = \frac{w\rho c_p V}{Ak}$$

$$A = 5 \times 3 \times 10 = 150\, m^2$$
$$V = 3 \times 10 \times 10 = 300\, m^3$$
$$w = 0.2\ \ m$$
$$k = 0.2\ \ \frac{J}{sm°C}\ \text{(wood)}$$
$$c_p = 700 \frac{J}{kg°C} \quad \rho = 1.3 \frac{kg}{m^3}$$

$$\text{characteristic time}, t_0 = \frac{1}{c} = \frac{w\rho c_p V}{Ak} = \frac{0.2 \times 1.3 \times 700 \times 300}{150 \times 0.2} \frac{m\, kg\, Jsm°C}{m^3 kg°Cm^2 mJ}$$
$$= 1820 \frac{m\, kg\, Jm^3 sm°C}{m^3 kg°Cm^2 J} = 1820s$$



$$V = \frac{4}{3}\pi R^3$$

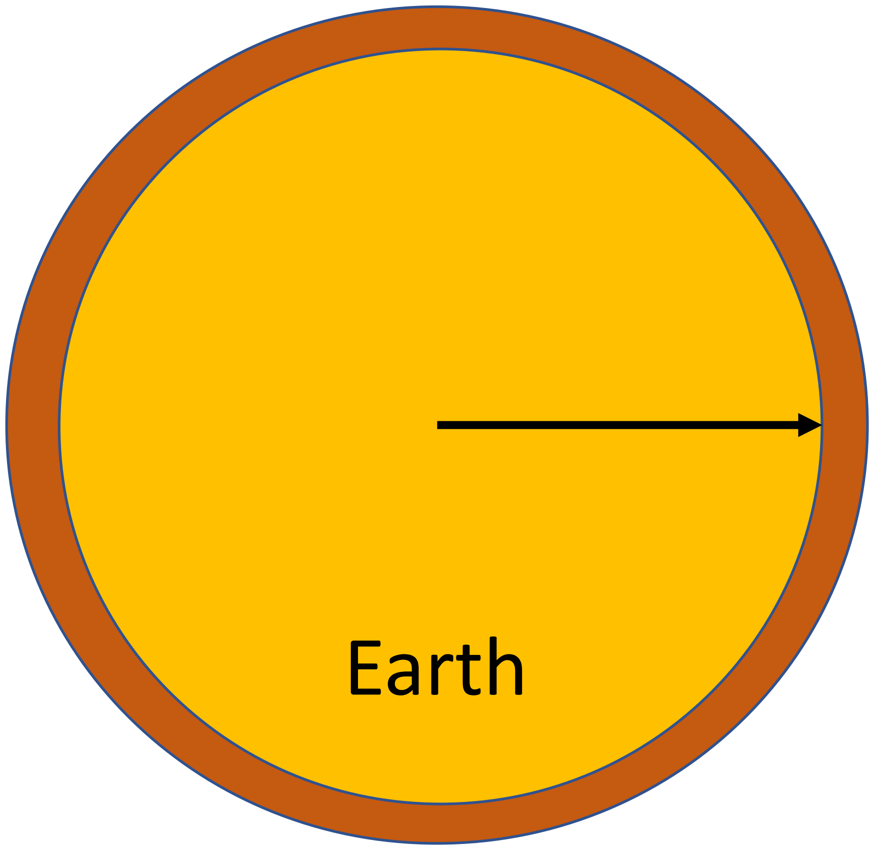
$$A = 4\pi R^2$$

$$\rho = 5000 \frac{kg}{R^3}$$

$$k = 0.27 \frac{J}{sm°C}$$

$$c_p = 950 \frac{J}{kg°C}$$

t_0: about 10 billion years



$$R = 6.4 \times 10^6 m$$
$$V = \frac{4}{3}\pi R^3$$
$$A = 4\pi R^2$$

size, shape of earth

$$\rho = 5000 \frac{kg}{R^3}$$

