Solid Earth Dynamics

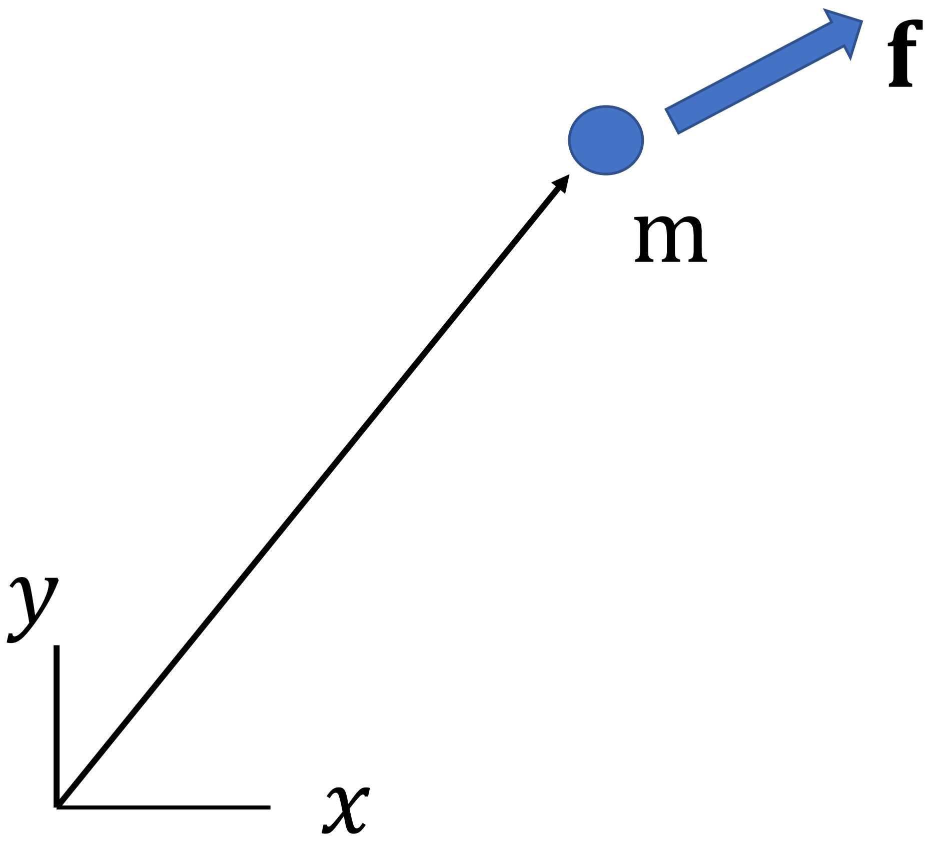
Bill Menke, Instructor

Lecture 7

Today:

Gravity:

- Newtonian orbits
- field of a point mass and sphere
- measuring gravity
- ocean surface



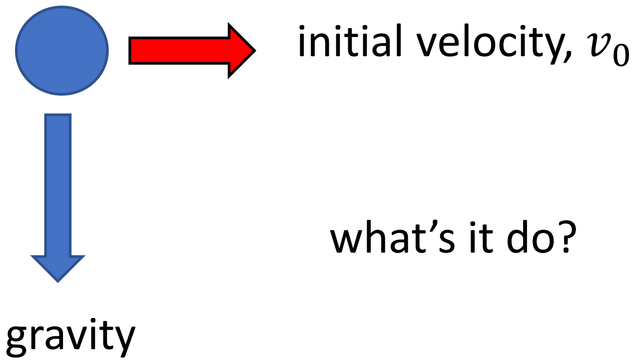
force = mass x acceleration

$$f_x = m\frac{d^2x}{dt^2}$$

$$f_y = m\frac{d^2y}{dt^2}$$

$$f_x = 0$$

$$f_y = -gm$$



what's it do?

