Solid Earth Dynamics

Bill Menke, Instructor

Lecture 5

Today:

Advection

Convection

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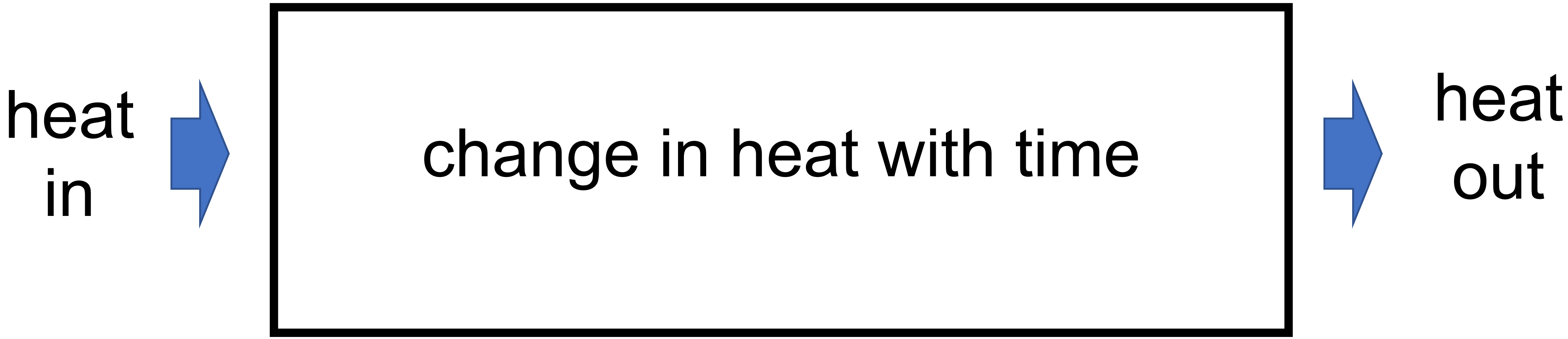
Advection

model of a hot spring

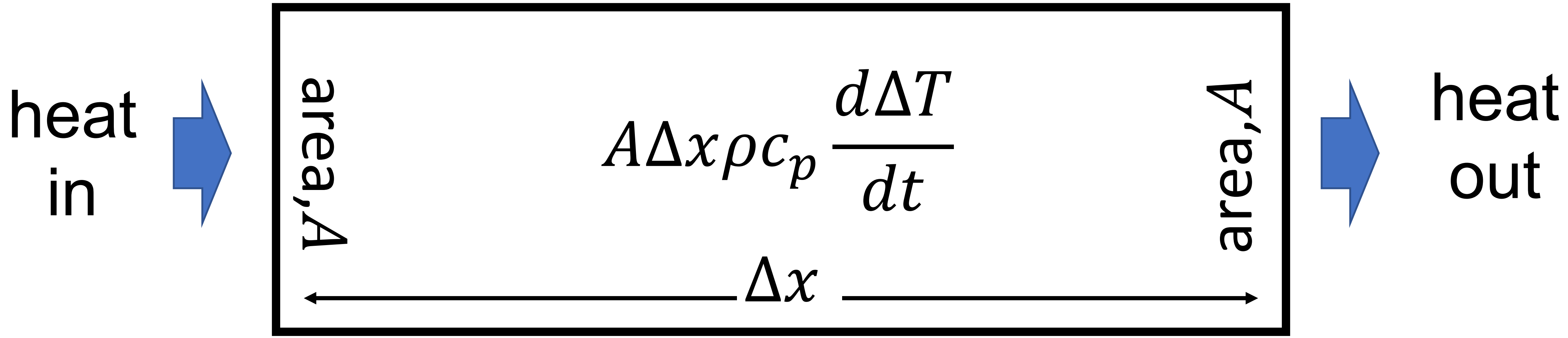
Convection

test of whether or not convection will occur

conservation of energy when fluid motion moves heat



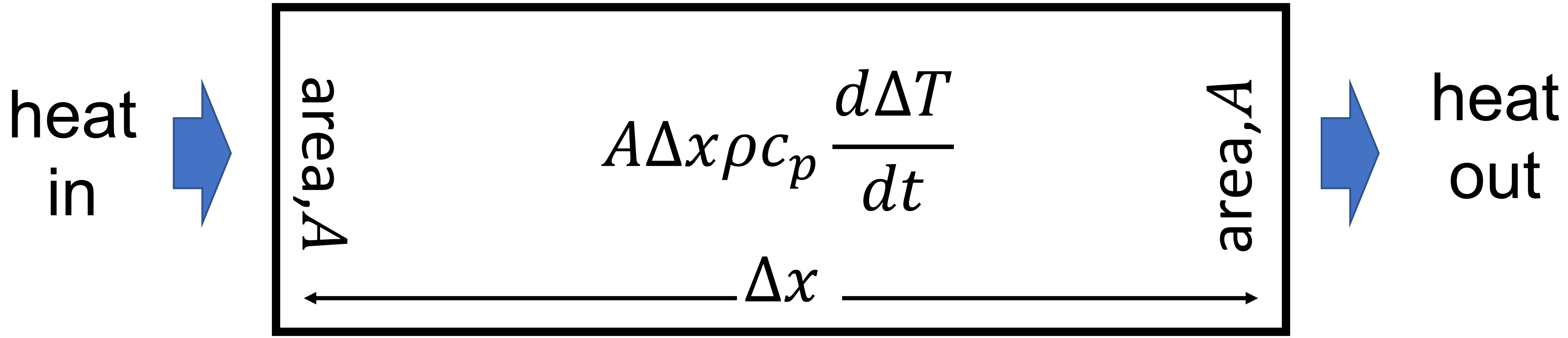
conservation of energy when fluid motion moves heat



$$A\rho c_p \Delta T(0) v(0)$$

$$A\rho c_p \Delta T(\Delta x)\, v(\Delta x)$$

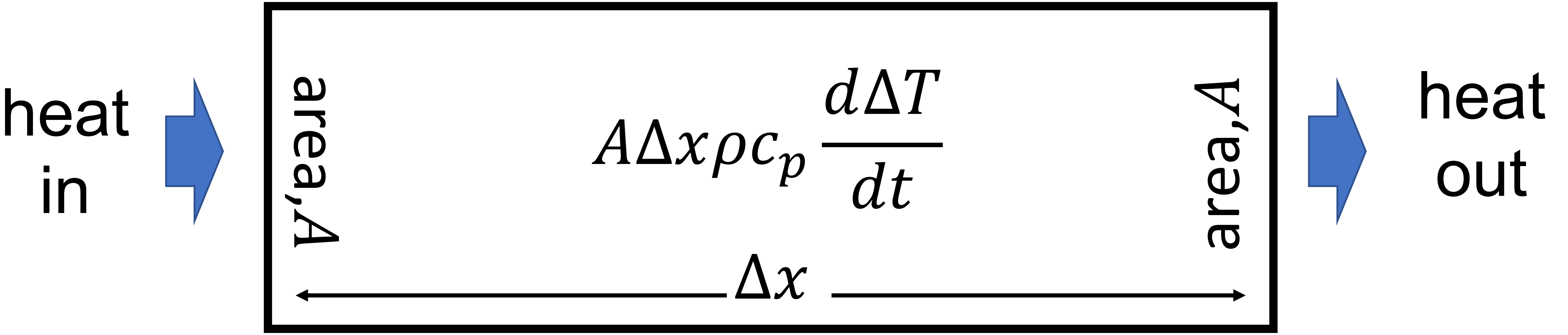
conservation of energy when fluid motion moves heat



$A\rho c_p \Delta T(0) v$ $\quad$ $A\rho c_p \Delta T(\Delta x)\, v$

incompressible assumption: $v(0) = v(\Delta x)$

conservation of energy when fluid motion moves heat

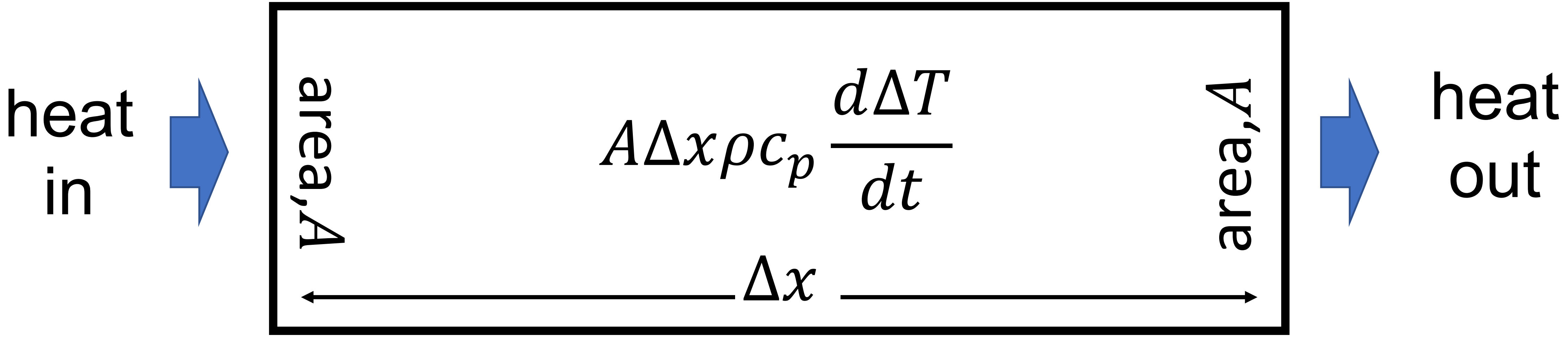


$A\rho c_p \Delta T(0) v$

$A\rho c_p \Delta T(\Delta x)\, v$

$$A\Delta x \rho c_p \frac{d\Delta T}{dt} = -A\rho c_p \Delta T(\Delta x)\ + vA\rho c_p \Delta T(0) v$$

conservation of energy when fluid motion moves heat

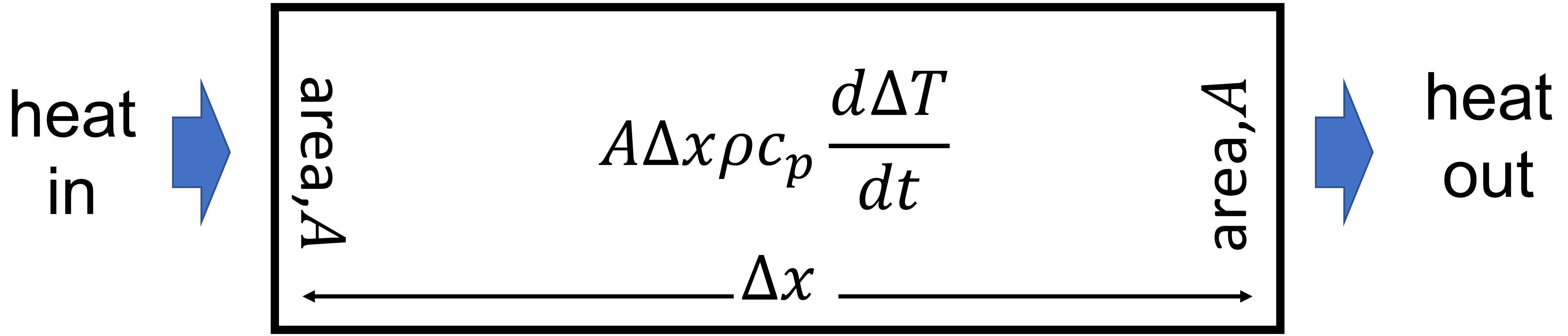


$A\rho c_p \Delta T(0) v$

$A\rho c_p \Delta T(\Delta x)\, v$

$$\rho c_p \frac{d\Delta T}{dt} = -\rho c_p v \frac{\Delta T(\Delta x) + \Delta T(0)}{\Delta x}$$

conservation of energy when fluid motion moves heat



$A\rho c_p \Delta T(0) v$

$A\rho c_p \Delta T(\Delta x)\, v$

$$\rho c_p \frac{d\Delta T}{dt} = -\rho c_p v \frac{d\Delta T}{dx}$$

conservation of energy when fluid motion moves heat

$$\rho c_p \frac{d\Delta T}{dt} = -\rho c_p v \frac{d\Delta T}{dx}$$

conservation of energy when conduction moves heat

$$\rho c_p \frac{d\Delta T}{dt} = k \frac{d^2\Delta T}{dx^2}$$

conservation of energy when fluid motion moves heat

$$\rho c_p \frac{d\Delta T}{dt} = -\rho c_p v \frac{d\Delta T}{dx}$$

conservation of energy when conduction moves heat

$$\rho c_p \frac{d\Delta T}{dt} = k \frac{d^2\Delta T}{dx^2}$$

conservation of energy when both move heat

$$\rho c_p \frac{d\Delta T}{dt} = -\rho c_p v \frac{d\Delta T}{dx} + k \frac{d^2\Delta T}{dx^2}$$