density of deep earth rocks

$$c_p = 950 \frac{J}{kg°C}$$

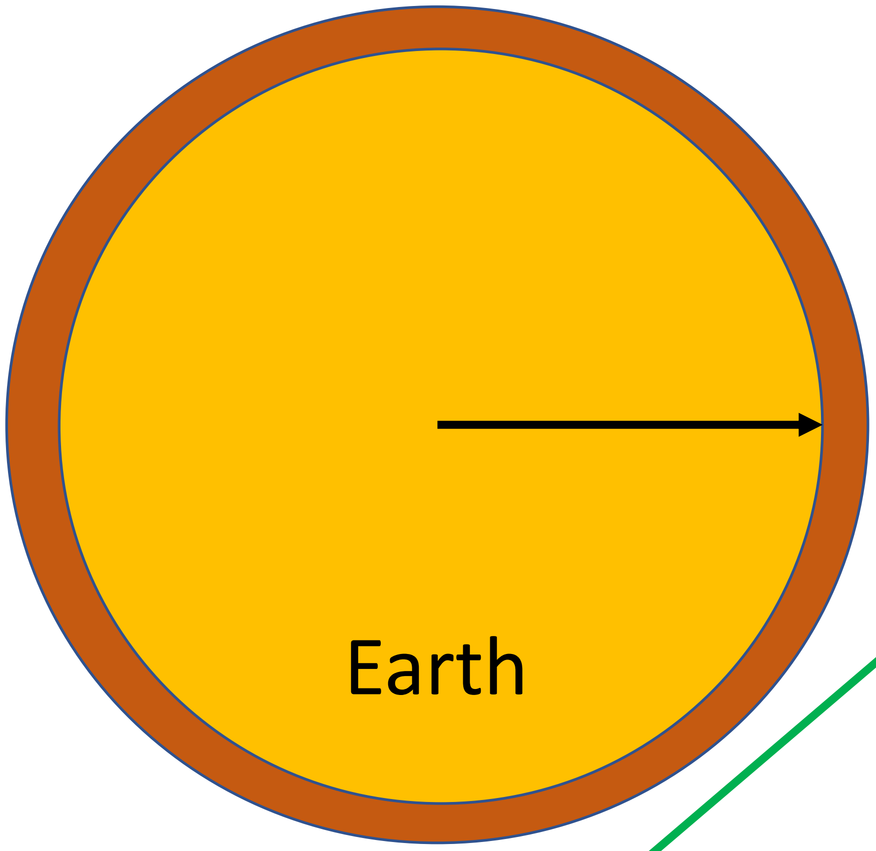
heat capacity of deep earth rocks

$$k = 0.27 \frac{J}{sm°C}$$

thermal conductivity of lithospheric rocks

$$w = 1 \times 10^5 m$$

thickness of lithosphere



$$R = 6.4 \times 10^6 m$$

$$V = \frac{4}{3}\pi R^3$$

$$A = 4\pi R^2$$

$$\rho = 5000 \frac{kg}{R^3}$$

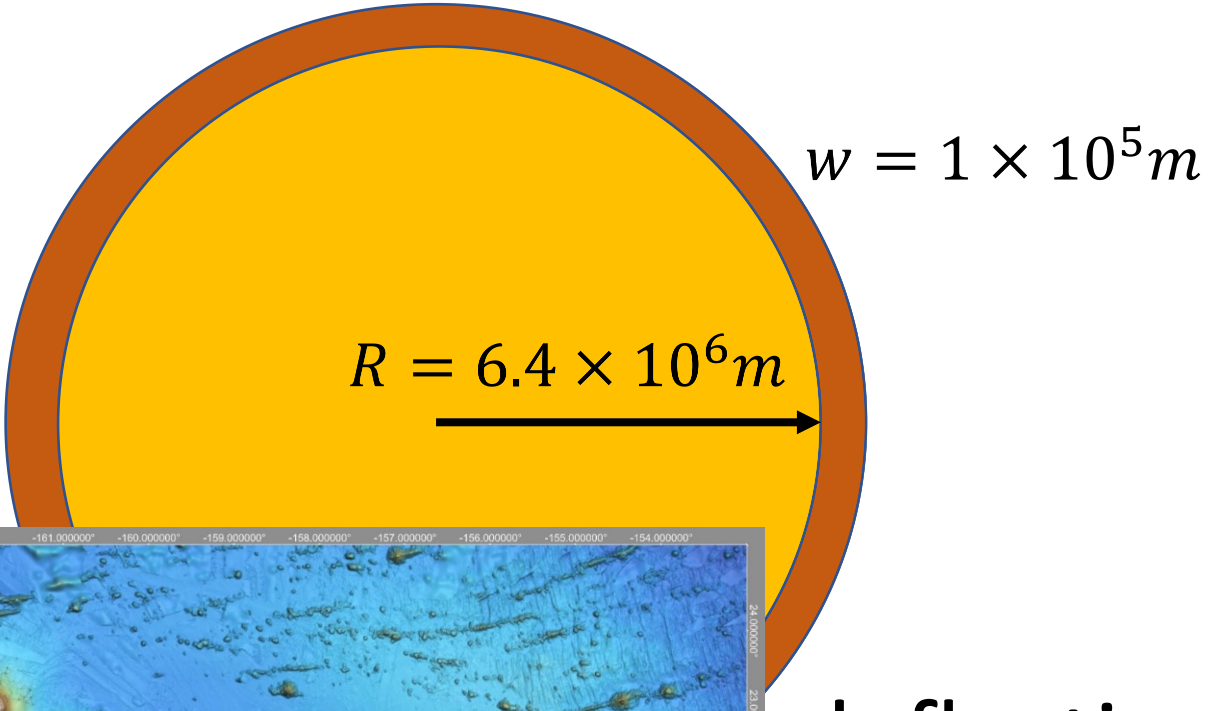
density of deep earth rocks

gravity

$$c_p = 950 \frac{J}{kg°\mathrm{C}}$$

$$k = 0.27 \frac{J}{sm°\mathrm{C}}$$

$$w = 1 \times 10^5 m$$



$$R = 6.4 \times 10^6 m$$

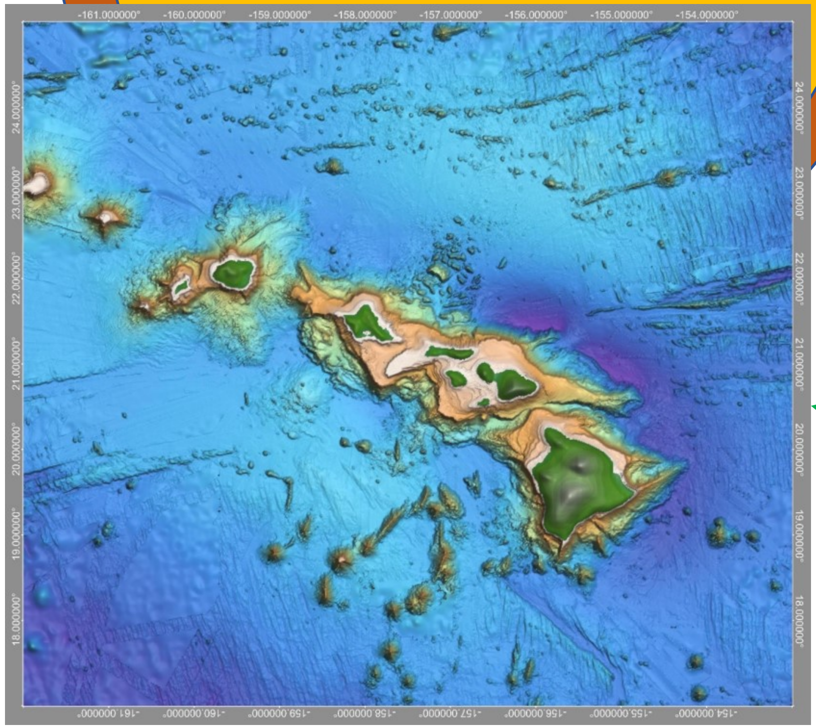
$$V = \frac{4}{3}\pi R^3$$

$$A = 4\pi R^2$$

$$\rho = 5000 \frac{kg}{R^3}$$

$$c_p = 950 \frac{J}{kg°C}$$

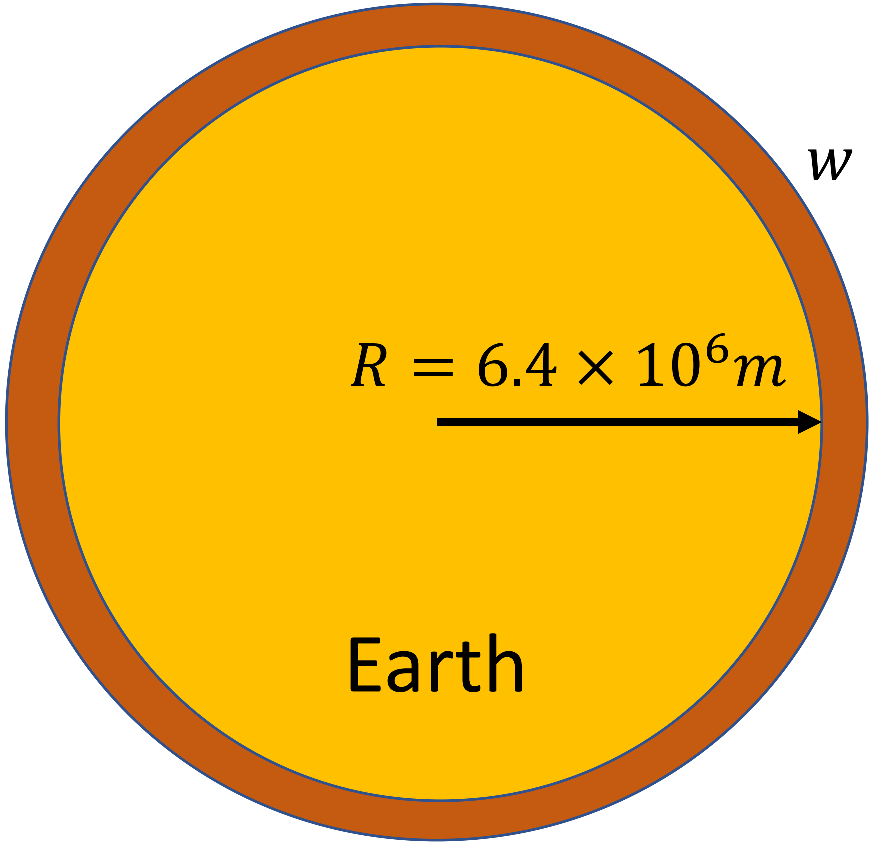