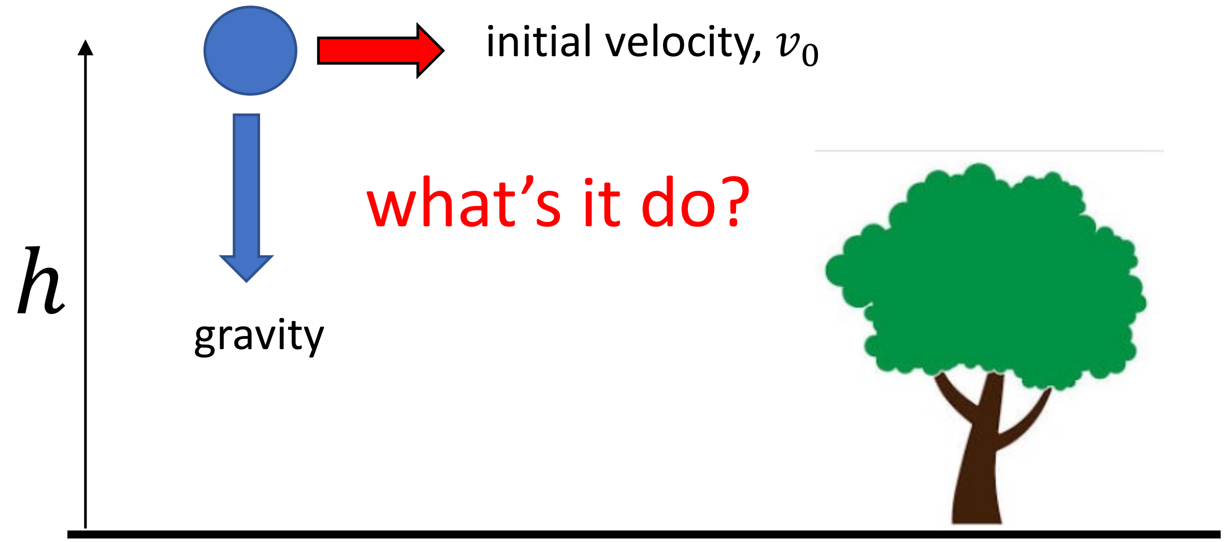
force = mass x acceleration

$$0 = m\frac{d^2x}{dt^2}$$

$$-gm = m\frac{d^2y}{dt^2}$$

$$f_x = 0$$

$$f_y = -mg$$



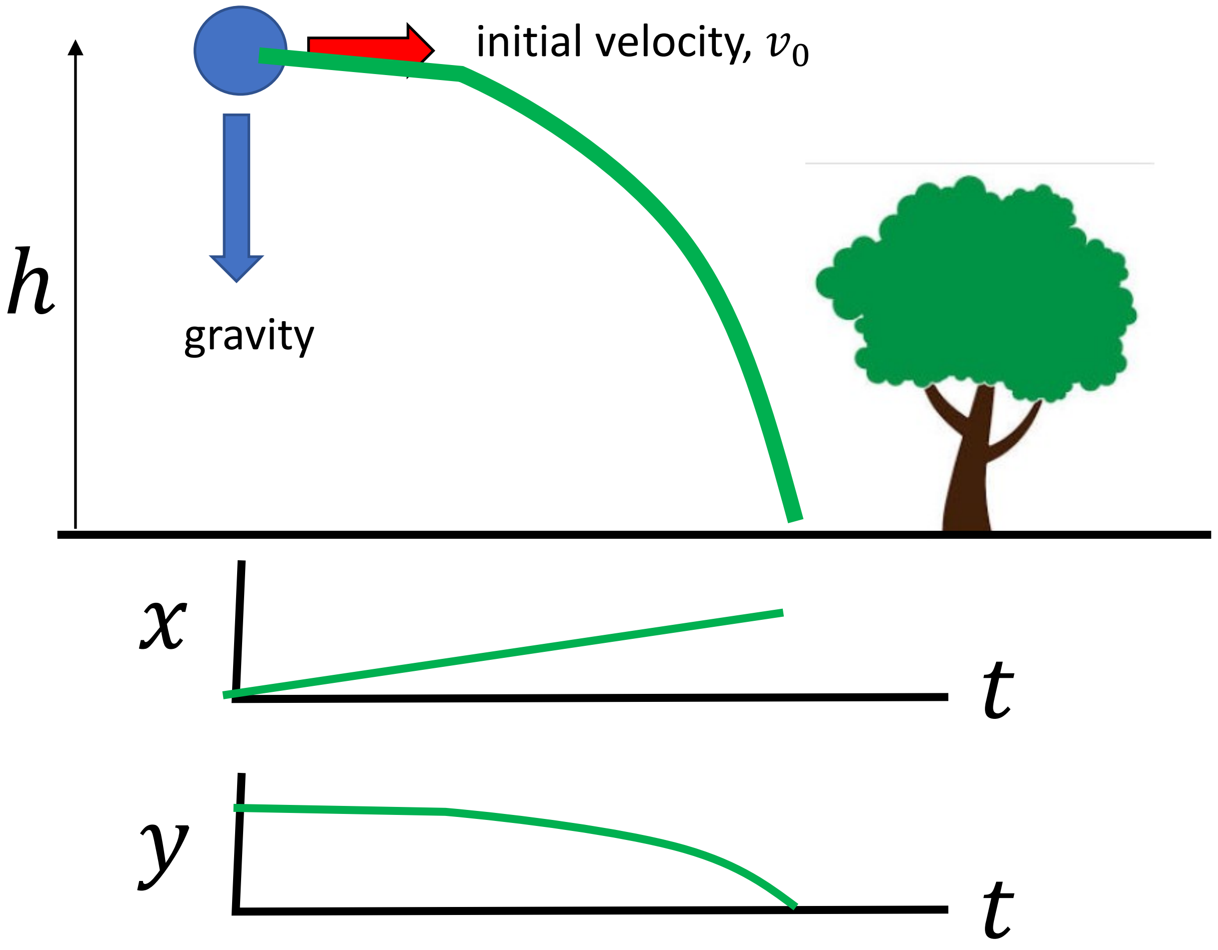
force = mass x acceleration

$$0 = \frac{d^2x}{dt^2}$$

$$-g = \frac{d^2y}{dt^2}$$

$$f_x = 0$$

$$f_y = -mg$$



force = mass x acceleration

$$0 = \frac{d^2x}{dt^2}$$

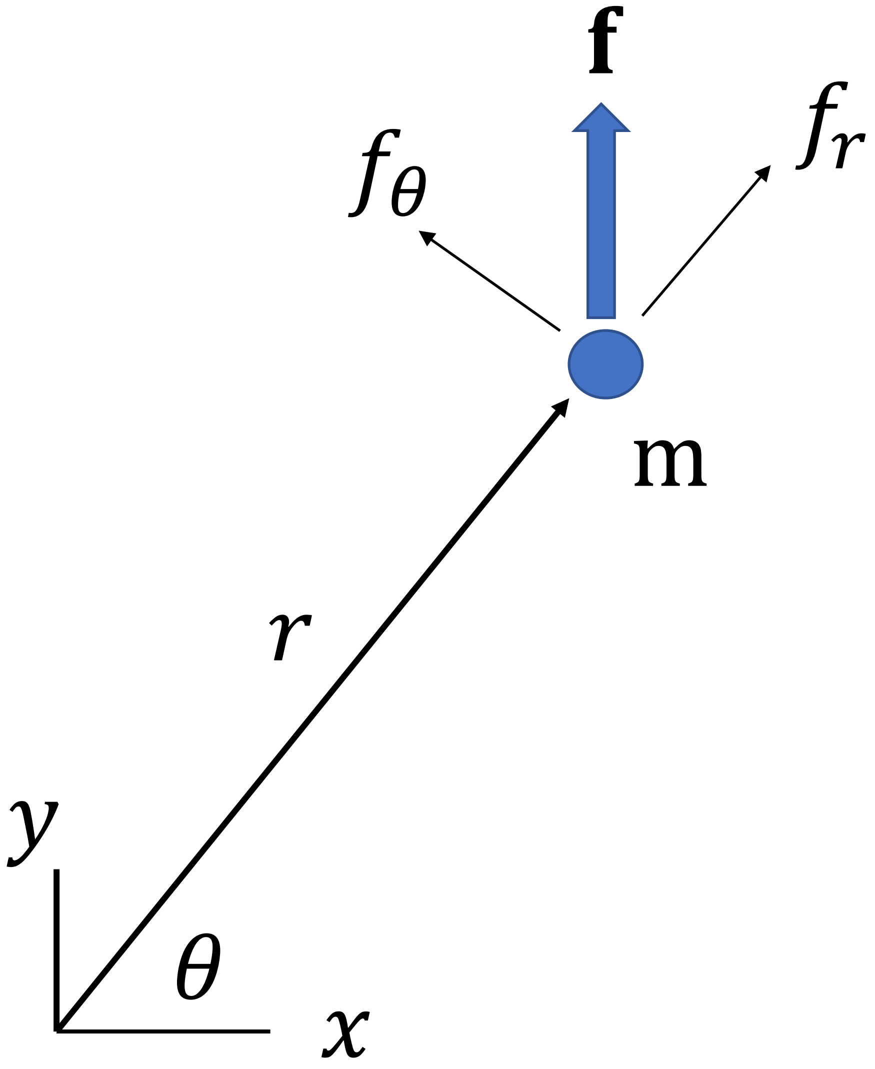
$$x = v_0 t$$

$$-g = \frac{d^2y}{dt^2}$$

$$y = h - ½gt^2$$

path doesn’t depend on mass

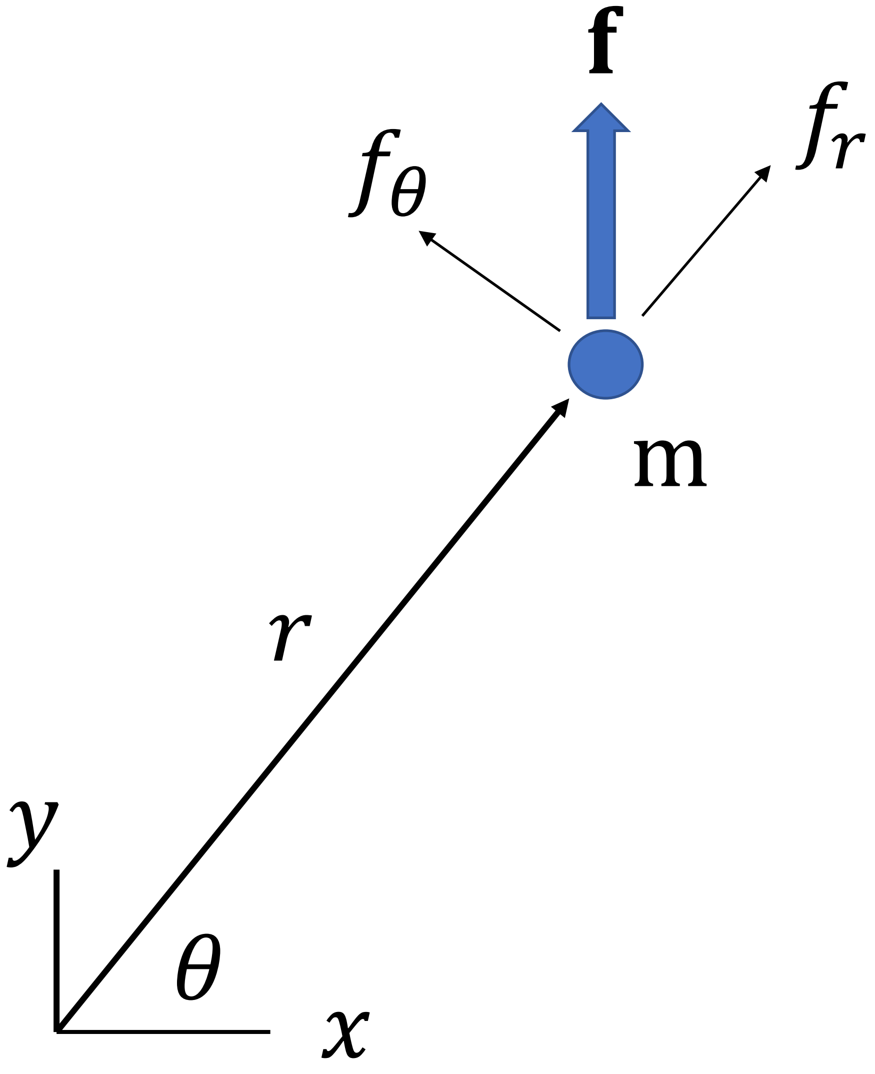

really need to work in polar coordinates



force = mass x acceleration

$$f_r = m\ ?$$

$$f_\theta = m\ ?$$

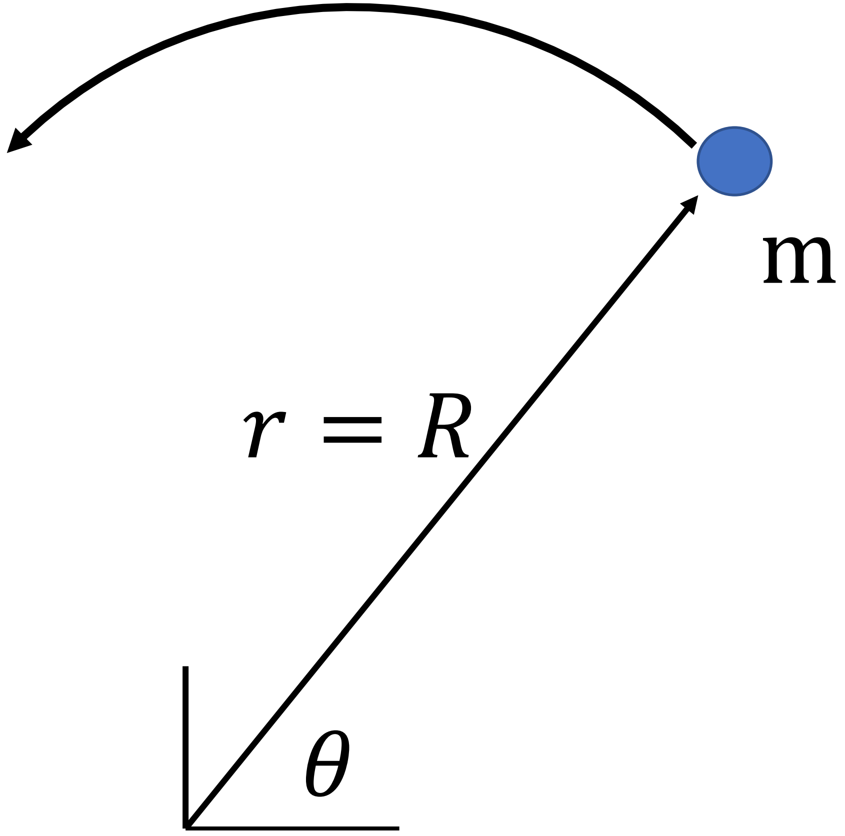