conservation of energy when fluid motion moves heat

$$\rho c_p \frac{d\Delta T}{dt} = -\rho c_p v \frac{d\Delta T}{dx}$$

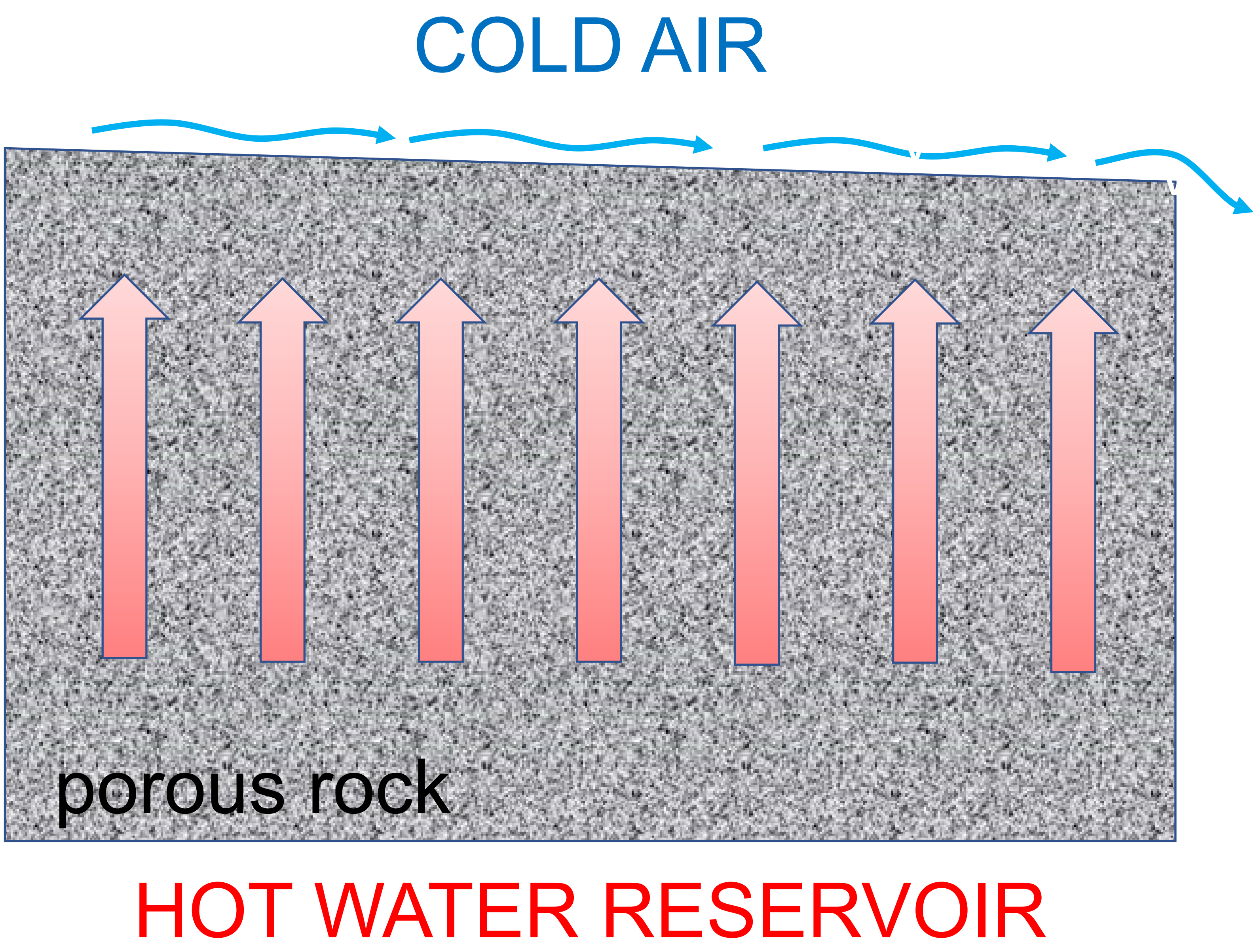
conservation of energy when conduction moves heat

$$\rho c_p \frac{d\Delta T}{dt} = k \frac{d^2\Delta T}{dx^2}$$

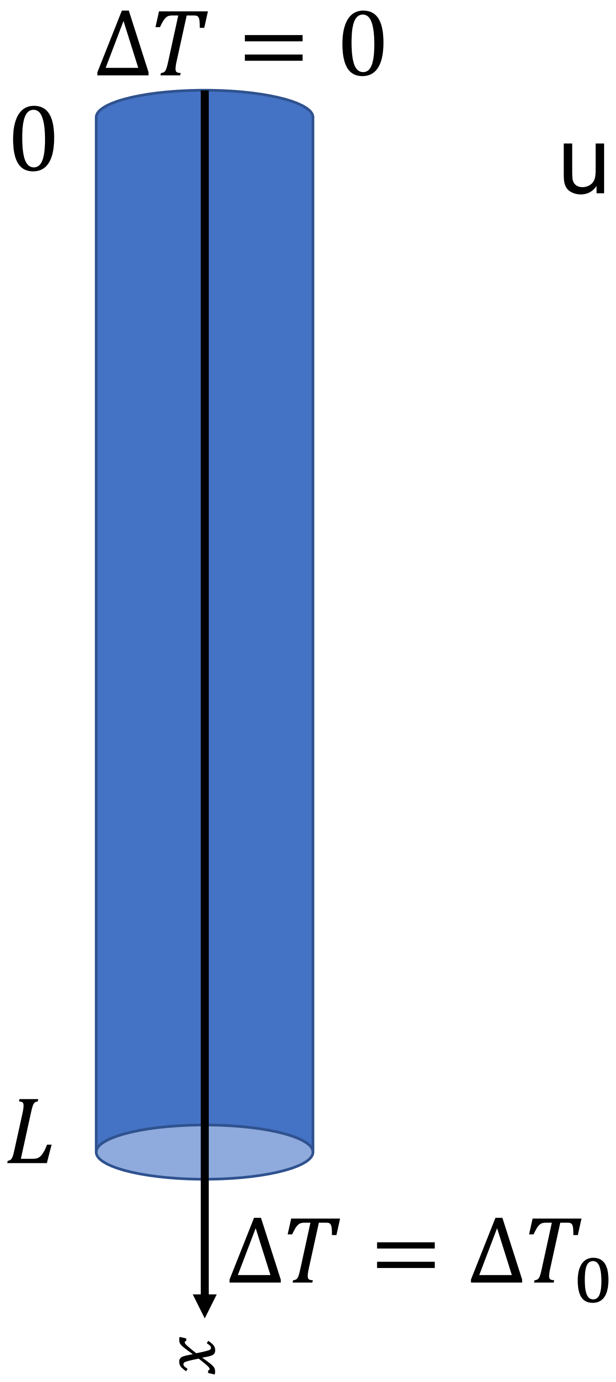
conservation of energy when both move heat

$$\rho c_p \left(\frac{d\Delta T}{dt} + v \frac{d\Delta T}{dx} \right) = k \frac{d^2\Delta T}{dx^2}$$

geothermal area in Iceland



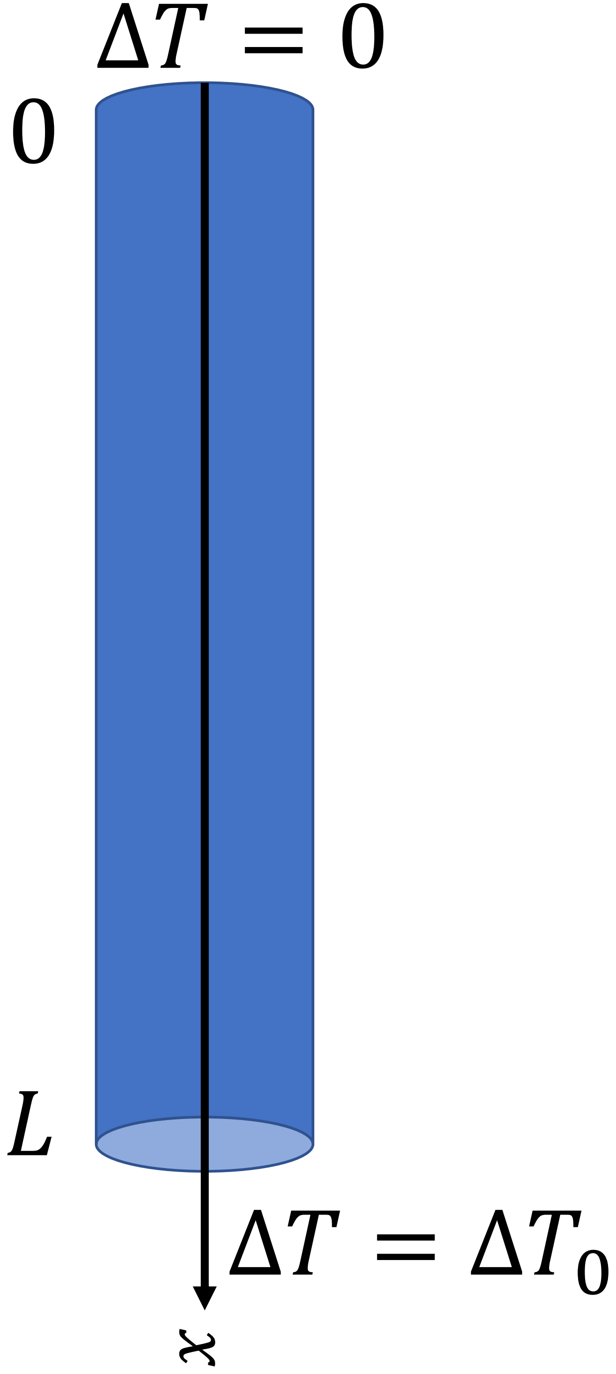
model of geothermal area in Iceland



upward conduction-advection of heat through porous rock

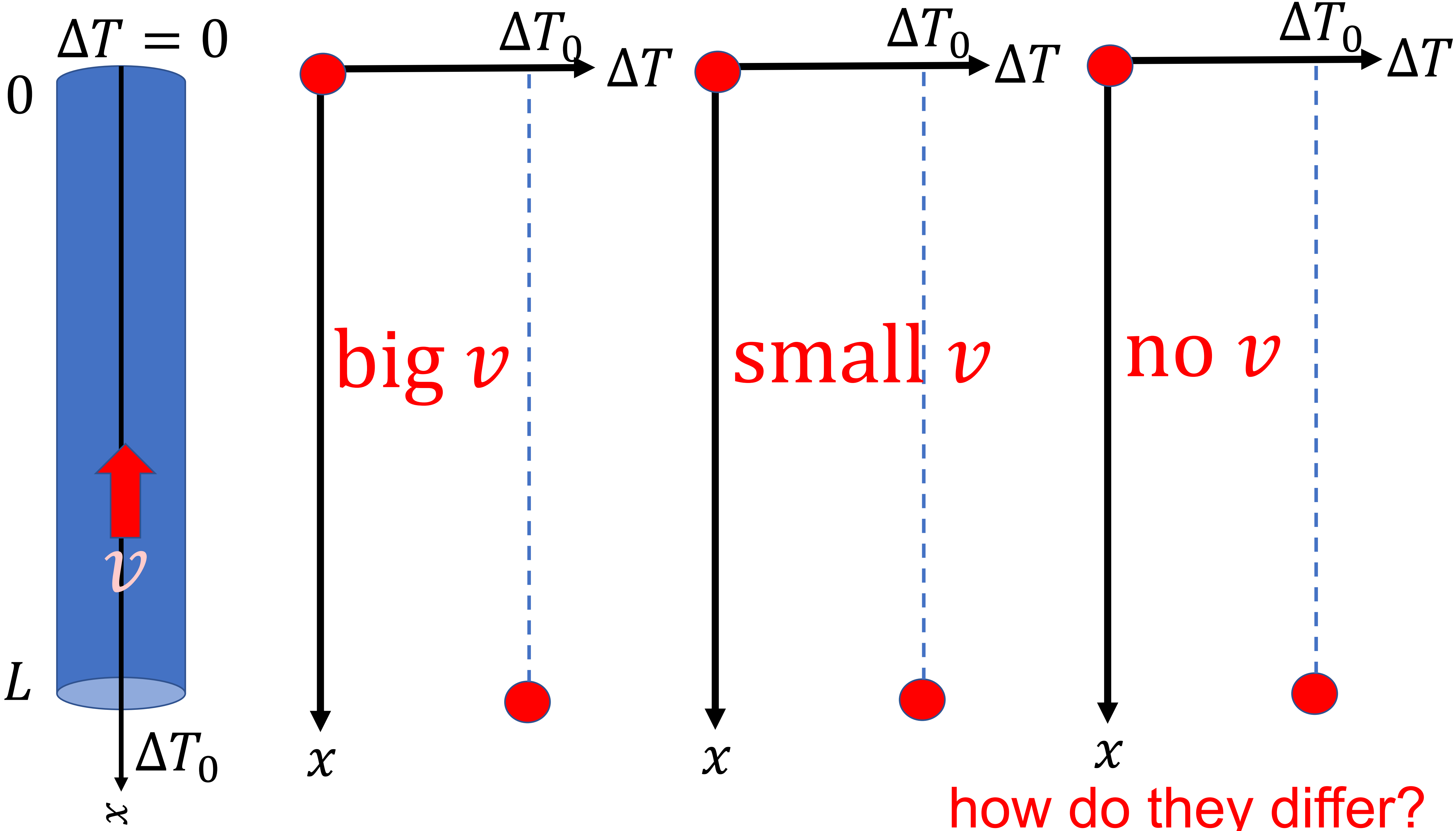
$$\rho c_p \left(\frac{d\Delta T}{dt} + v' \frac{d\Delta T}{dx} \right) = k \frac{d^2 \Delta T}{dx^2}$$