$$k = 0.27 \frac{J}{sm°C}$$



deflection of seafloor from volcano

$$w = 1 \times 10^5 m$$

thickness of lithosphere



$w = 1 \times 10^5 m$

$$R = 6.4 \times 10^6 m$$
$$V = \frac{4}{3}\pi R^3$$
$$A = 4\pi R^2$$

size, shape of earth

$$\rho = 5000 \frac{kg}{R^3}$$

density of deep earth rocks

$$c_p = 950 \frac{J}{kg°C}$$

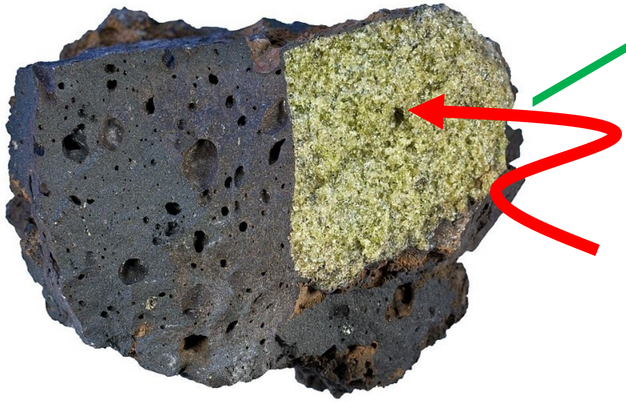
heat capacity of deep earth rocks

$$k = 0.27 \frac{J}{sm°C}$$

thermal conductivity of lithospheric rocks

$$w = 1 \times 10^5 m$$

thickness of lithosphere

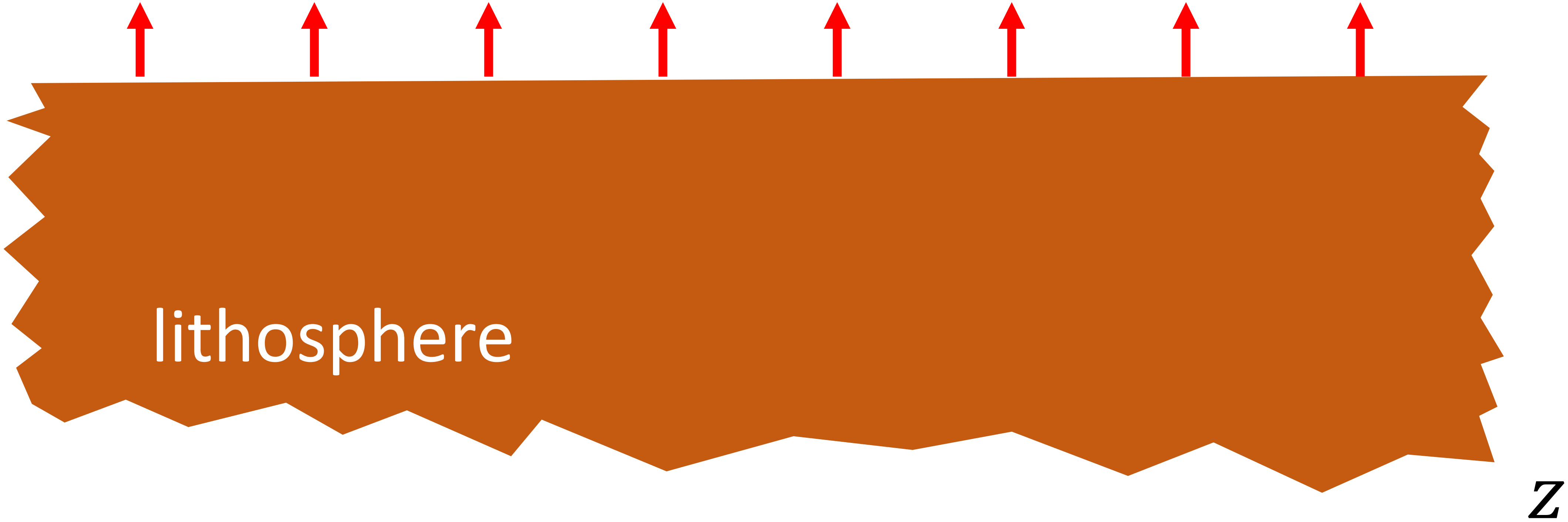


volcanic rock from volcano

Xenolith piece of the mantle

measuring heat flow

escaping heat, q $in - k$ $\frac{W}{m^2}$



measuring heat flow

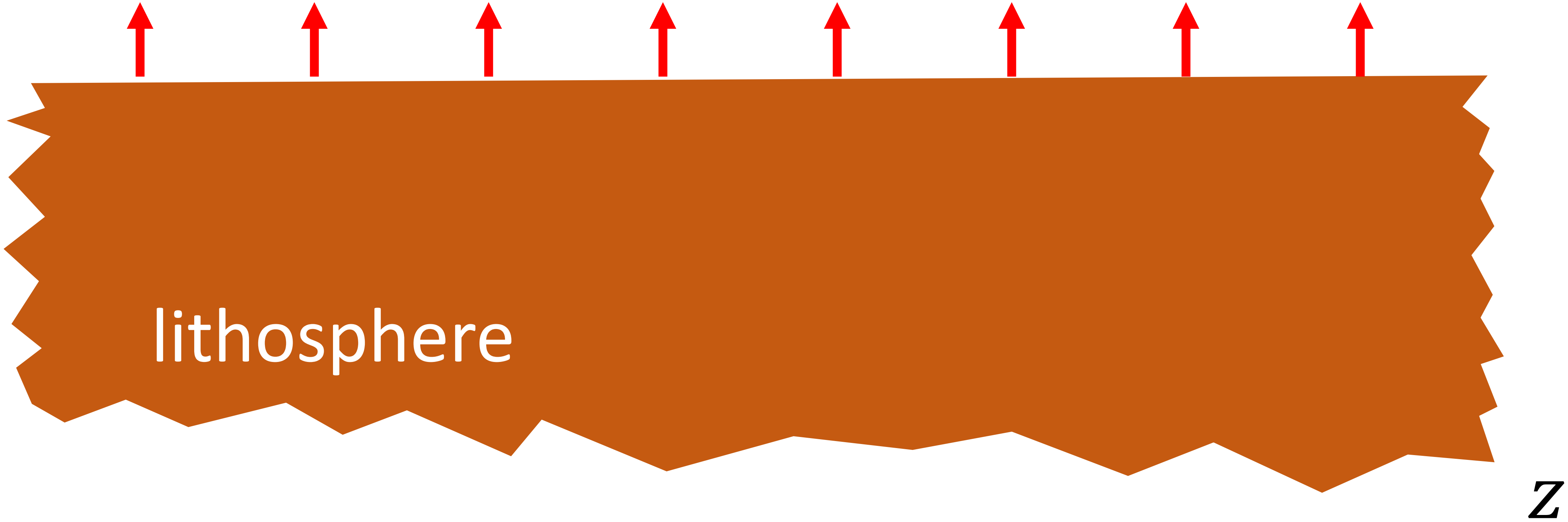
escaping heat, $q = -k \dfrac{d\Delta T}{dz}$