force = mass x acceleration

$$f_r = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$$

$$f_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)$$

constant rate in circular orbit



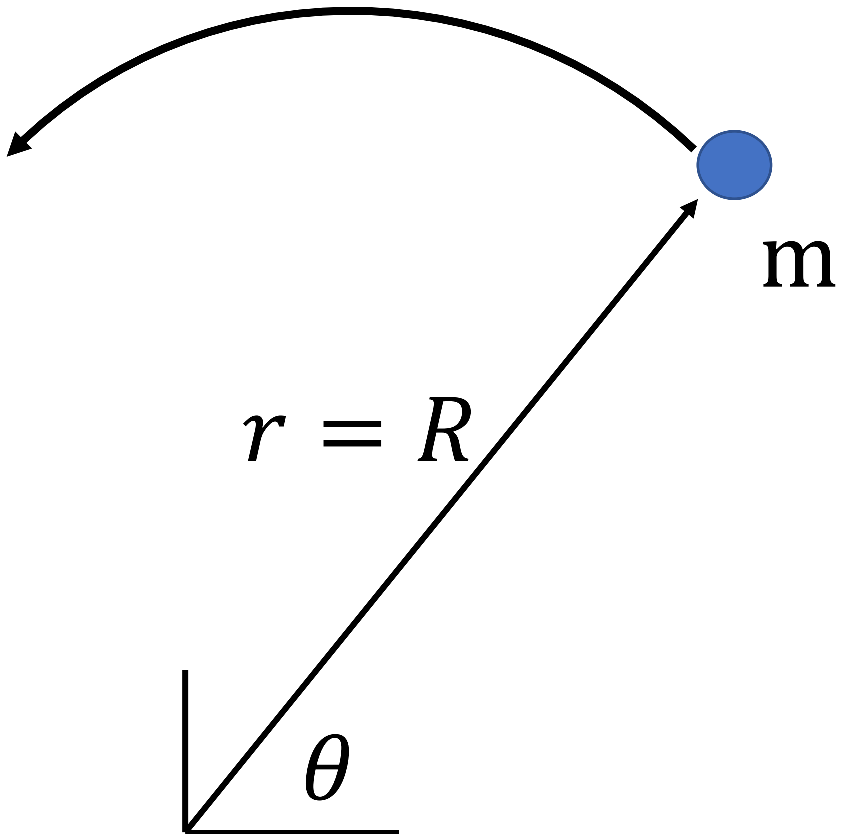
$$r = R = \text{constant}$$

$$\theta = \frac{2\pi t}{T} \text{ with period}, T$$

$$f_r = m\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$$

$$f_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)$$

constant rate in circular orbit



$$r = R = \text{constant}$$

$$\theta = \frac{2\pi t}{T} \text{ with period}, T$$

$$f_r = -mR\left(\frac{2\pi}{T}\right)^2$$

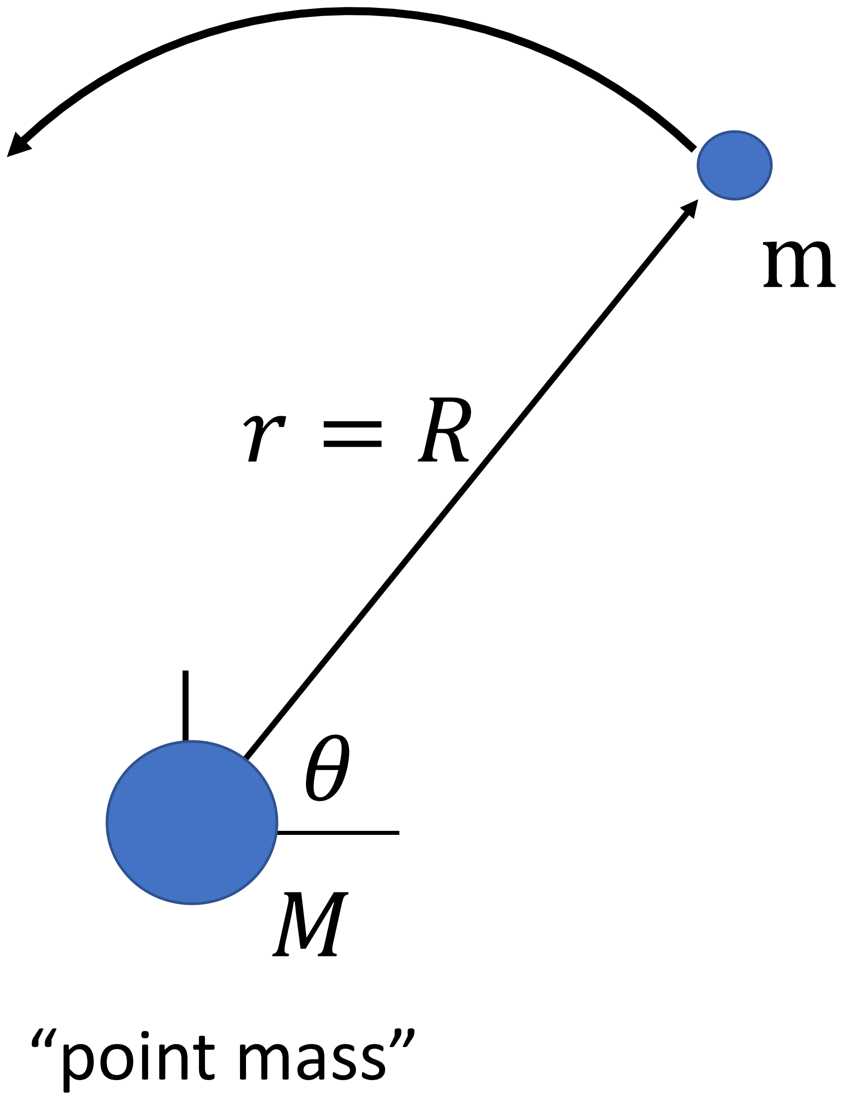
$$f_\theta = 0$$

Kepler's "Law"