effective velocity, v’=φv,
porosity, φ

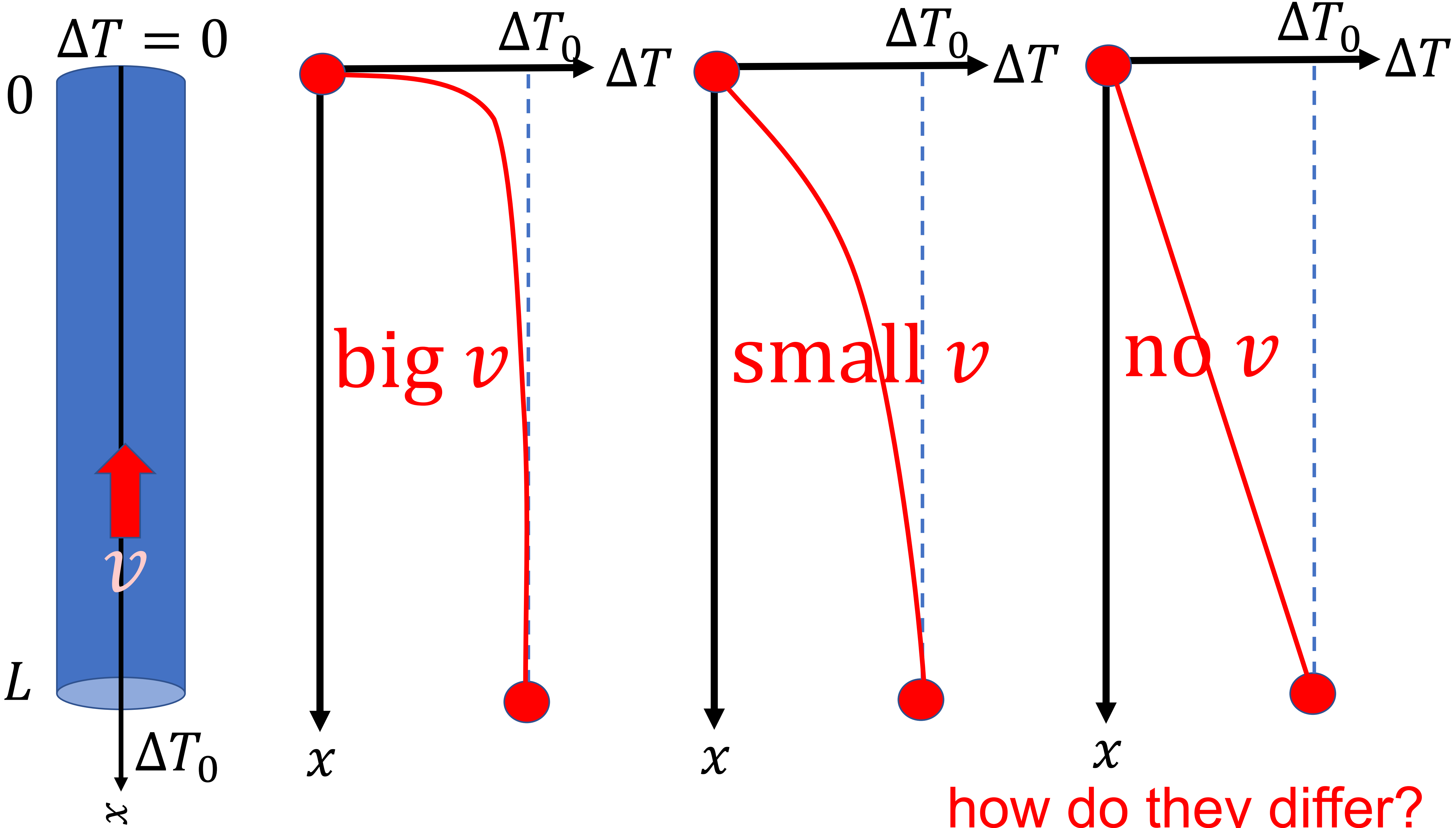


steady state $\frac{d\Delta T}{dt} = 0$

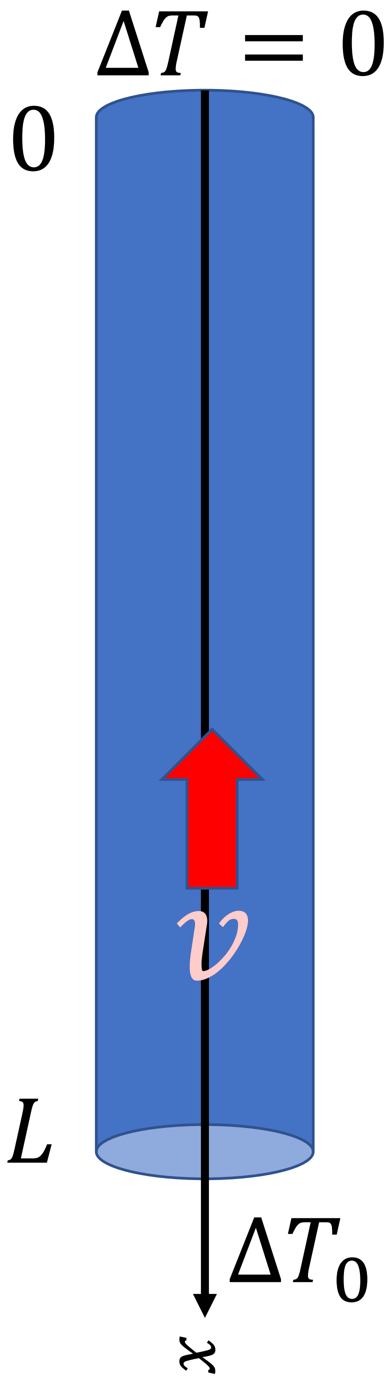
$$v' \frac{d\Delta T}{dx} = \kappa \frac{d^2 \Delta T}{dx^2}$$



how do they differ?



how do they differ?



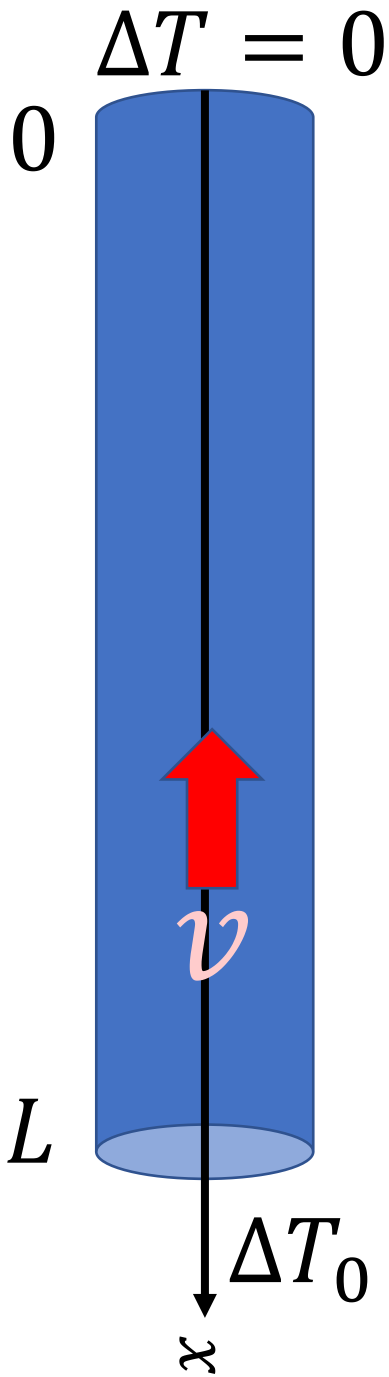
water
moving
up

$$-v'\frac{d\Delta T}{dx} = \kappa\frac{d^2\Delta T}{dx^2}$$

$$f = \frac{d\Delta T}{dx}$$

$$-\frac{v'}{\kappa}f = \frac{d}{dx}f$$

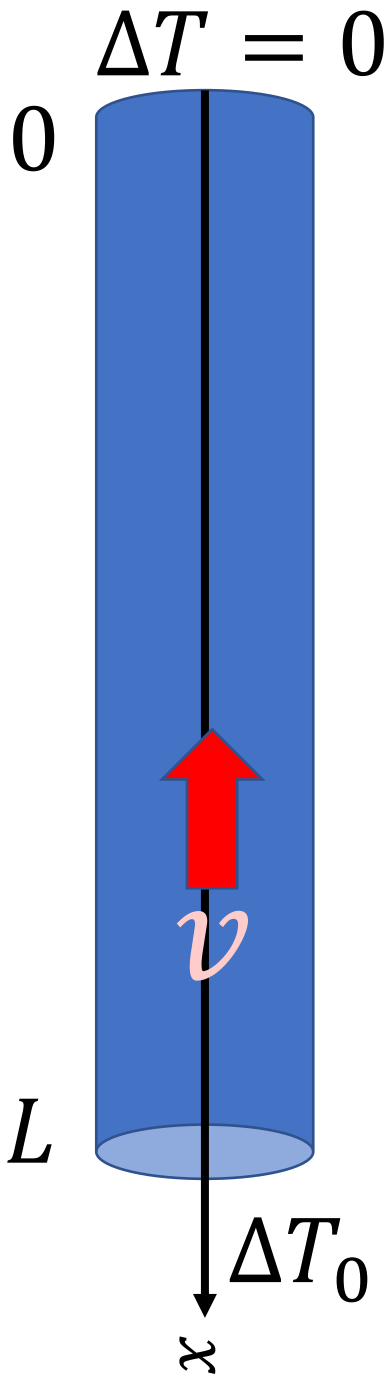
the derivative of a function
is proportional to the function
suggests an exponential



$$-c\frac{d\Delta T}{dx} = \frac{d^2\Delta T}{dx^2}$$

$$\Delta T = B\{\exp(-cx) + C\}$$

$$\text{with } c = \frac{v'}{\kappa}$$



$$-c\frac{d\Delta T}{dx} = \frac{d^2\Delta T}{dx^2}$$

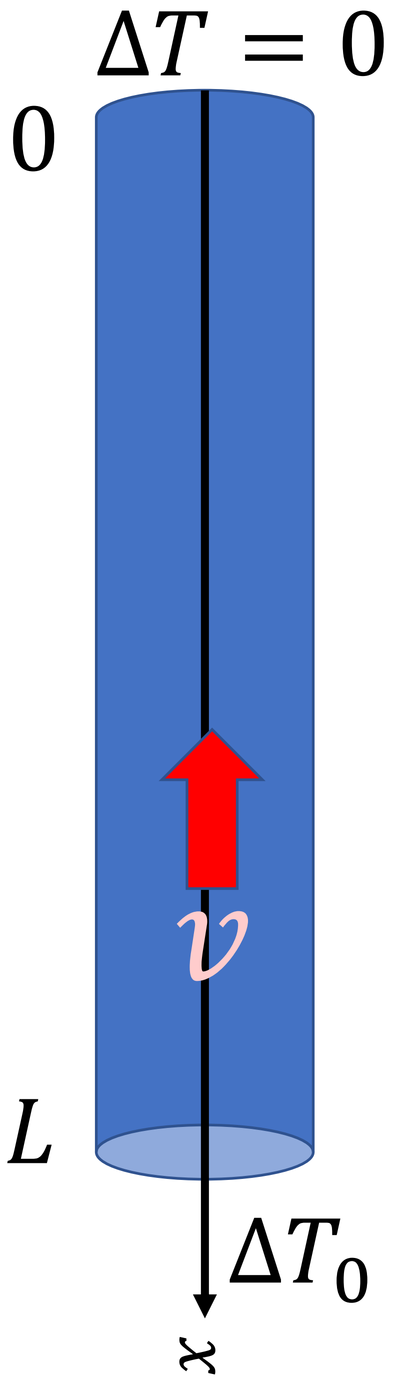
$$\Delta T = B\{\exp(-cx) + C\}$$

with $c = \frac{v'}{\kappa}$

$$\frac{d\Delta T}{dx} = -Bc\exp(-cx)$$

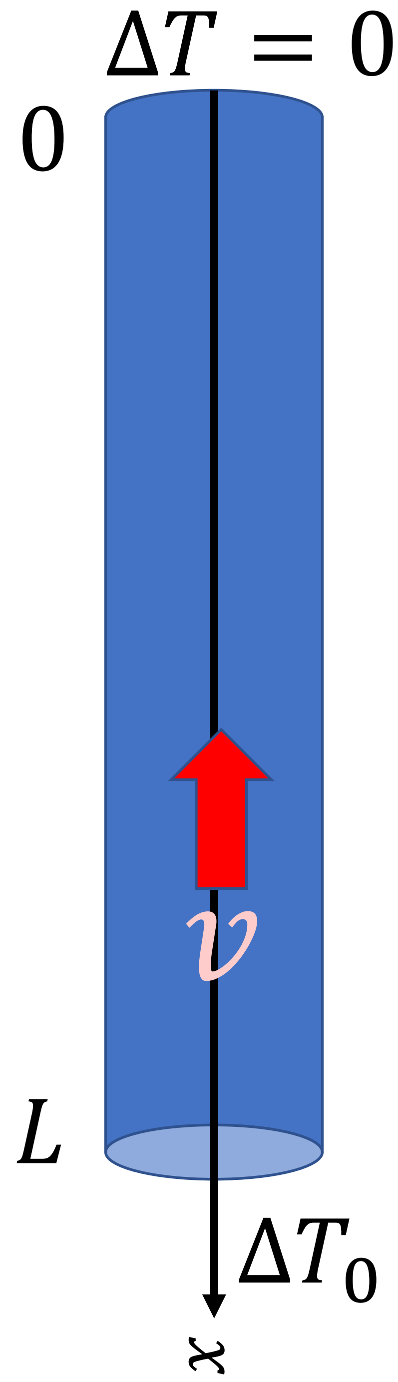
$$\frac{d^2\Delta T}{dx^2} = Bc^2\exp(-cx)$$