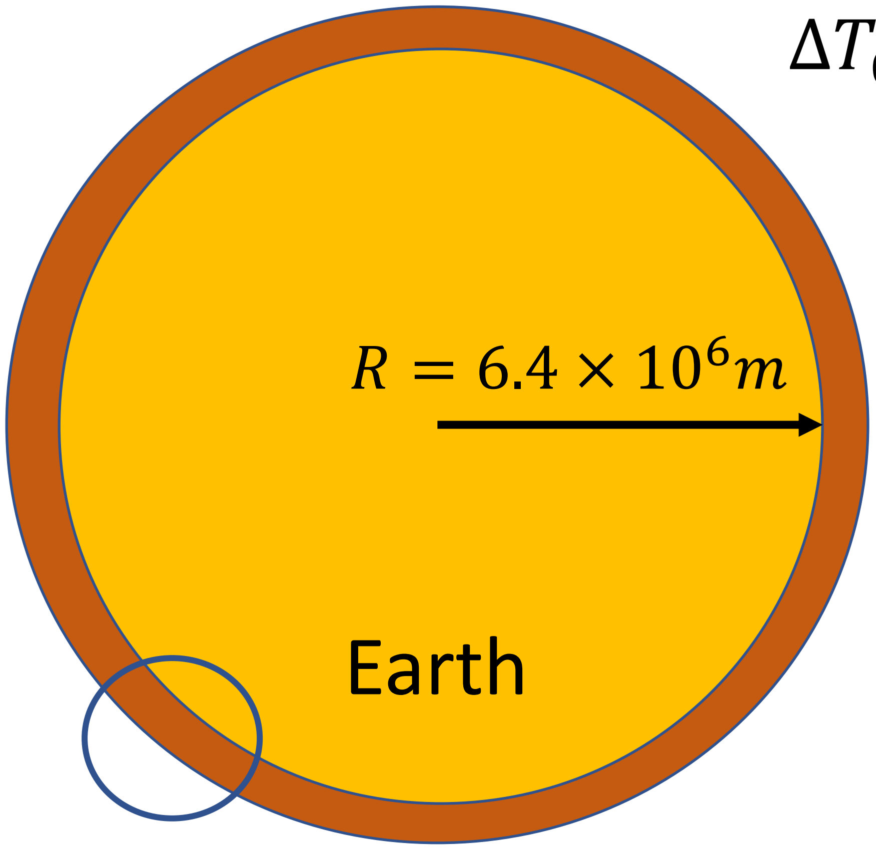
measuring heat flow

escaping heat, $q = -k \dfrac{d\Delta T}{dz}$

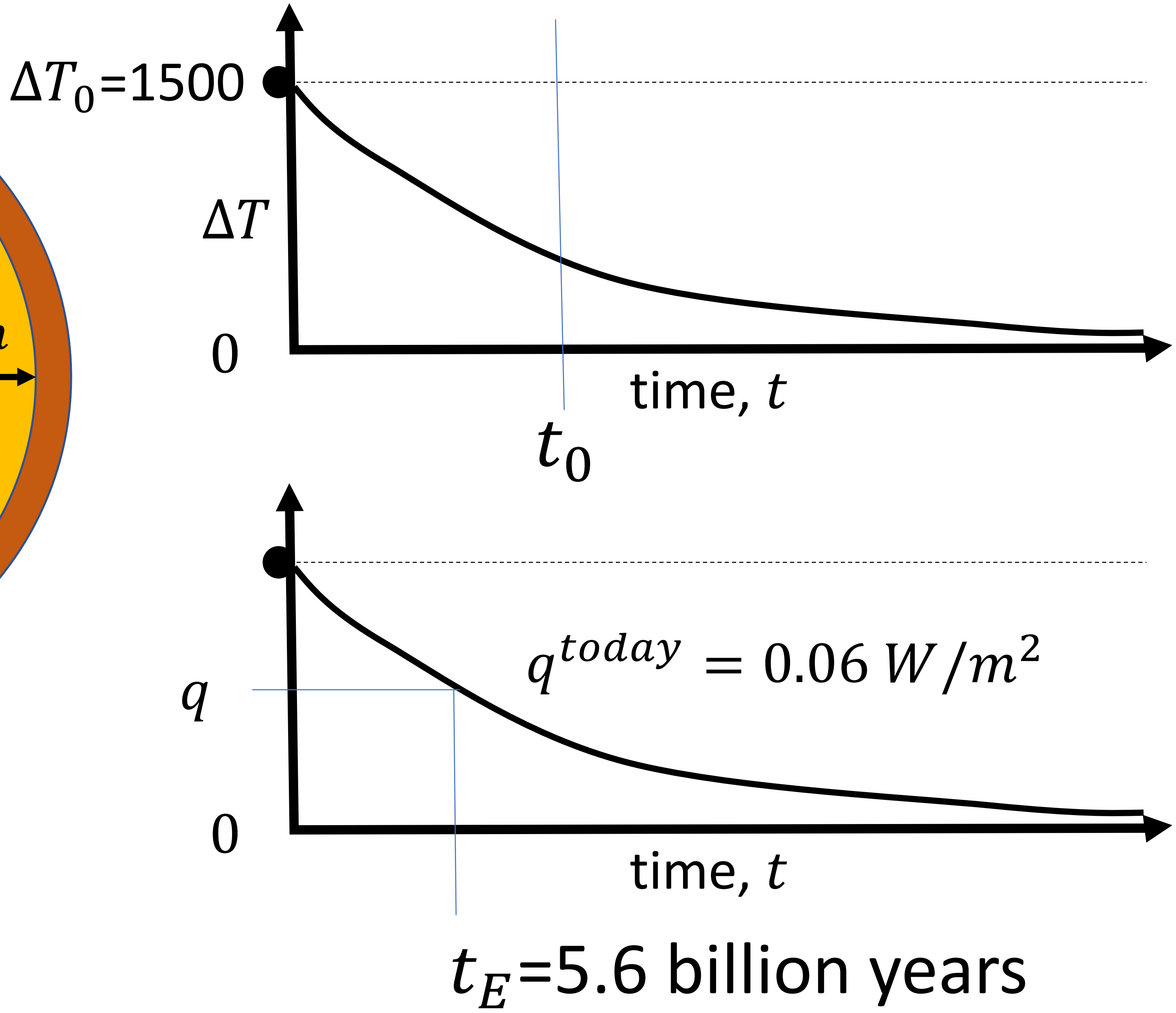
average heat flow of Earth

$$q = 0.06 \frac{W}{m^2}$$





$$q = \frac{k\Delta T}{w} = \frac{k\Delta T_0}{w} e^{-ct}$$



t_E=5.6 billion years

back of the envelope estimate of age of the Earth

neglects radioactive heating

dependent on ΔT_0=1500 (melting point of mantle rocks)