based on planetary observations

$$T^2 \propto R^3$$

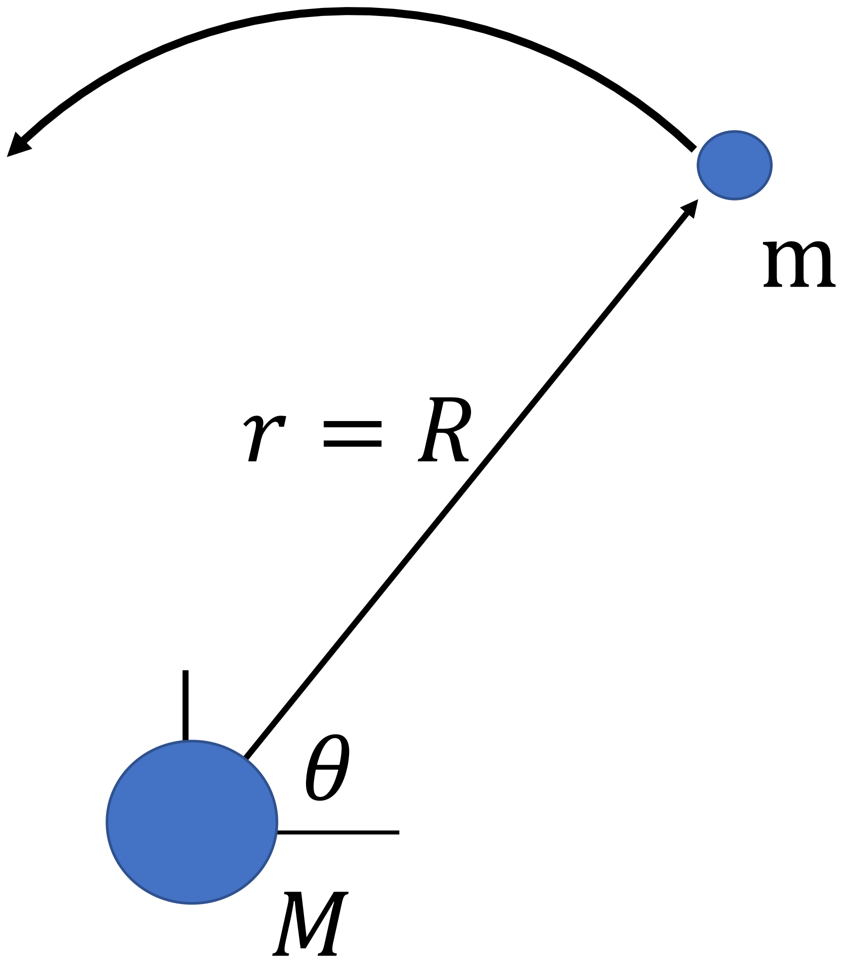
insert Kepler's 3rd Law



$$f_r = -mR\left(\frac{2\pi}{T}\right)^2$$

$$= -mR\frac{(2\pi)^2}{T^2}$$

$$\propto -mR\frac{(2\pi)^2}{R^3} = -\frac{\gamma mM}{R^2}$$



Newton's "Law" of gravity

$$f_\theta = 0$$

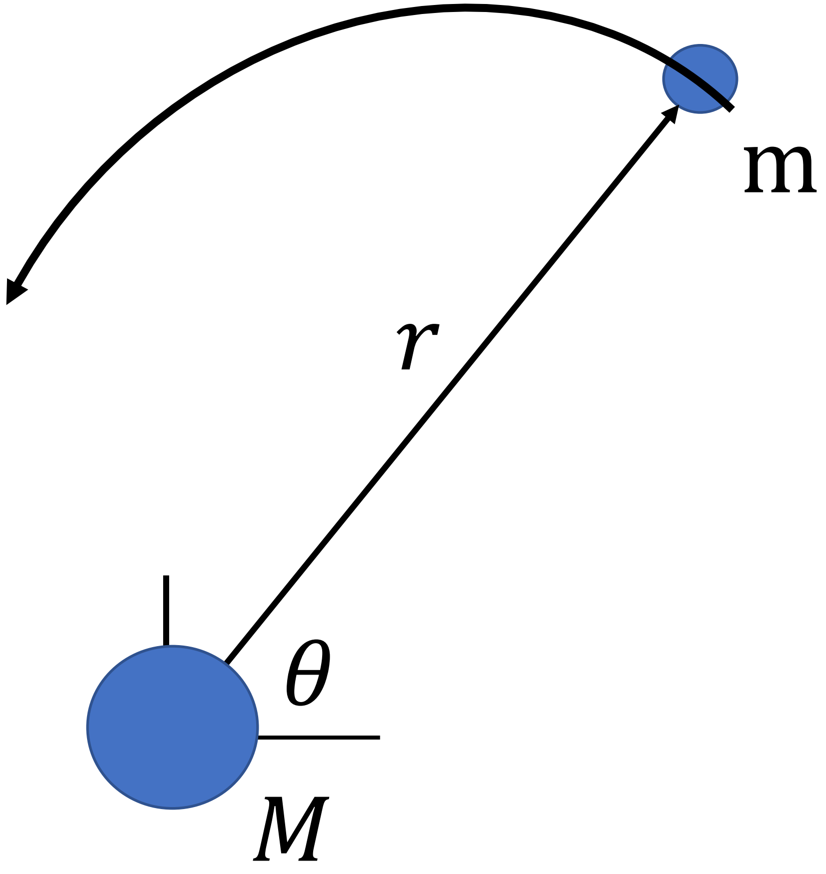
ε

$$f_r = -mR\left(\frac{2\pi}{T}\right)^2$$

$$= -mR\frac{(2\pi)^2}{T^2}$$

$$\propto -mR\frac{(2\pi)^2}{R^3} = -\frac{\gamma mM}{R^2}$$

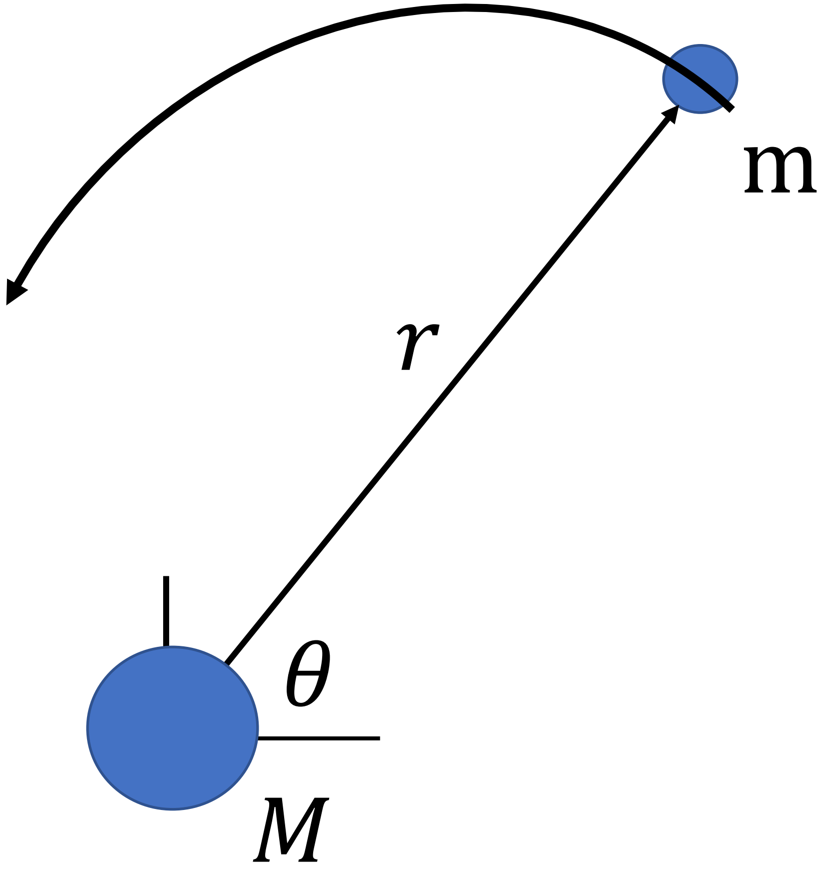
non-circular orbit



$$C = \gamma M$$

$$-\frac{Cm}{r^2} = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$$

$$0 = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)$$

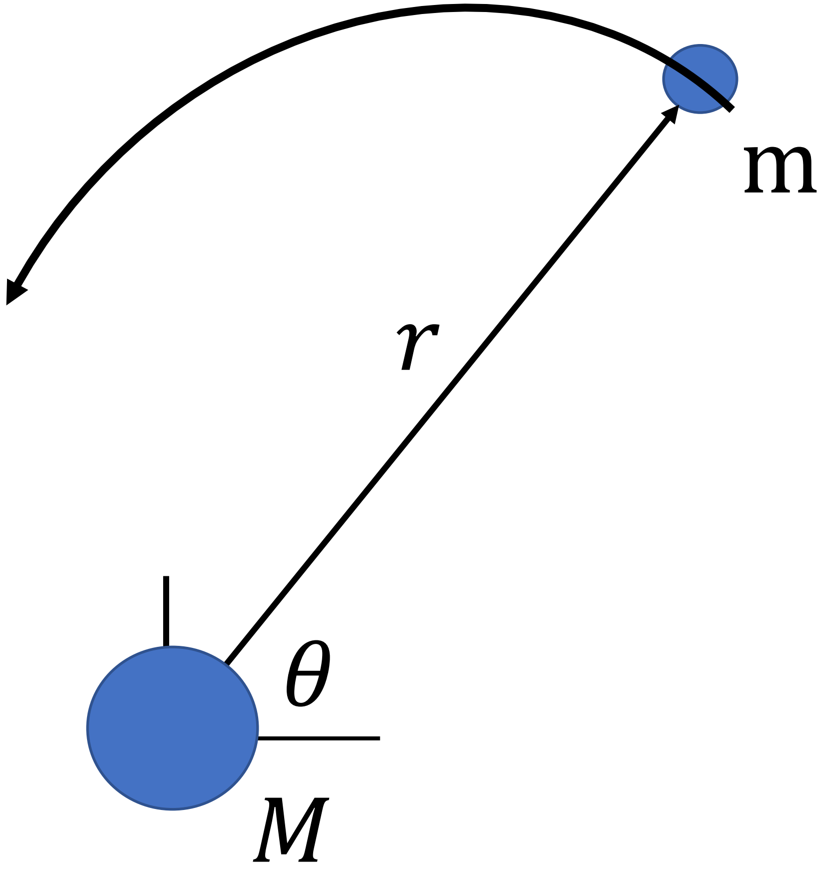


chain rule

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{d\theta}{dt}$$

multiply by r

$$r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