compatible? Yes



chose B and C so temperature correct at ends

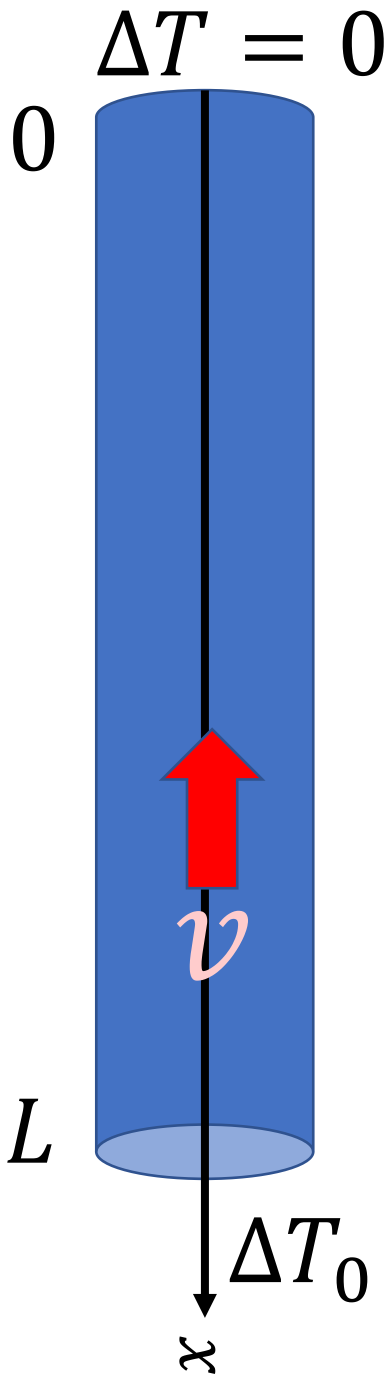
$$\Delta T = B\{\exp(-cx) + C\}$$



chose B and C so temperature correct at ends

$$\Delta T = B\{\exp(-cx) - 1\}$$

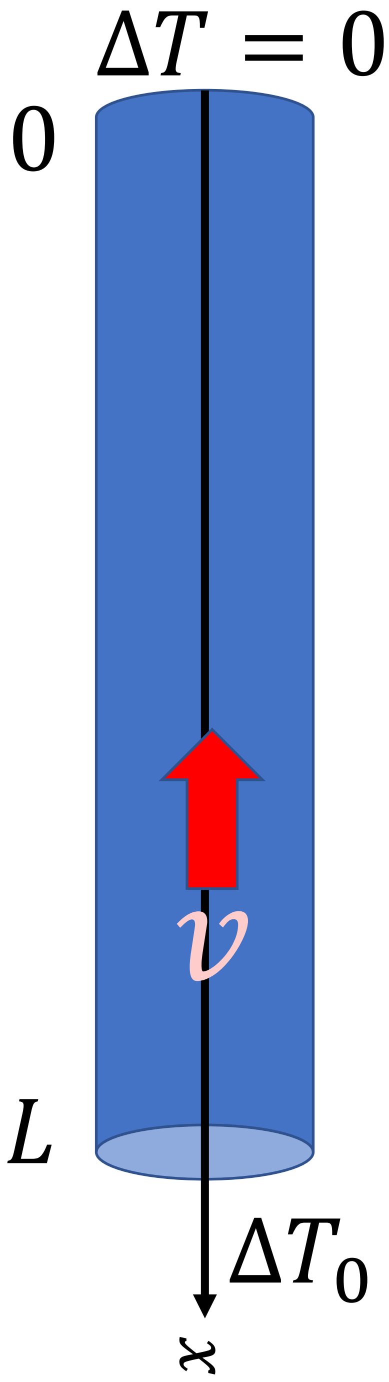
$$\Delta T(x = 0) = 0$$



chose B and C so temperature correct at ends

$$\Delta T = \frac{\Delta T_0}{\{\exp(-cL) - 1\}}\{\exp(-cx) - 1\}$$

$$\Delta T(x = L) = \Delta T_0$$



chose B and C so temperature correct at ends

$$\Delta T = \Delta T_0 \frac{\{\exp(-cx) - 1\}}{\{\exp(-cL) - 1\}}$$

with $c = \dfrac{v'}{\kappa}$

whew!

Convection

temperature causes buoyancy

buoyancy causes flow

flow changes temperature

Convection

temperature causes buoyancy

buoyancy causes flow

flow changes temperature

Is this positive of negative feedback?