assumes lithosphere doesn't thicken with time

Back to the rod

$$\rho c_p \frac{d\Delta T}{dt} = k \frac{d^2 \Delta T}{dx^2}$$

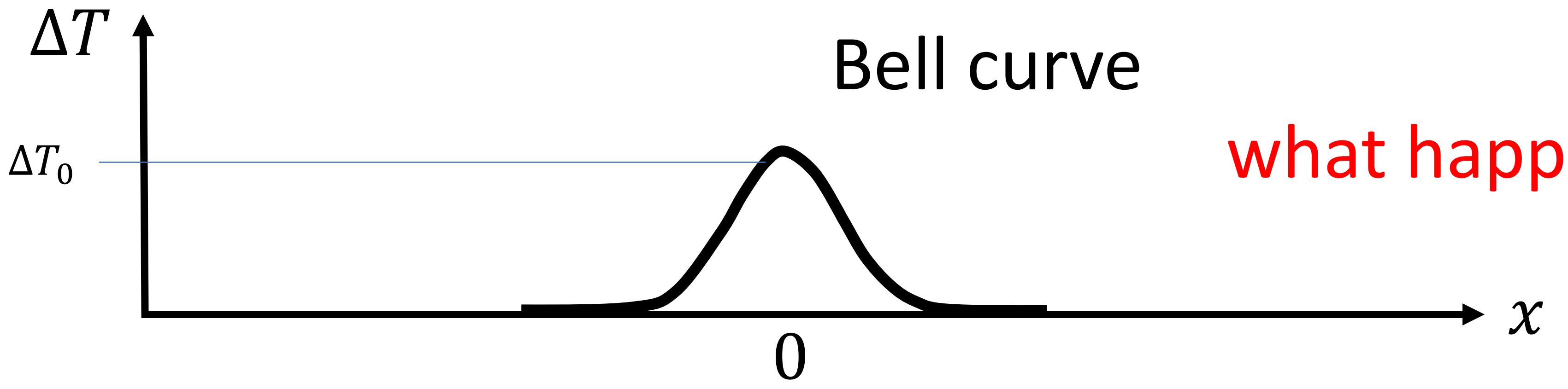
$$\frac{d\Delta T}{dt} = \kappa \frac{d^2 \Delta T}{dx^2} \qquad \kappa = \frac{k}{\rho c_p}$$

thermal diffusivity

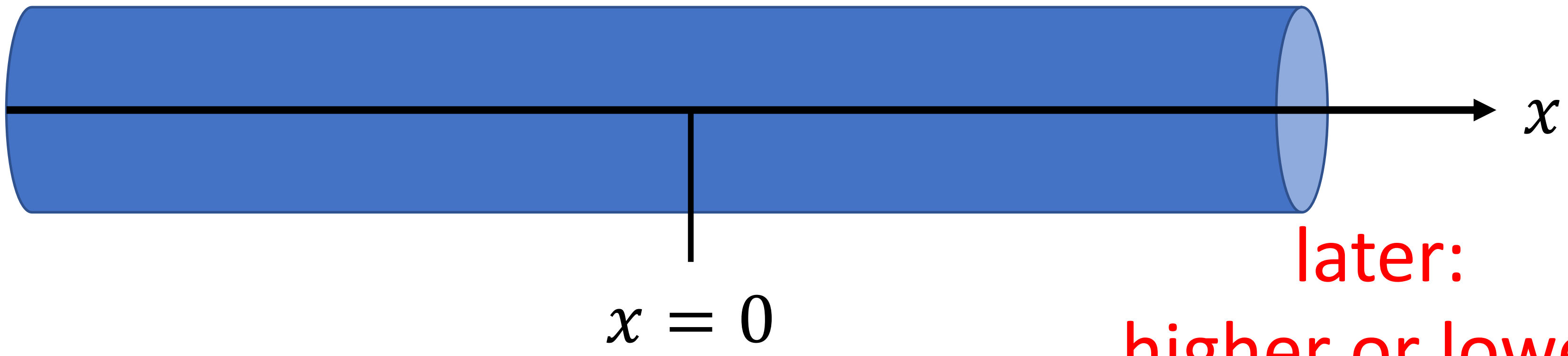
Back to the rod

$$\frac{d\Delta T}{dt} = \kappa \frac{d^2\Delta T}{dx^2}$$

away from the ends of the bar, at time, $t = now$



what happens?