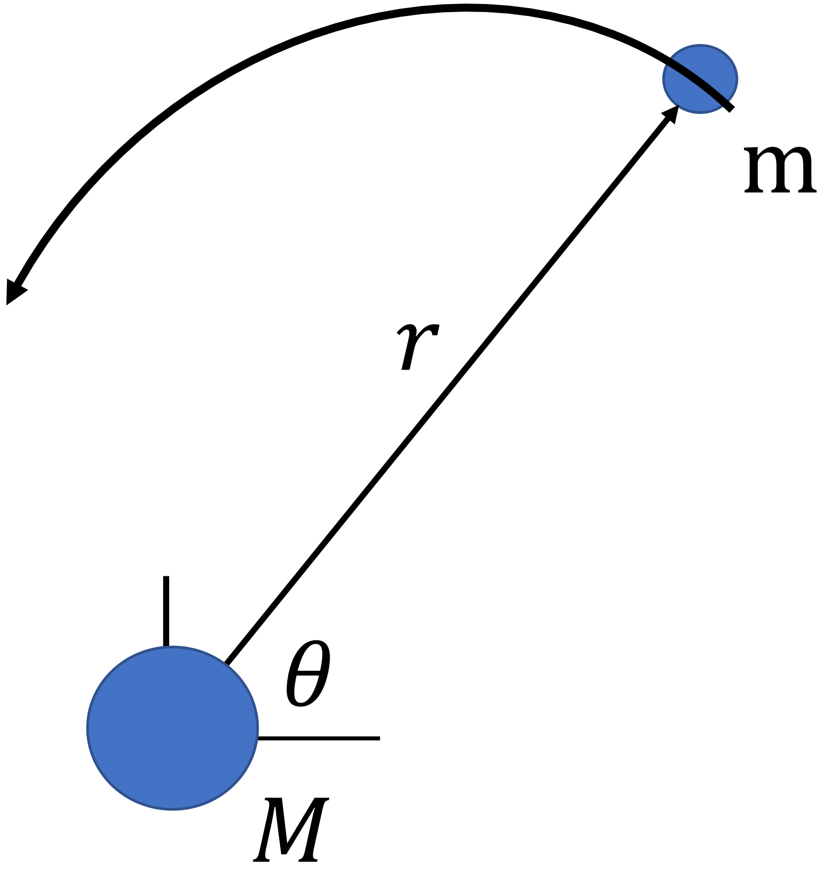


chain rule

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{d\theta}{dt}$$

multiply by r

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

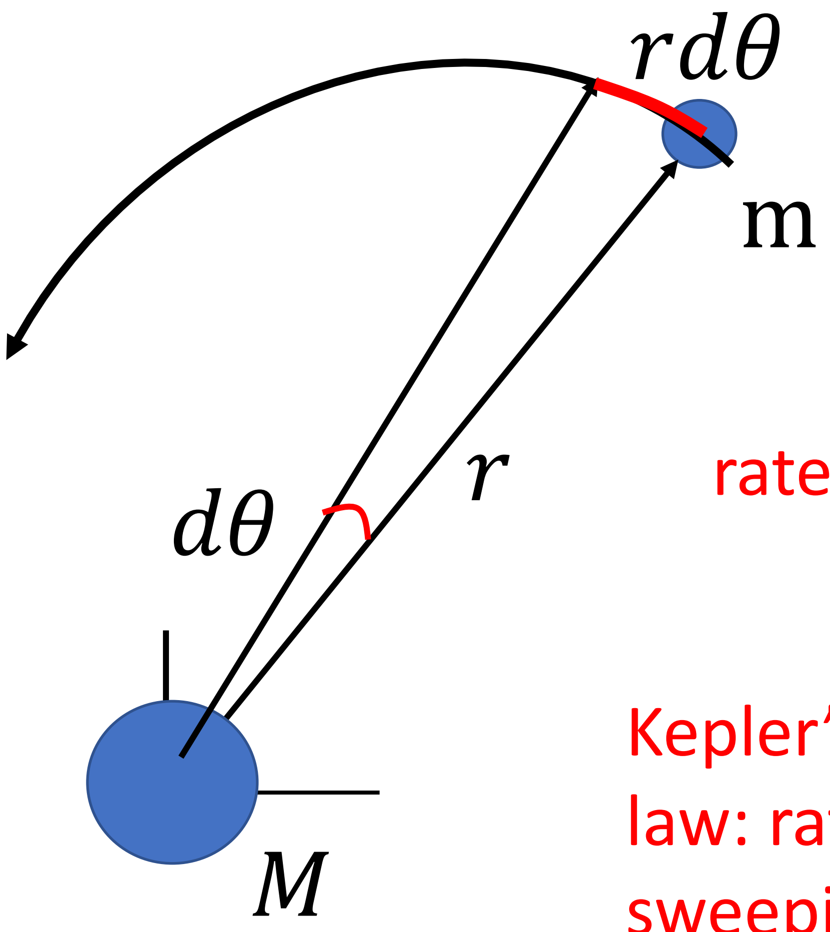


chain rule

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{d\theta}{dt}$$

so

$$r^2\frac{d\theta}{dt} = \text{constant}$$



chain rule

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{d\theta}{dt}$$

rate of change of area = $r^2\frac{d\theta}{dt}$

Kepler's second law: rate of sweeping out area is constant

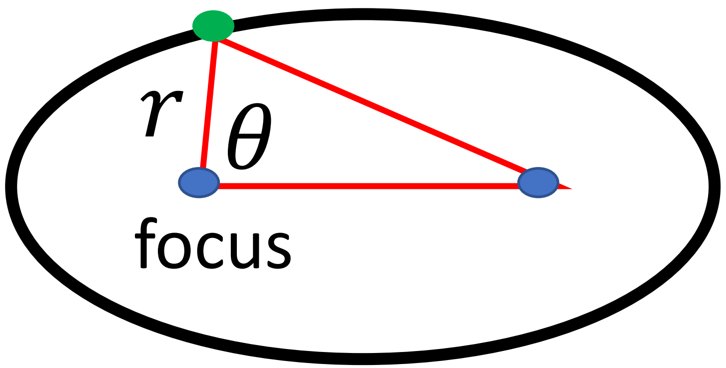
$$r^2\frac{d\theta}{dt} = \text{constant}$$

further analysis shows that the shape of an orbit is an ellipse

formula for ellipse

$$r = \frac{p}{1 + \varepsilon \cos\theta}$$

p and ε constants



ellipse: length of red line constant as green dot moves around circumfernce

But the Earth is not a “point mass”