Convection

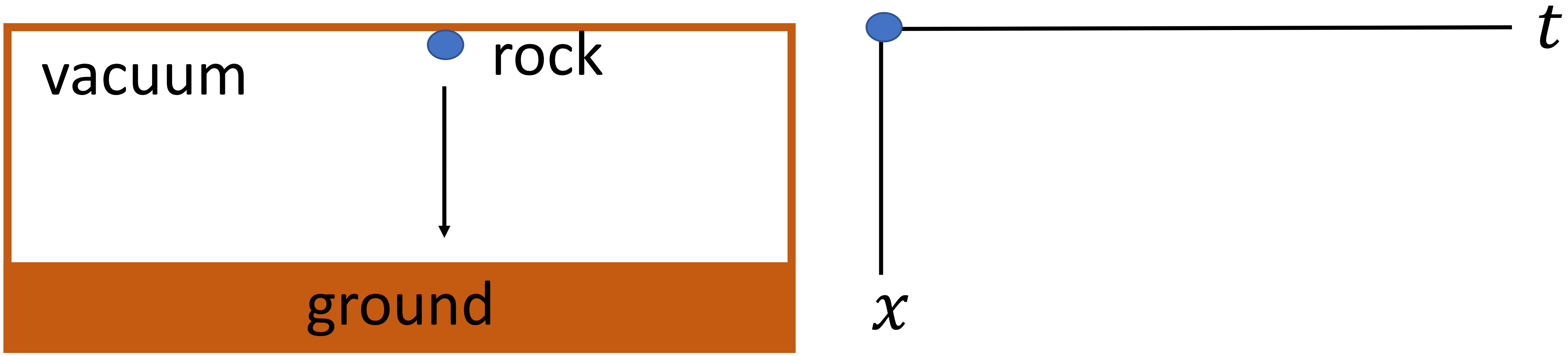
temperature causes buoyancy

buoyancy causes flow

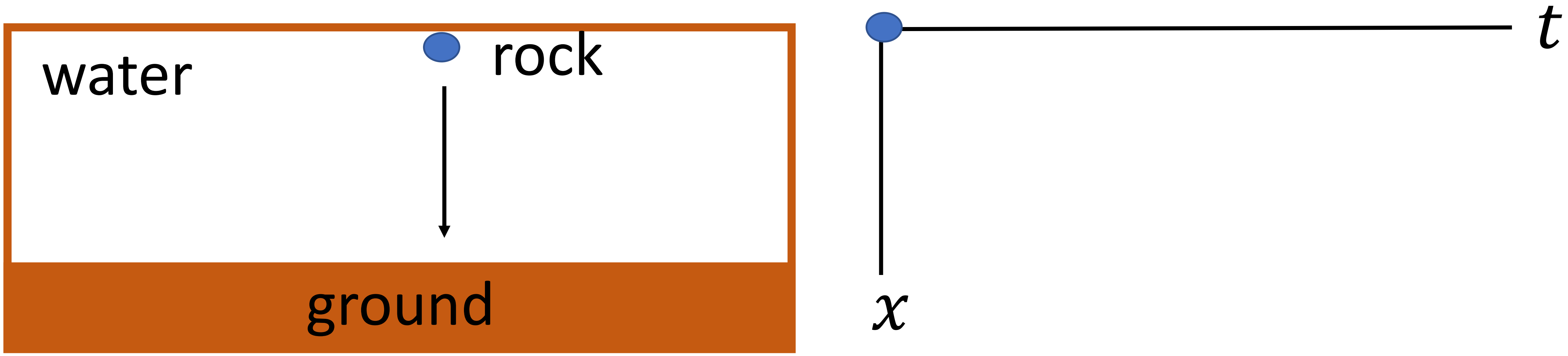
negative
feedback

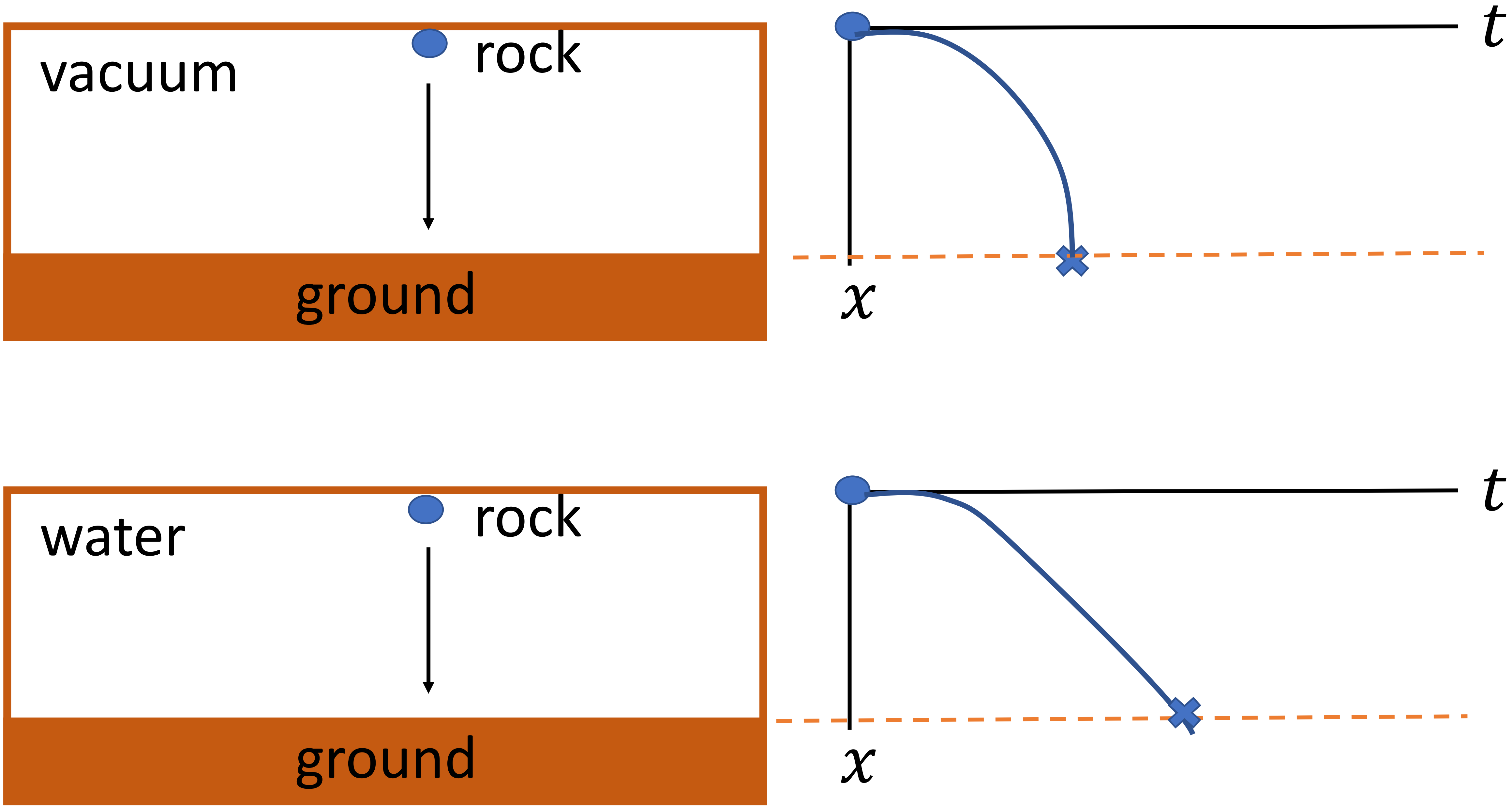
flow speeds up reduction
of temperature

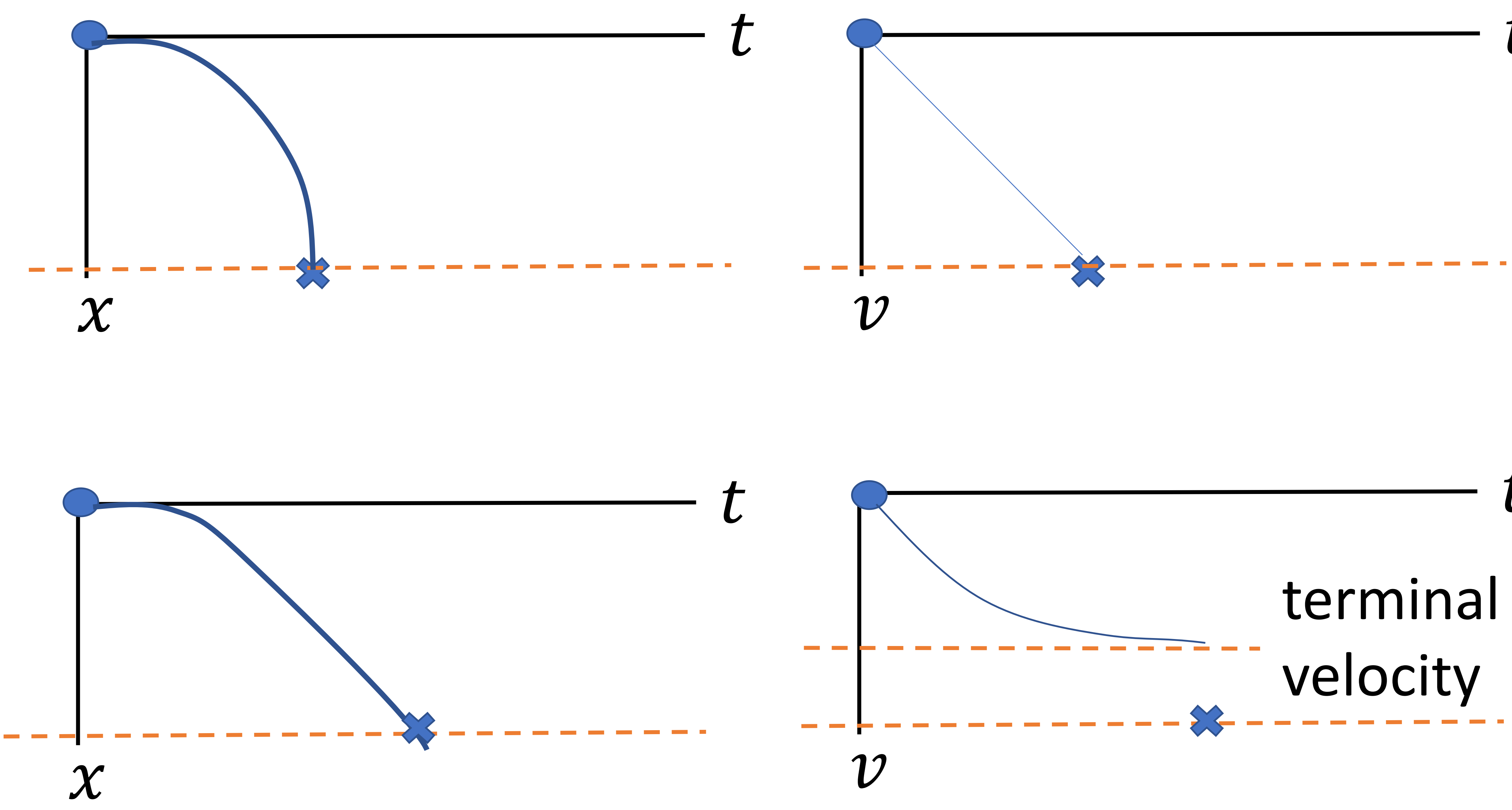
less buoyancy, less flow



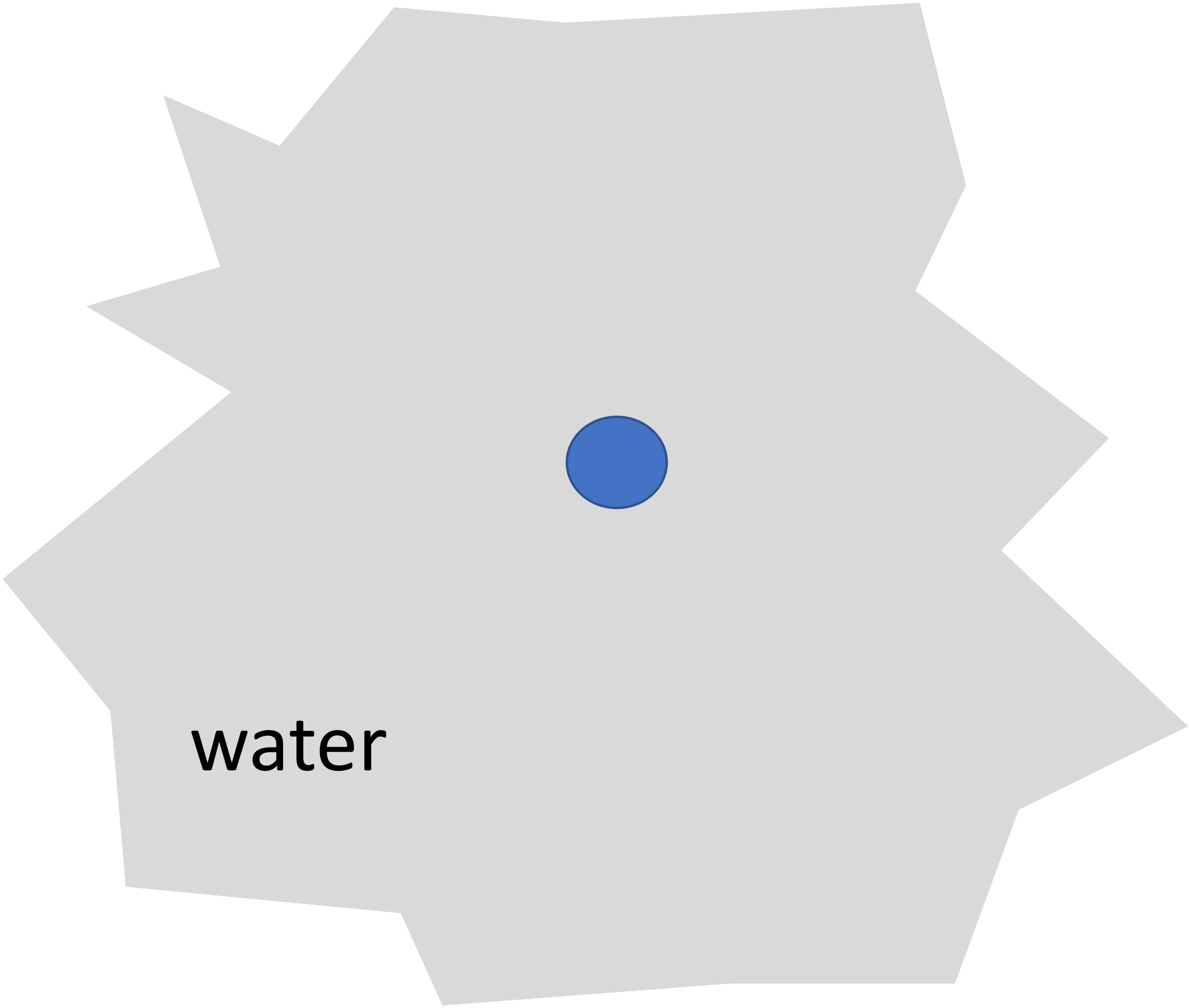
which takes longer to get to the ground?