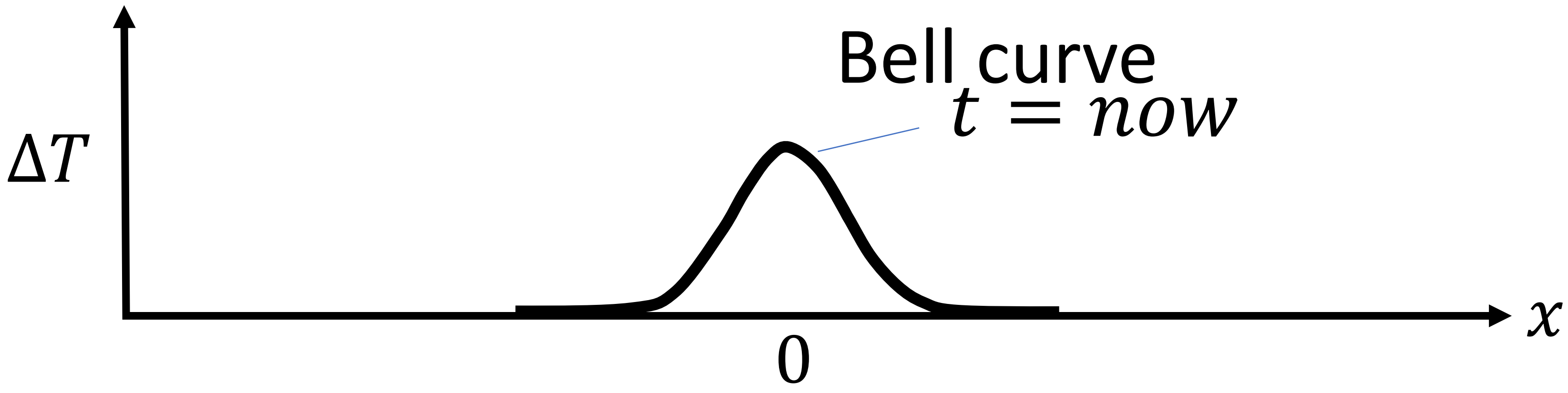


later:

higher or lower?

narrower or wider?

center position?



higher or lower?

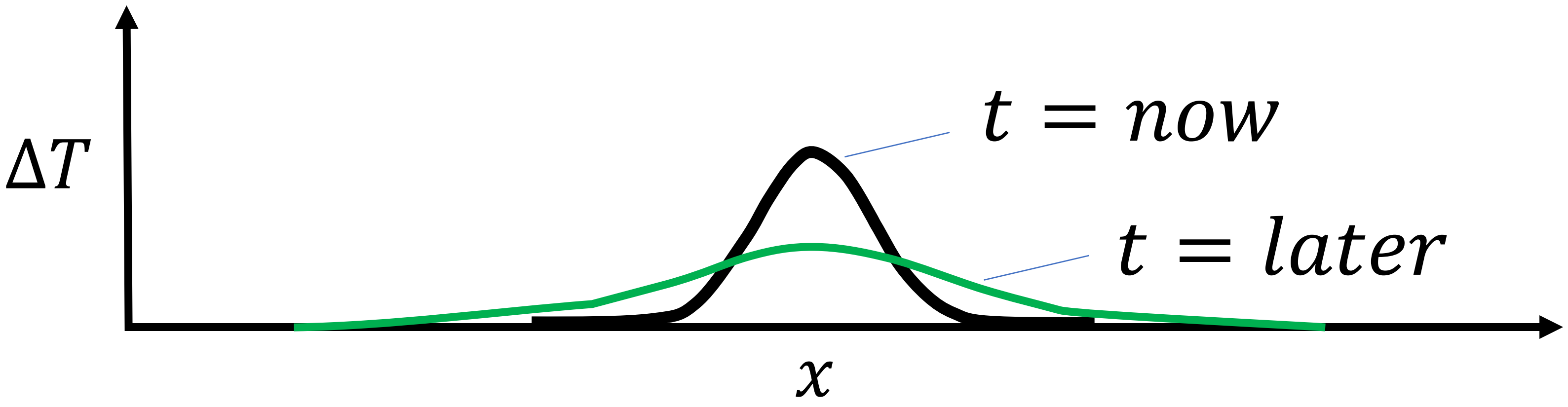
lower

narrower or wider?

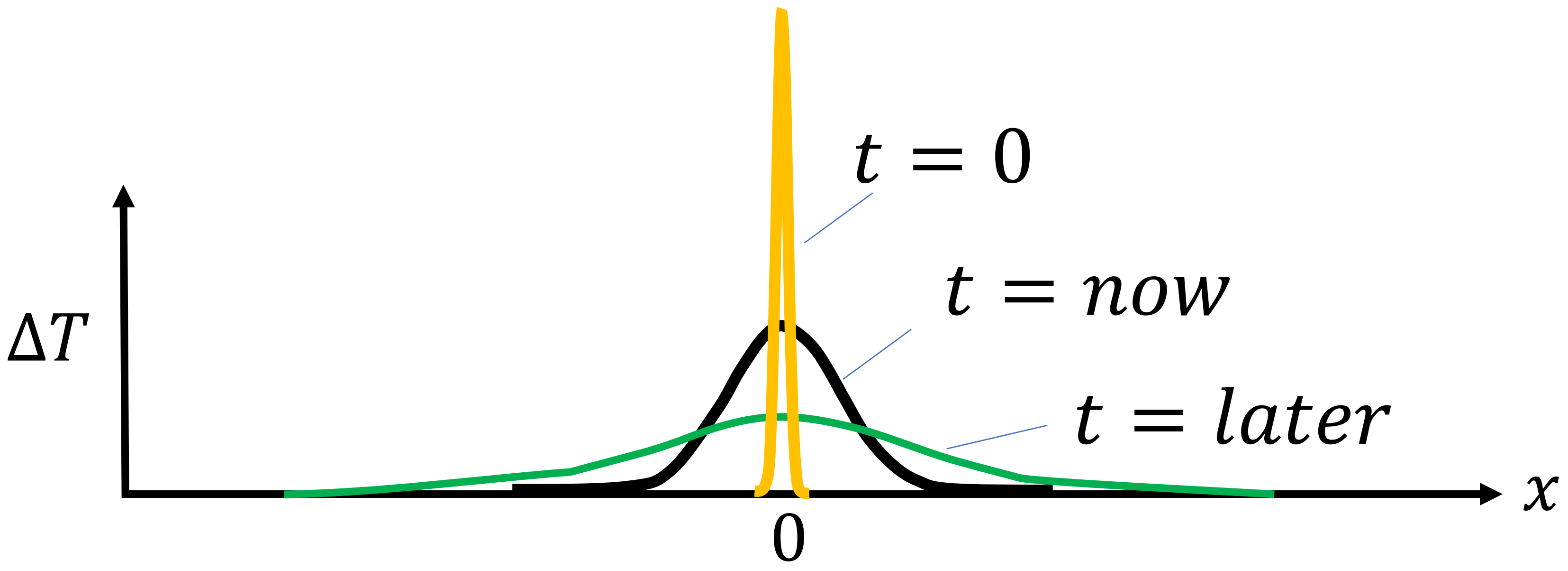
wider

center position?