This issue stymied Newton for 20 years!

what Newton discovered:

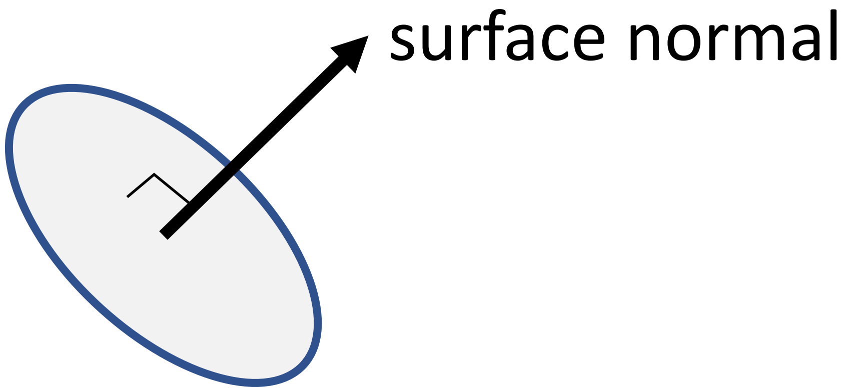
The gravitational force outside a spherically-symmetric object is the same as for a point mass located at the object's center what Newton discovered:

The gravitational force outside a spherically-symmetric object
is the same as for a point mass located at the object's center

I'll show you its true
but using modern concepts
that you are familiar with
(but Newton wasn't)

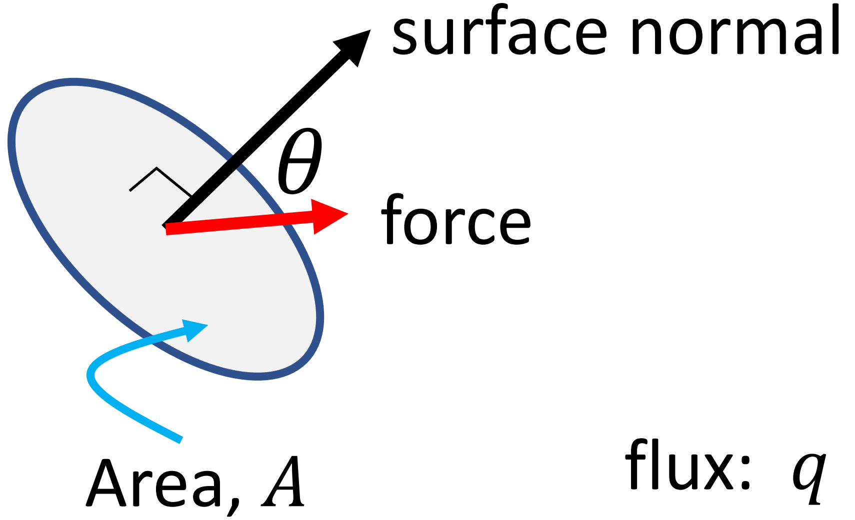
But first

I need to introduce the concept of
flux of force through a surface



(this is the same as
water flux thru a surface
heat flux thru a surface)