vacuum
rock
ground
t
x
water
rock
ground
t
x



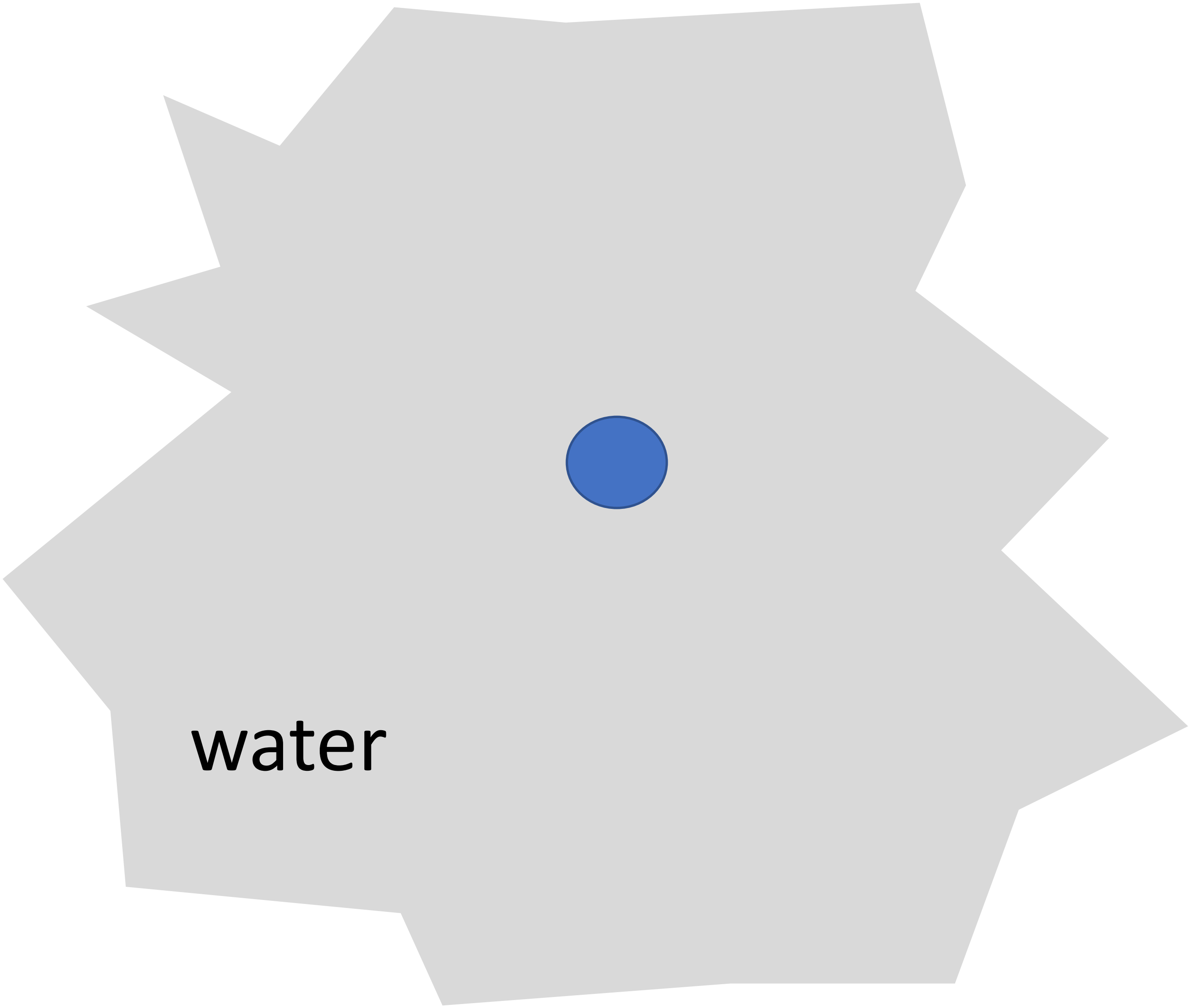
t
x
t
v
t
x
t
terminal
velocity
v



conservation of momentum

Newton's Force Law

$$m\frac{dv}{dt} = f$$

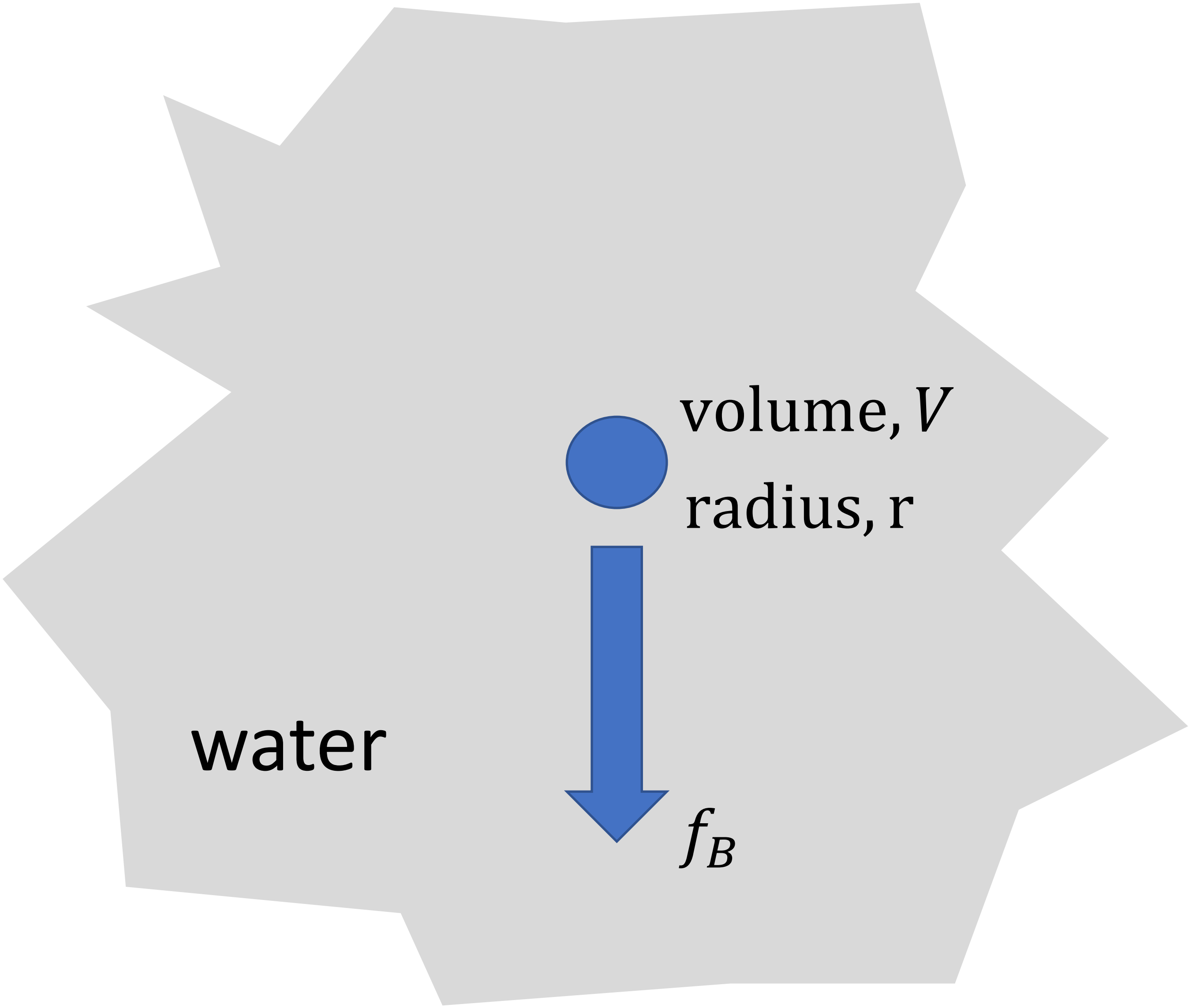


conservation of momentum

Newton's Force Law

$$m\frac{dv}{dt} = f$$

zero
so net force
zero, too



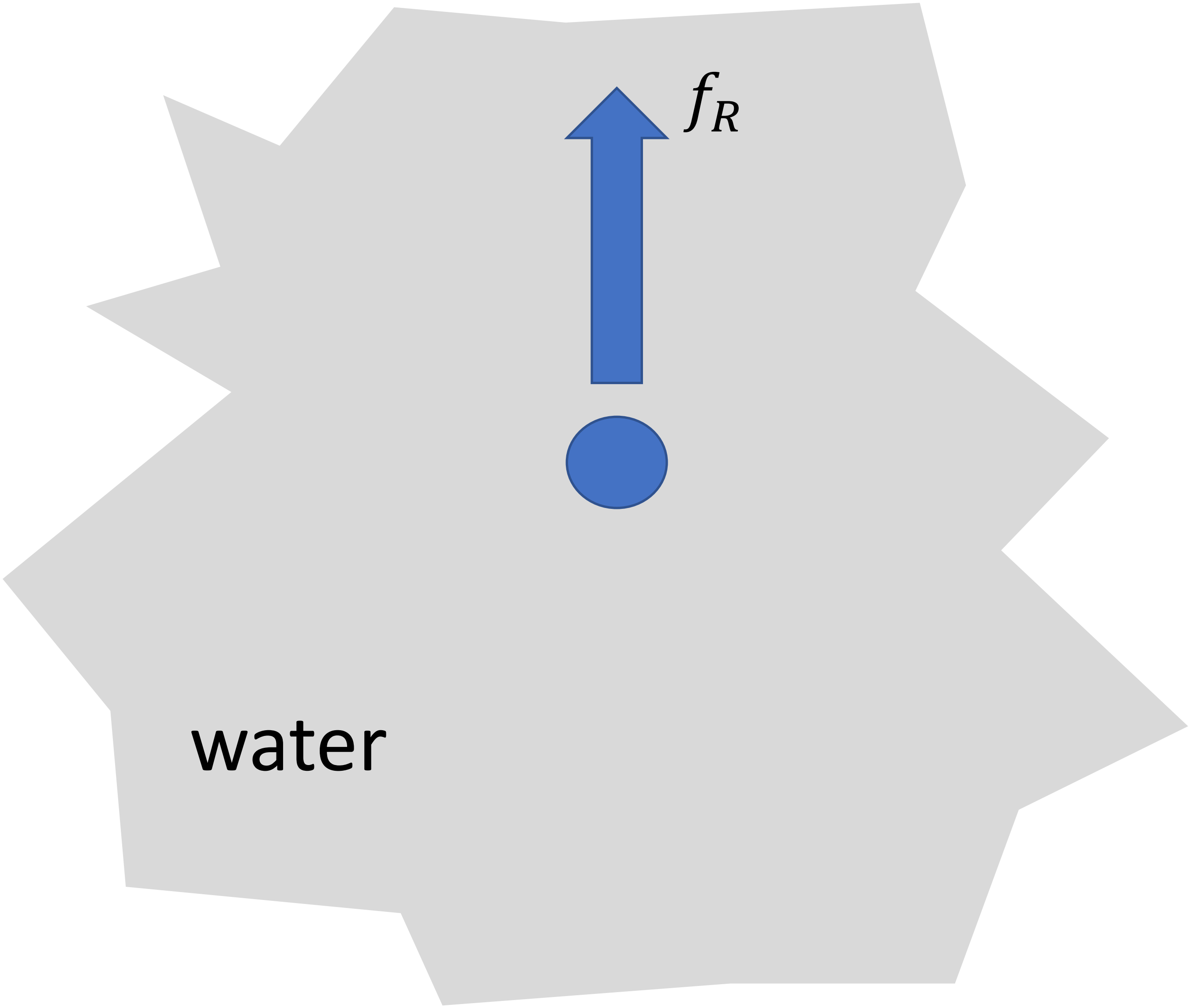
conservation of momentum

Newton's Force Law

$$m\frac{dv}{dt} = f$$

buoyancy force

$$f_B = \Delta\rho\ g\ V = \frac{4}{3}\pi r^3\ \Delta\rho\ g$$



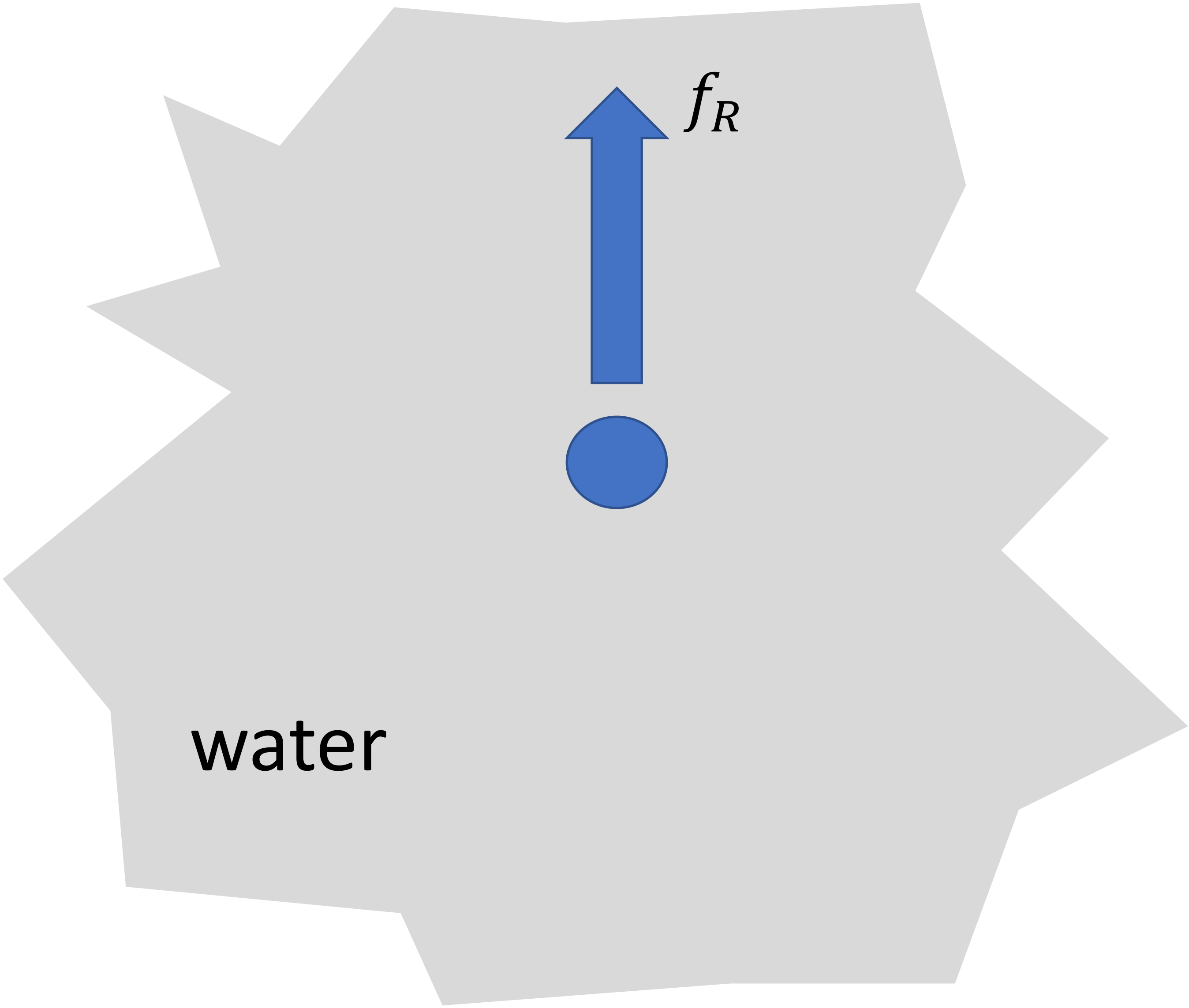
conservation of momentum

Newton's Force Law

$$m\frac{dv}{dt} = f$$

resistive force, f_R

velocity of object
size of object
viscosity of water



conservation of momentum

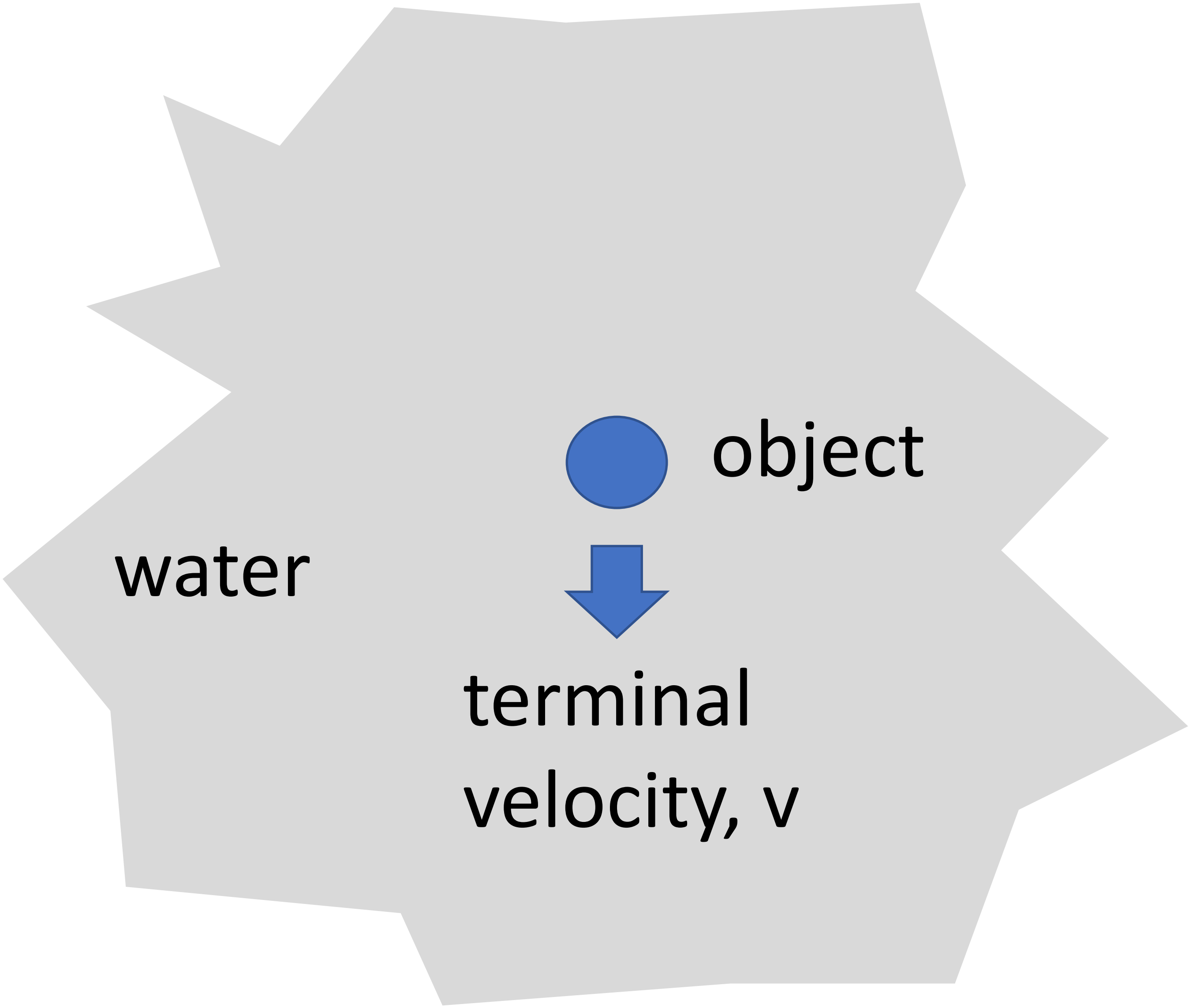
Newton’s Force Law

$$m\frac{dv}{dt} = f$$

resistive force

$$f_R = c\,\mu\,r\,v$$

viscosity, μ
constant, $c = 6\pi$



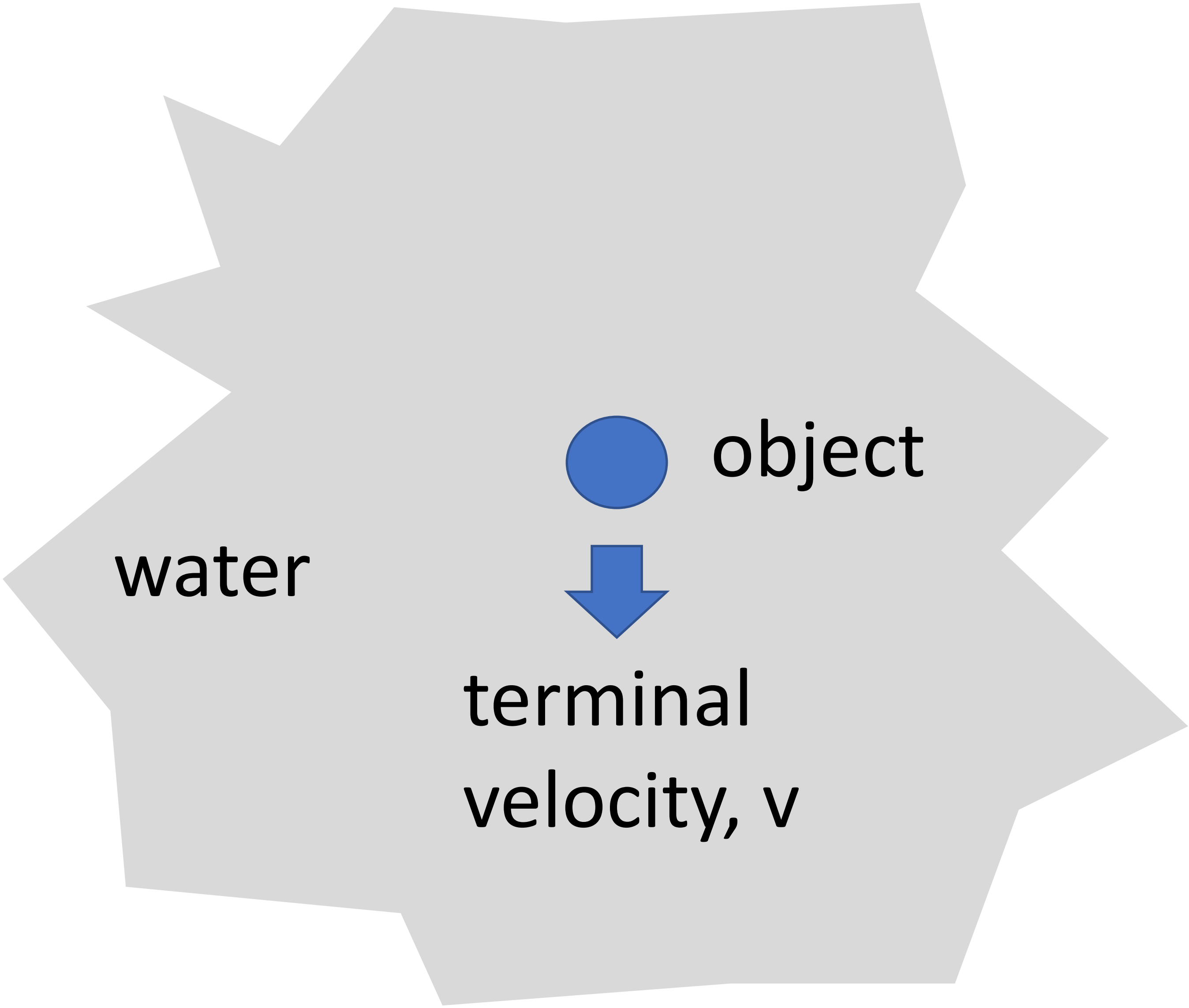
conservation of momentum

Newton's Force Law

$$0 = f_B - f_R$$

$$0 = \frac{4}{3}\pi r^3 \, \Delta\rho \, g - 6\pi \, r \, \mu \, v$$