same

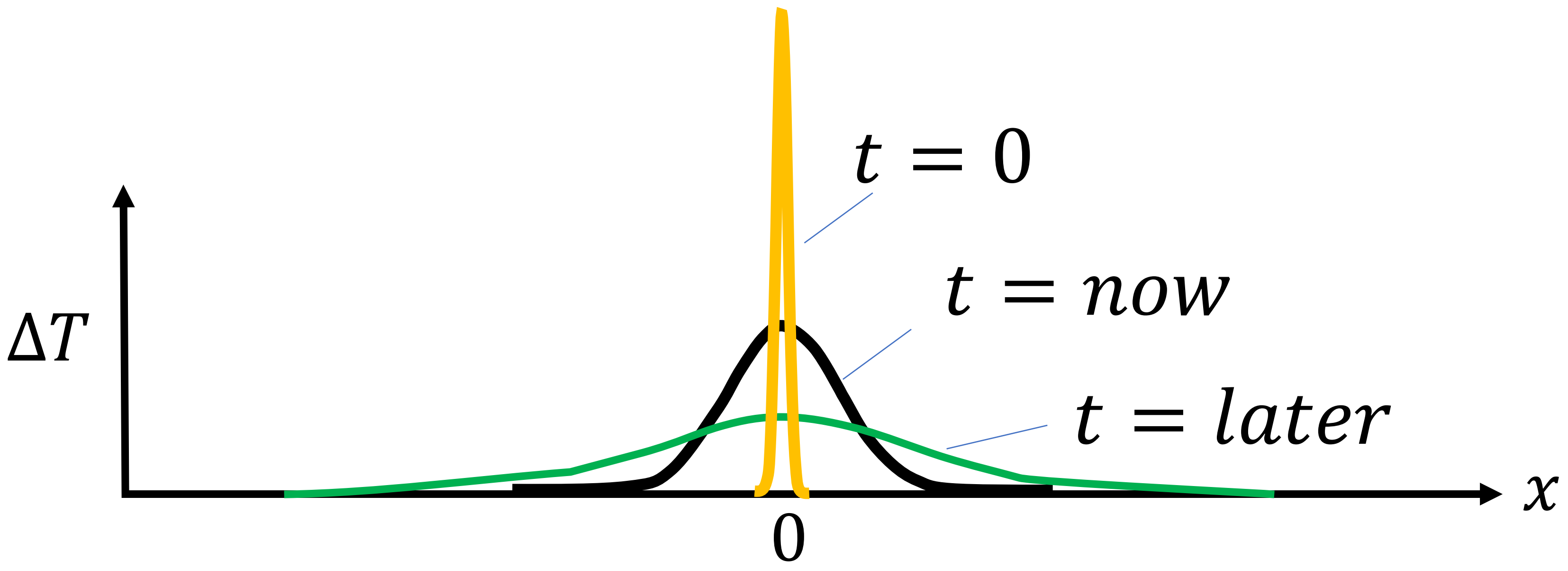


$$\Delta T = \frac{Q_0}{2\pi\rho c_p\sqrt{2\kappa t}} \exp\left\{-\frac{x^2}{4\kappa t}\right\}$$



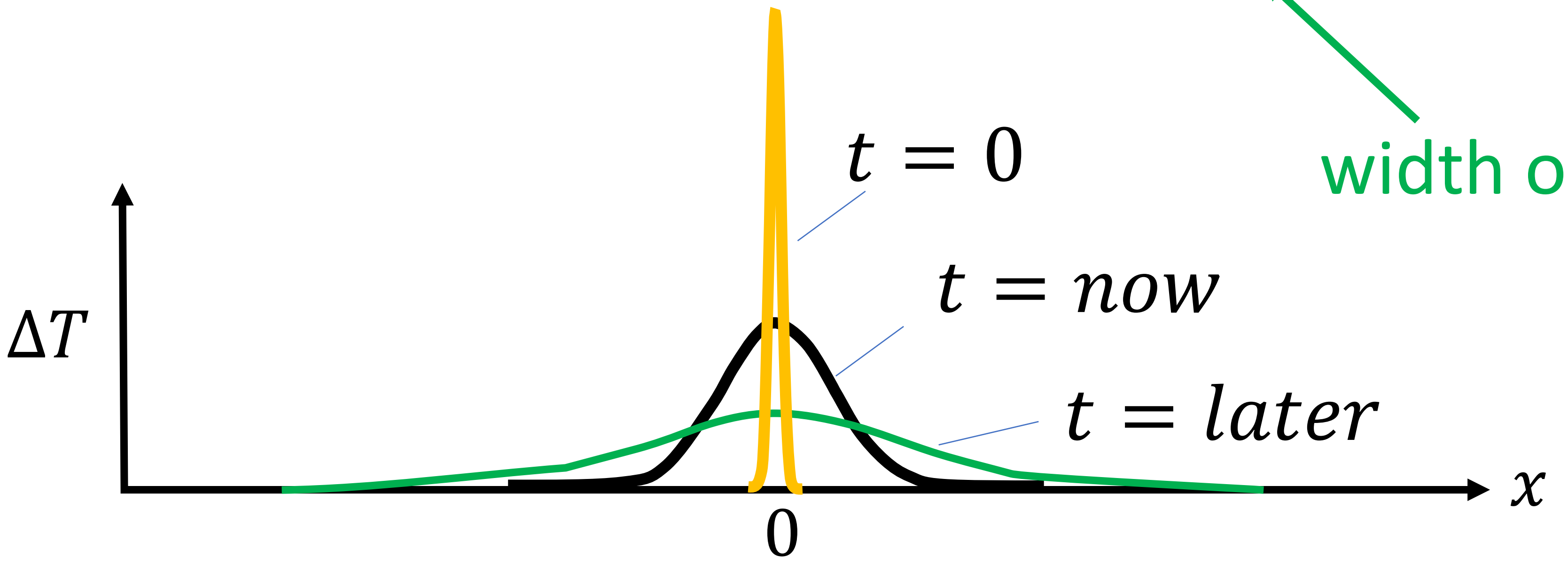
total amount of heat → Q_0

$$\Delta T = \frac{Q_0}{2\pi\rho c_p\sqrt{2\kappa t}}\,\exp\left\{-\frac{x^2}{4\kappa t}\right\}$$



$$\Delta T = \frac{Q_0}{2\pi\rho c_p\sqrt{2\kappa t}}\exp\left\{-\frac{x^2}{4\kappa t}\right\}$$