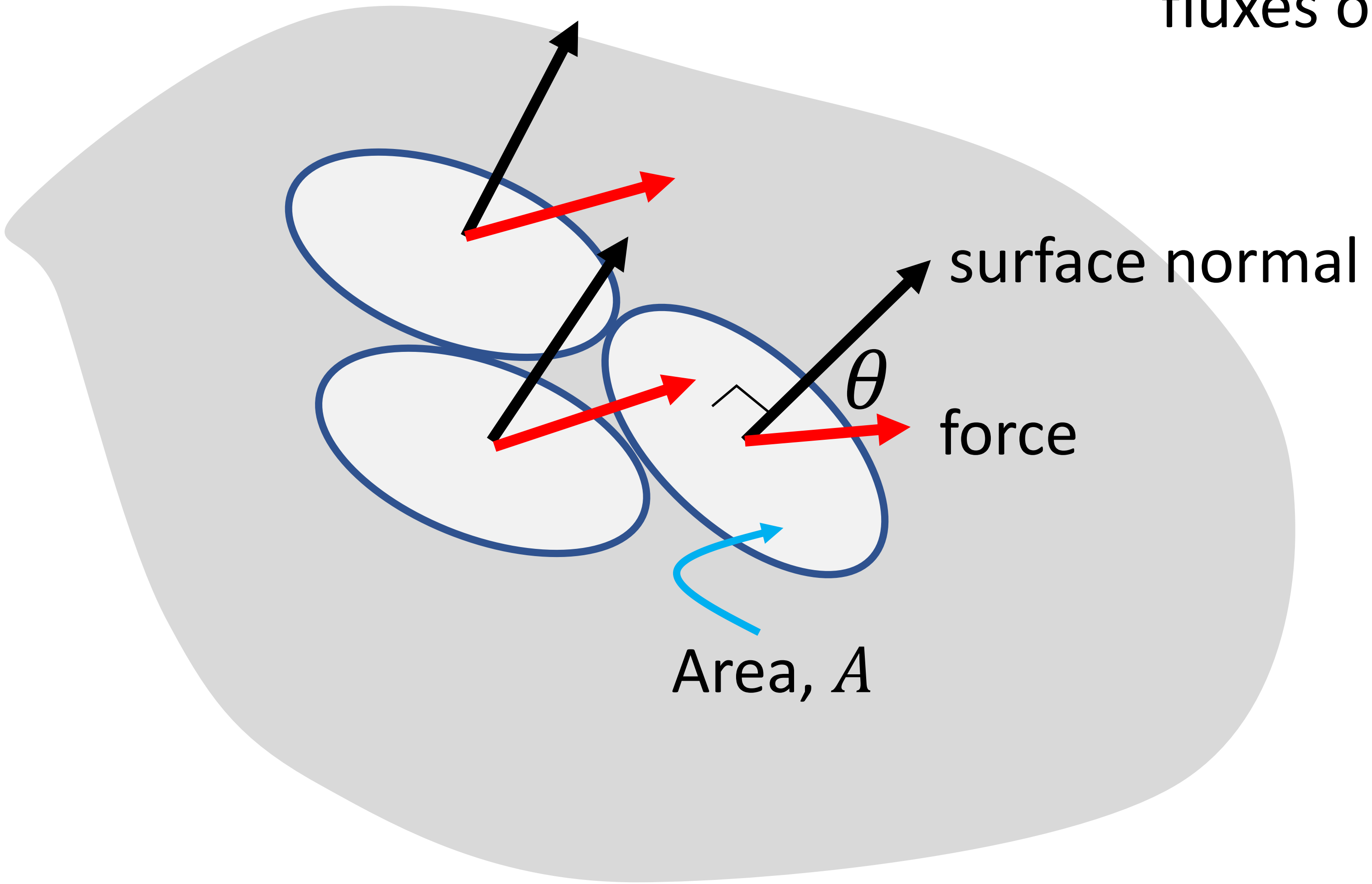
for a small surface



flux: $q = A\ |f|\ \cos\theta$

for a big surface

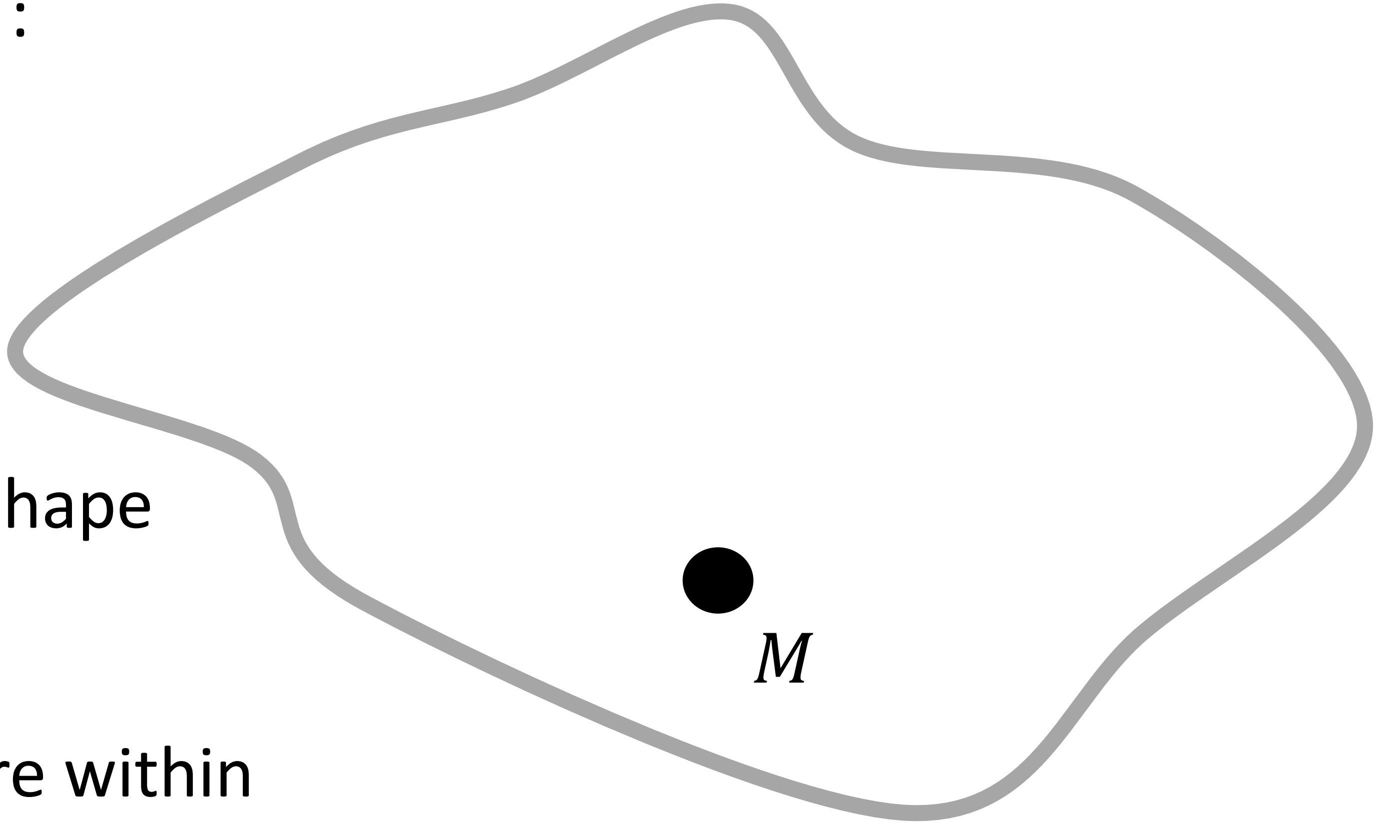
just break into small surfaces and add up the fluxes of each



flux: $q = \sum A \ |f| \ \cos\theta$

Step 1:

The flux through a surface enclosing a point mass, M, is always $q = -4\pi\gamma M$:



irrespective of the shape
of the surface
and
irrespective of where within
the enclosed volume the point mass is located

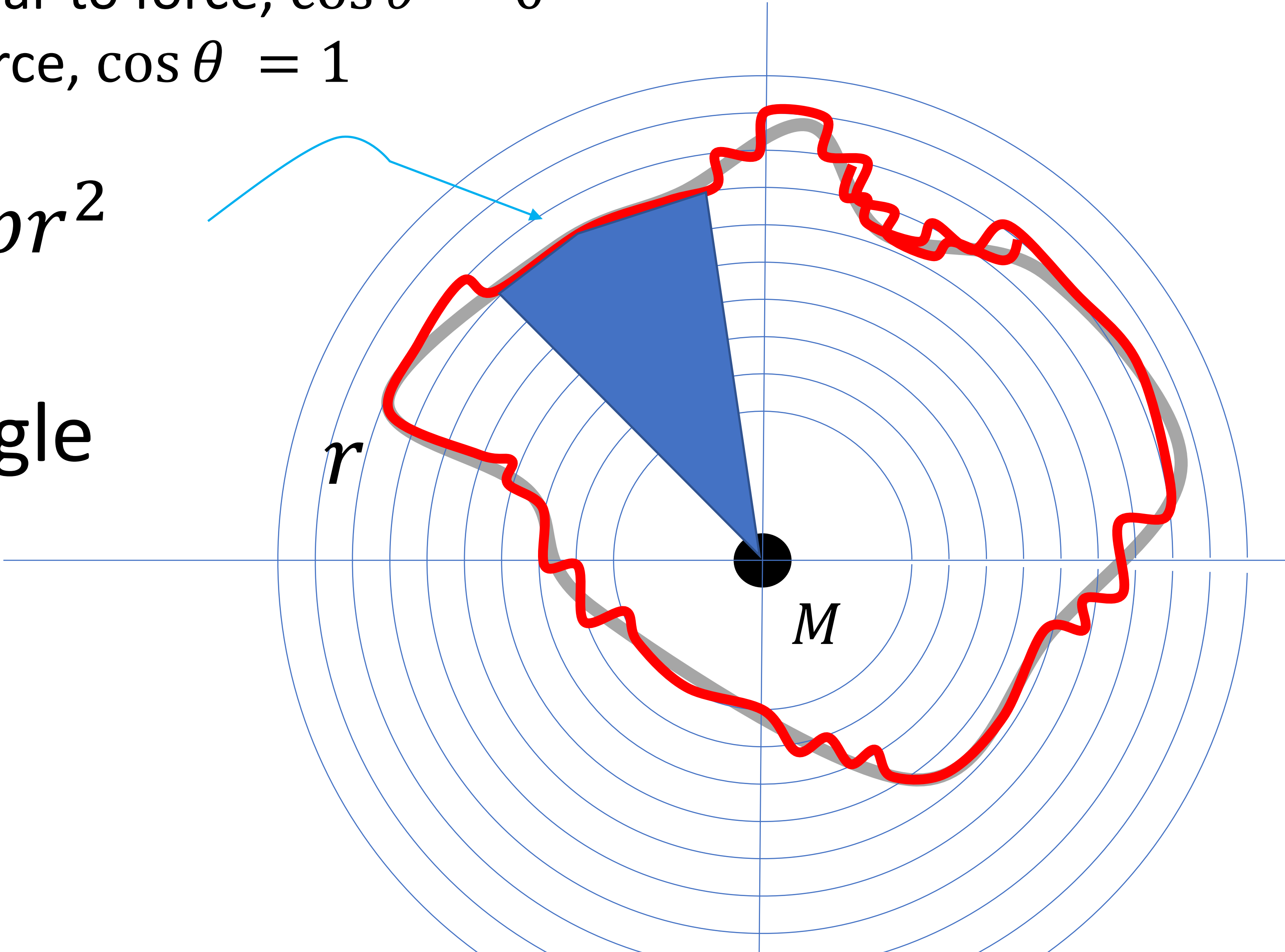
approximate the surface as sum of many “sides” and “caps”

“sides” normal perpendicular to force, $\cos\theta = 0$

“caps” normal parallel to force, $\cos\theta = 1$

area of a cap , $4\pi p r^2$

p= fraction of total spherical angle



approximate the surface as sum of many "sides" and "caps"