$$v = \frac{\frac{4}{3}\pi r^3 \, \Delta\rho \, g}{6\pi \, r \, \mu} = \frac{2r^2 \, \Delta\rho \, g}{9 \, \mu}$$



conservation of momentum

Newton's Force Law

$$0 = f_B - f_R$$

$$0 = \frac{4}{3}\pi r^3 \, \Delta\rho \, g - 6\pi \, r \, \mu \, v$$

$$v = \frac{\frac{4}{3}\pi r^3 \, \Delta\rho \, g}{6\pi \, r \, \mu} = \frac{2r^2 \, \Delta\rho \, g}{9 \, \mu}$$

Stokes' law

viscosity, μ

water, $\mu = 10^{-3}$ Pa-s

honey, $\mu = 5$ Pa–s

basaltic magma, $\mu = 100$ Pa–s

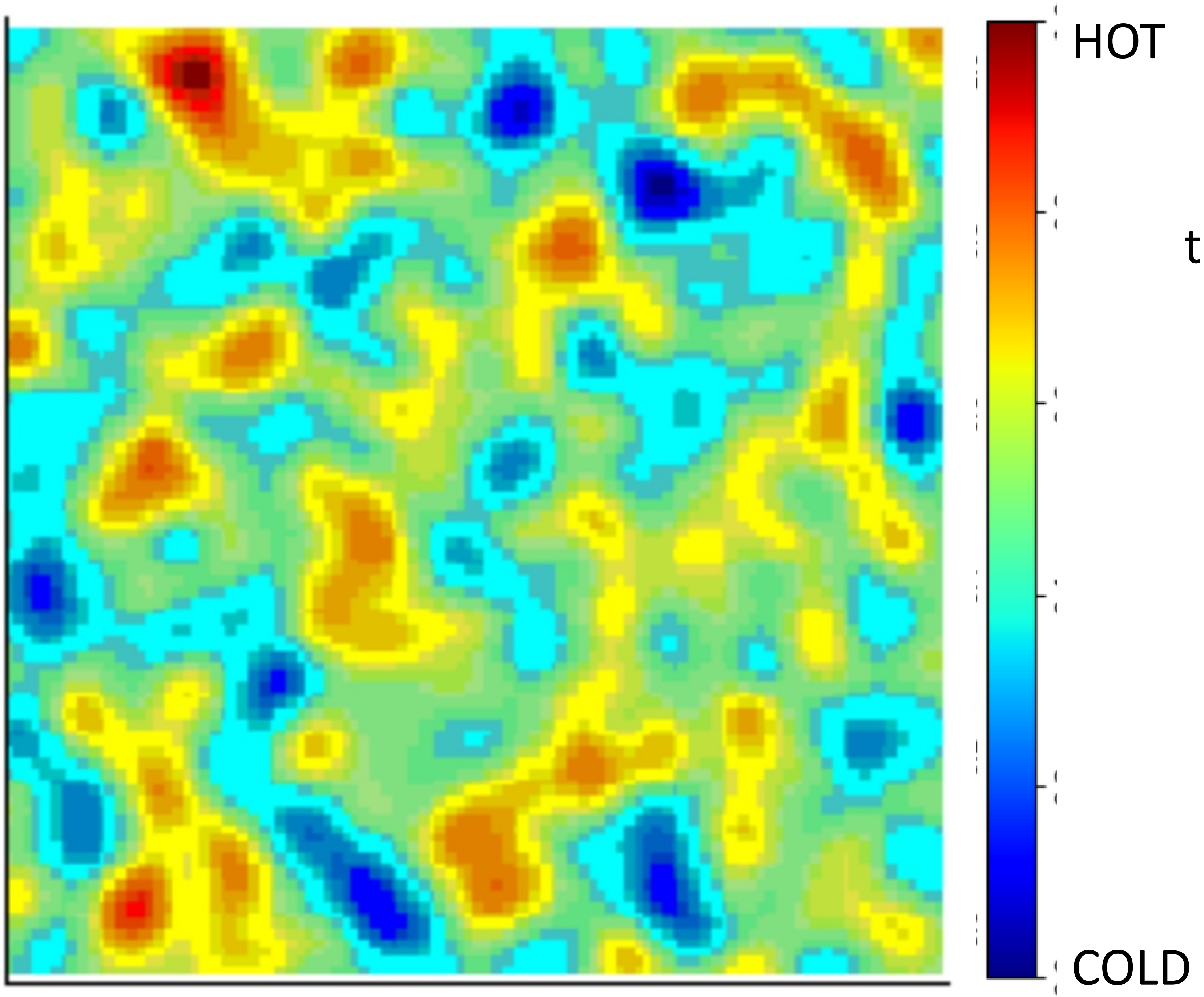
upper mantle rock, $\mu = 4 \times 10^{19}$ Pa–s

Pascal-second, Pa-s: $1 \frac{\mathrm{kg}}{\mathrm{ms}}$

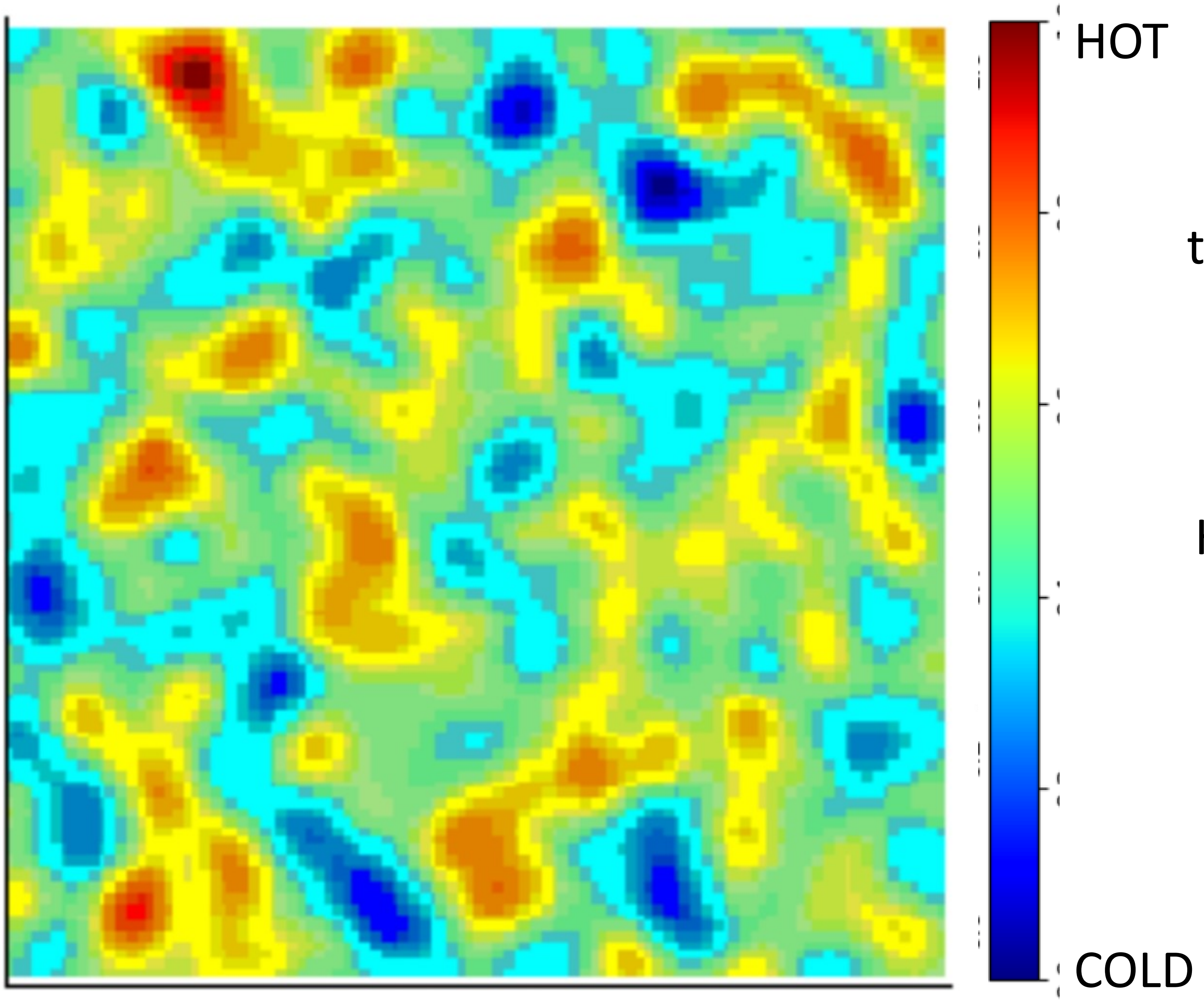
terminal velocity of a 0.001-meter sphere of granite in water

$$\Delta\rho = (2500 - 1000)\ \mathrm{kg/m^3}$$
$$r = 0.001\ \mathrm{m}$$
$$g = 9.81\ m/s^2$$
$$\mu = 10^{-3}\ \frac{\mathrm{kg}}{ms}$$

$$v = \frac{2r^2\ \Delta\rho\ g}{9\ \mu} = \frac{2\times(0.001)^2\times 1500 \times 9.81}{9\ \times 10^{-3}} \frac{\mathrm{m^2 kg\ m\ ms}}{\mathrm{m^3\ s^2\ kg}} = 3.3\,\frac{\mathrm{m}}{\mathrm{s}}$$