width of bell curve



Bell Curve or Gaussian Curve or Normal Curve

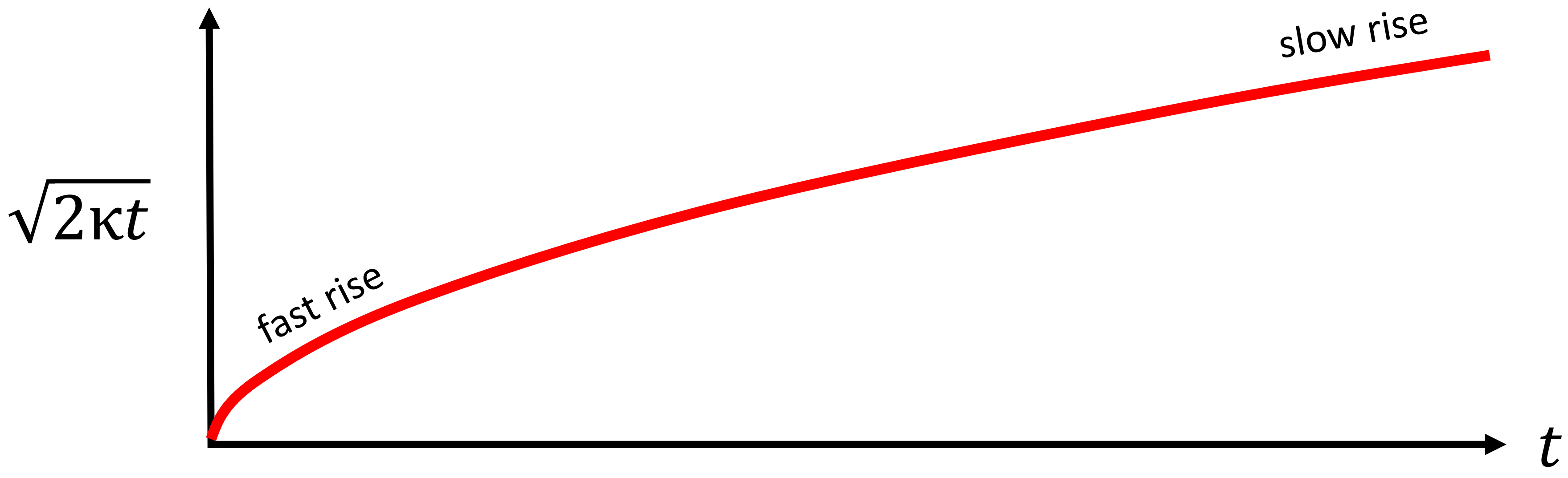
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}$$

width or standard deviation

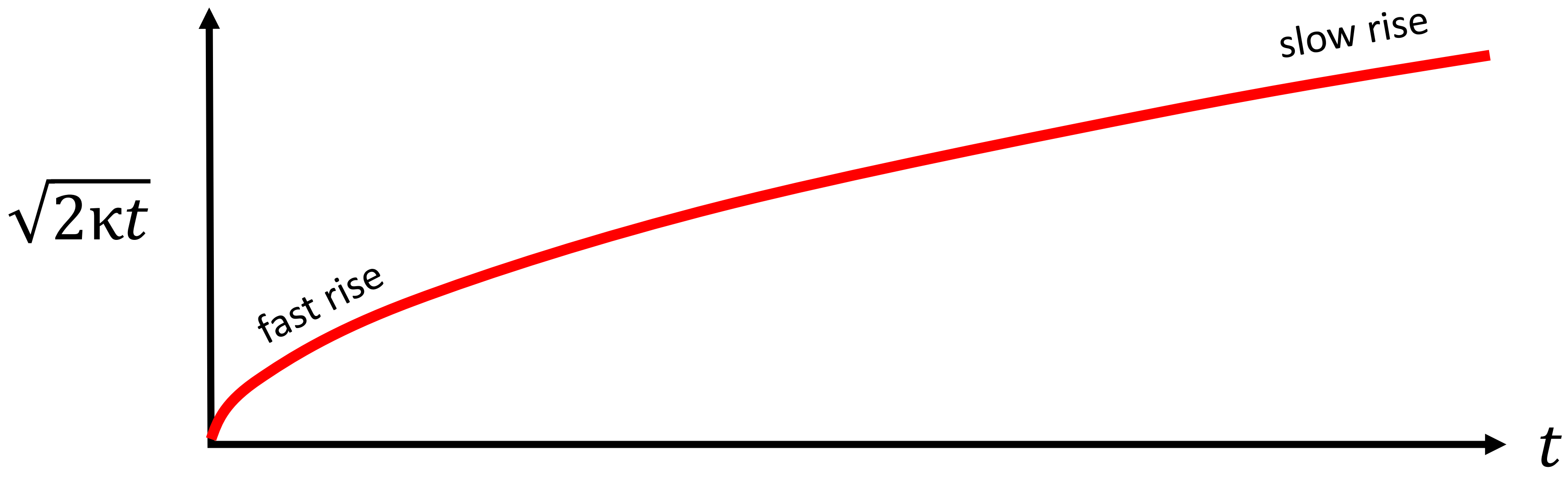
in the cooling formula, width grows as $\sigma = \sqrt{2\kappa t}$

area under Bell Curve $f(x)$ is 1

area under $\rho c_p \Delta T(x)$ is Q_0 is constant; "heat is conserved"



put in words?



put in words?

initially widens very quickly, then slows down

then slows way down


volcanic dikes crossing a hillside
(Iceland)

Time to double width ... proxy for time to cool significantly

$$\sigma_1 = \sqrt{2\kappa t_1} \qquad \sigma_1^2 = 2\kappa t_1 \qquad t_1 = \frac{\sigma_1^2}{2\kappa}$$

$$2\sigma_1 = \sqrt{2\kappa t_2} \qquad 4\sigma_1^2 = 2\kappa t_2 \qquad t_2 = \frac{2\sigma_1^2}{\kappa}$$

$$t_2 - t_1 = \frac{2\sigma_1^2}{\kappa} - \frac{\sigma_1^2}{2\kappa} = \frac{3\sigma_1^2}{2\kappa}$$

Time to double width

$$\Delta t = t_2 - t_1 = \frac{3\rho c_p \sigma_1^2}{2k}$$

for $\sigma_1 = 1$ m Bell Curve of hot rock (a “dike”)

Time to double width

$$\Delta t = t_2 - t_1 = \frac{3\rho c_p \sigma_1^2}{2k}$$

for $\sigma_1 = 1$ m Bell Curve of hot rock (a “dike”)

$\rho = 2500\ kg/m^3$

$k = 3.1\ J/sm°\mathrm{C}$

$c_p = 800\ J/kg°\mathrm{C}$

$$\Delta t = \frac{3\rho c_p \sigma_1^2}{2k} = \frac{3 \times 2500 \times 800 \times 1}{2 \times 3.1}$$

$$\frac{kg \times J \times m^2 \times sm°\mathrm{C}}{m^3 kg°\mathrm{C} \times J}$$

$$\Delta t = 968000\ \mathrm{s}$$

about 11 days