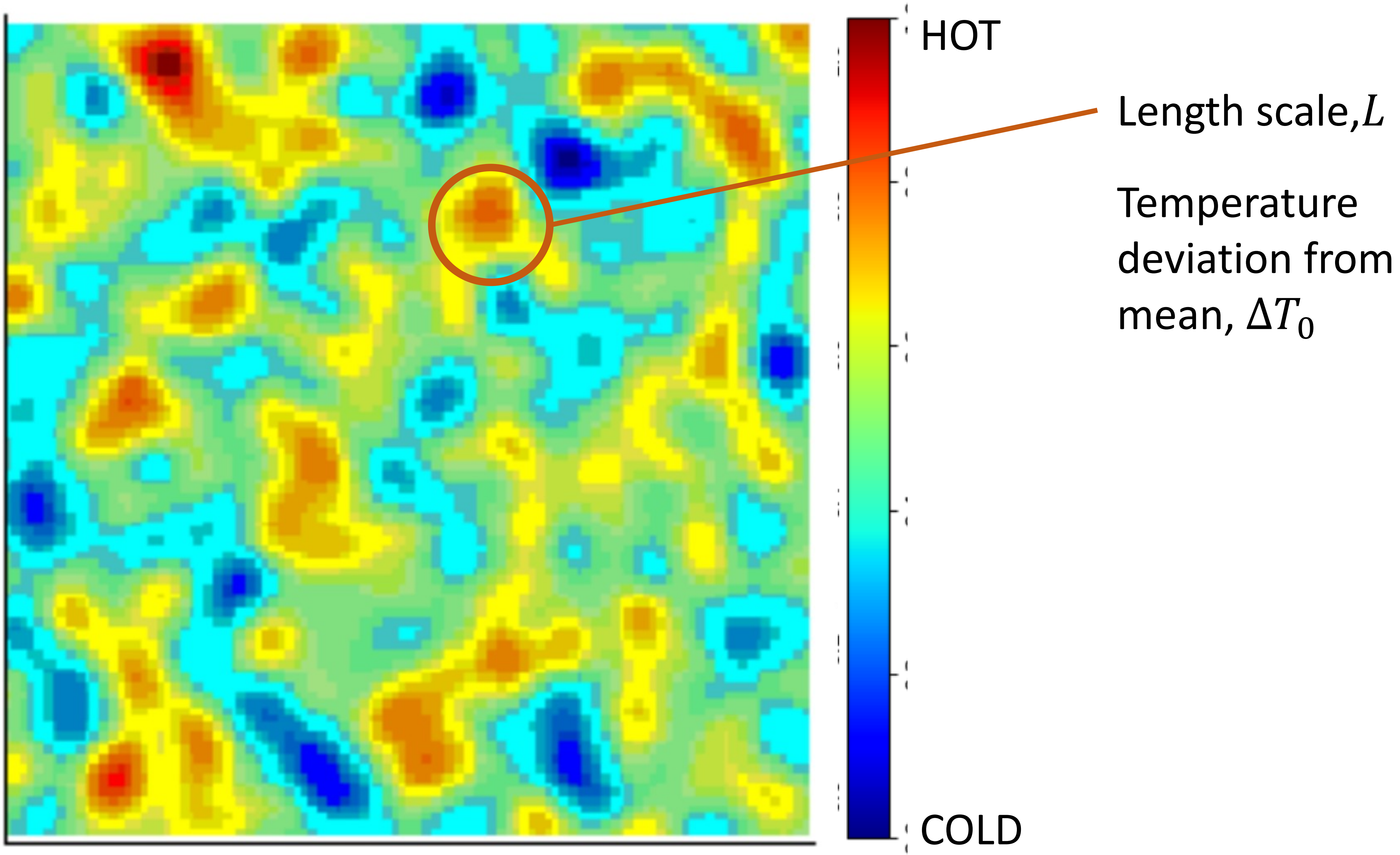


Hypothetical temperatures in the Earth



Hypothetical temperatures in the Earth

How important will convection be in this material


HOT
COLD
Length scale, L
Temperature deviation from mean, ΔT_0

What is the time scale for thermal transport via conduction (quick-and-dirty)

heat $Q = \rho c_p \Delta T_0 L$

heat flux $q = k\Delta T_0 / L$

time scale $t_C = Q/q = L^2/\kappa$

What is the time scale for advection of object by one length scale (quick-and-dirty)

distance L

velocity v

$$t_A = \frac{L}{v}$$

What is the time scale for thermal transport via advection (quick-and-dirty)

$$t_A = \frac{L}{v}$$

velocity $$v = \frac{L^2\ \Delta\rho\ g}{\mu}$$ (Stokes' Law, ignoring the $2/9$)

density $$\Delta\rho\ = \rho_0 \alpha \Delta T_0$$ (thermal expansion)

$$t_A = \frac{L}{v} = \frac{L\mu}{L^2 \rho_0 \alpha \Delta T_0 g} = \frac{\mu}{L \rho_0 \alpha \Delta T_0 g}$$

Ratio of time scales

$$t_C = L^2/\kappa \qquad t_A = \frac{\mu}{L\rho_0 \alpha \Delta T_0 g}$$

$$\frac{t_C}{t_A} = \frac{L^2}{\kappa}\frac{L\rho_0 \alpha \Delta T_0 g}{\mu} = \frac{L^3 \rho_0 \alpha \Delta T_0 g}{\mu\kappa}$$

Ratio of time scales

$$t_C = L^2/\kappa \qquad\qquad t_A = \frac{\mu}{L\rho_0 \alpha \Delta T_0 g}$$

$$R_a = \frac{t_C}{t_A} = \frac{L^3 \rho_0 \alpha \Delta T_0 g}{\mu\kappa}$$

Rayleigh number

dimensionless (no units)

Rayleigh number $R_a = \frac{t_C}{t_A} = \frac{L^3 \rho_0 \alpha \Delta T_0 g}{\mu \kappa}$

Low Rayleigh number: Conduction wins, heterogeneities dissipate before they move significantly

$$R_a = \frac{t_C}{t_A} = \frac{small}{big}$$

High Rayleigh number: Advection wins, heterogeneities move before dissipate significantly

$$R_a = \frac{t_C}{t_A} = \frac{big}{small}$$

so what's the Rayleigh number of the upper mantle?

Rayleigh number $R_a = \frac{t_C}{t_A} = \frac{7.7 \times 10^{14}\ s}{3.4 \times 10^{15}\ s} = \frac{25\ \text{my}}{0.146\ \text{my}} = 170$

$L = 10^5\ \text{m}$ — **100 km, comparable to the thickness of the lithosphere**

$\alpha = 3 \times 10^{-5}\ \frac{1}{°\text{C}}$

$\Delta T_0 = 100°\text{C}$

$g = 9.81\ \frac{\text{m}}{\text{s}^2}$

$\rho_0 = 3000\ \frac{\text{kg}}{\text{m}^3}$

$\mu = 4 \times 10^{19} \frac{\text{kg}}{\text{ms}}$

$k = 3.1\, J/sm°\text{C}$

$c_p = 800\, J/kg°\text{C}$

$\kappa = \frac{k}{\rho_0 c_p} = 1.3 \times 10^{-5} \frac{m^2}{s}$

Rayleigh number $R_a = \frac{t_C}{t_A} = \frac{7.7 \times 10^{14}\ s}{3.4 \times 10^{15}\ s} = \frac{25\ \text{my}}{0.146\ \text{my}} = 170$

$L = 10^5\ \text{m}$

$\alpha = 3 \times 10^{-5}\ \frac{1}{°\text{C}}$

$\Delta T_0 = 100°\text{C}$

mantle under volcanic area needs to be about 100 ℃ hotter than normal to make magma

$g = 9.81\ \frac{\text{m}}{\text{s}^2}$

$\rho_0 = 3000\ \frac{\text{kg}}{\text{m}^3}$

$\mu = 4 \times 10^{19} \frac{\text{kg}}{\text{ms}}$

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$L = 10^5$ m

$\alpha = 3 \times 10^{-5}\ \frac{1}{°C}$

$\Delta T_0 = 100°C$

$g = 9.81\ \frac{m}{s^2}$

$\rho_0 = 3000\ \frac{kg}{m^3}$

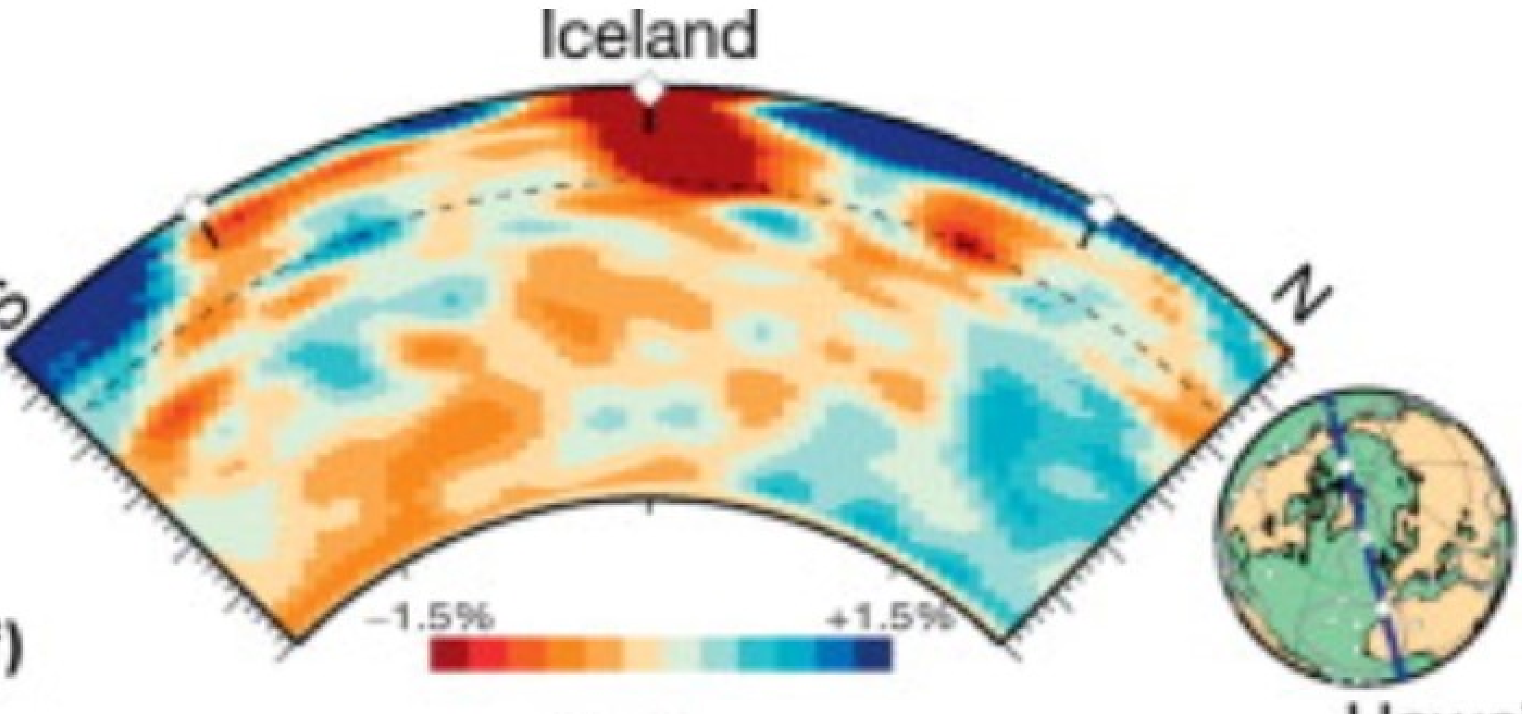
$\mu = 4 \times 10^{19} \frac{kg}{ms}$

IF there are 100 km heterogeneities in the mantle, they will convect

$k = 3.1\ J/sm°C$

$c_p = 800\ J/kg°C$

$\kappa = \frac{k}{\rho_0 c_p} = 1.3 \times 10^{-5}\ \frac{m^2}{s}$



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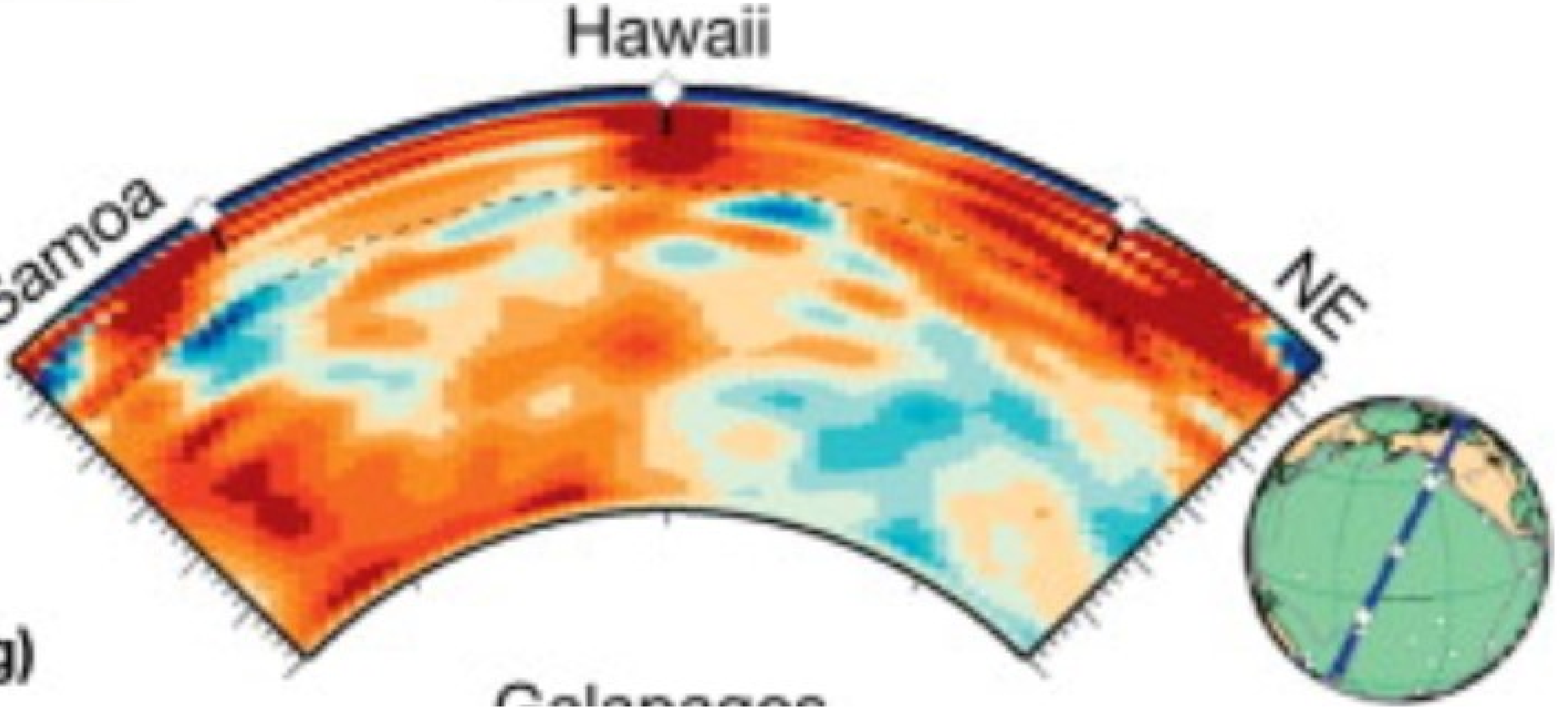

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