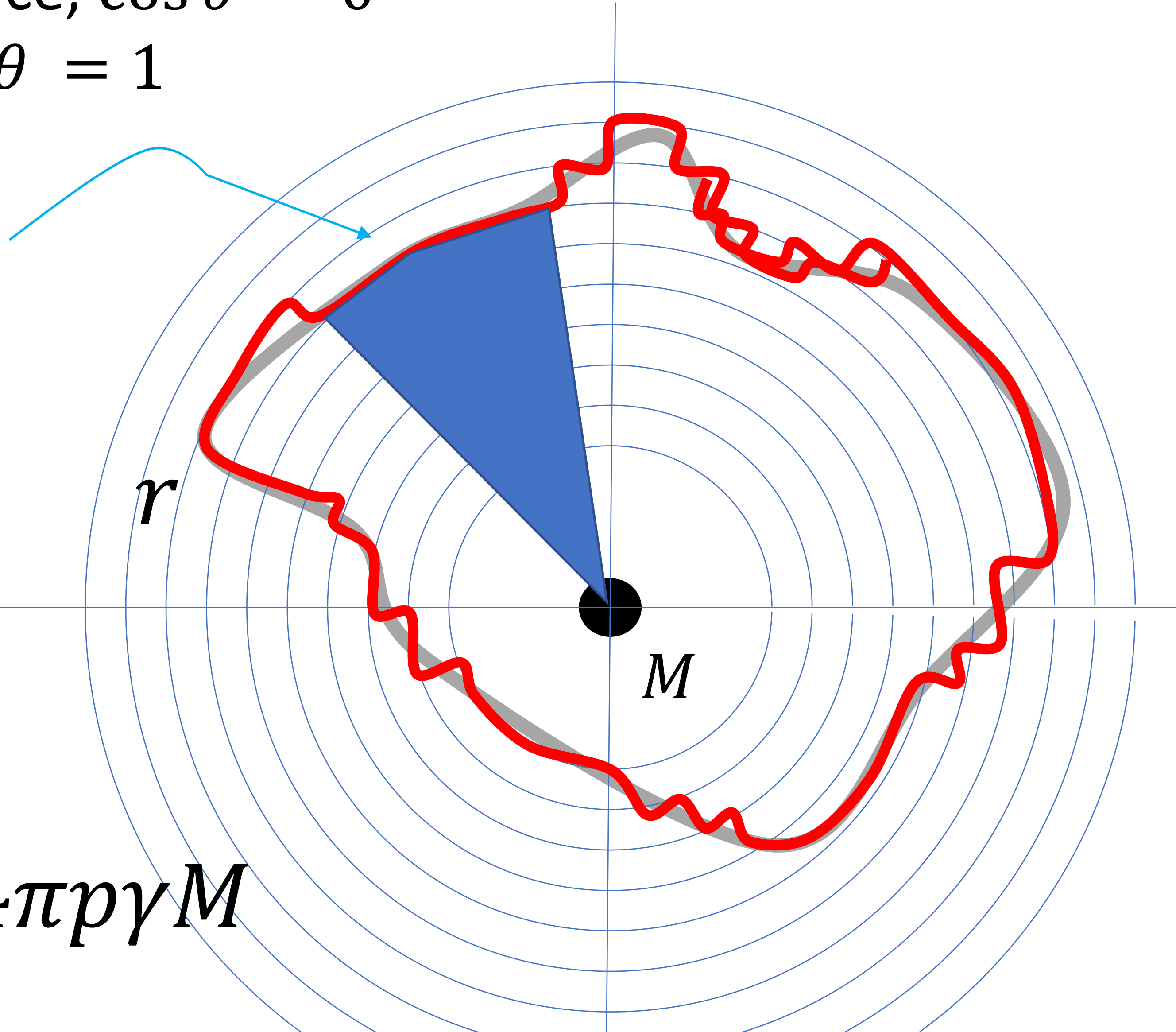
"sides" normal perpendicular to force, $\cos\theta = 0$
"caps" normal parallel to force, $\cos\theta = 1$

area of a cap , $4\pi p r^2$
p= fraction of
 total spherical angle

force on cap , $-\frac{\gamma M}{r^2}$

flux thru cap

$$q = (4\pi p r^2)\left(-\frac{\gamma M}{r^2}\right) = -4\pi p\gamma M$$



approximate the surface as sum of many “sides” and “caps”

“sides” normal perpendicular to force, $\cos\theta = 0$

“caps” normal parallel to force, $\cos\theta = 1$

area of a cap , $4\pi p r^2$

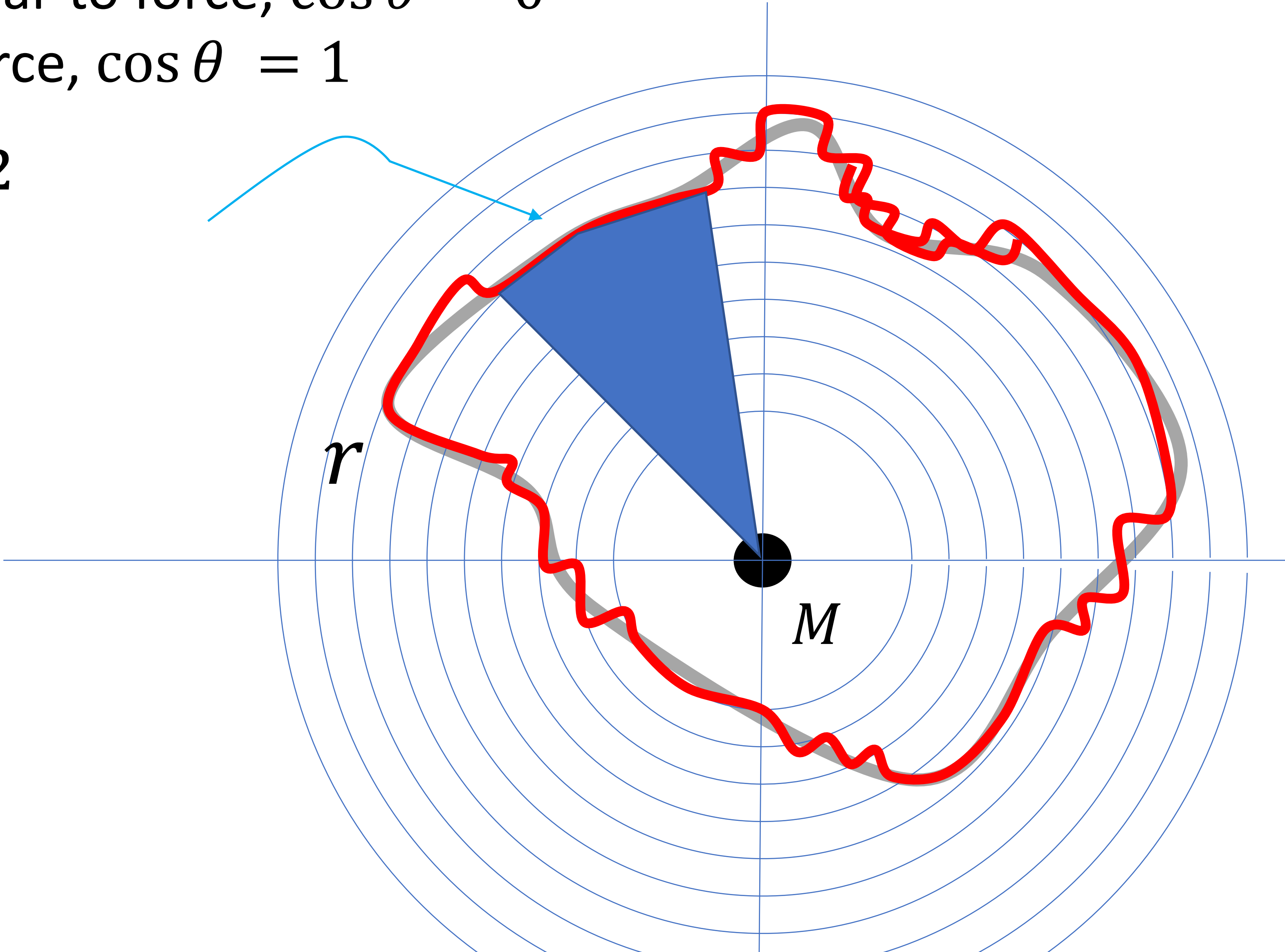
p= fraction of total spherical angle

force on cap , $-\frac{\gamma M}{r^2}$

flux thru all caps

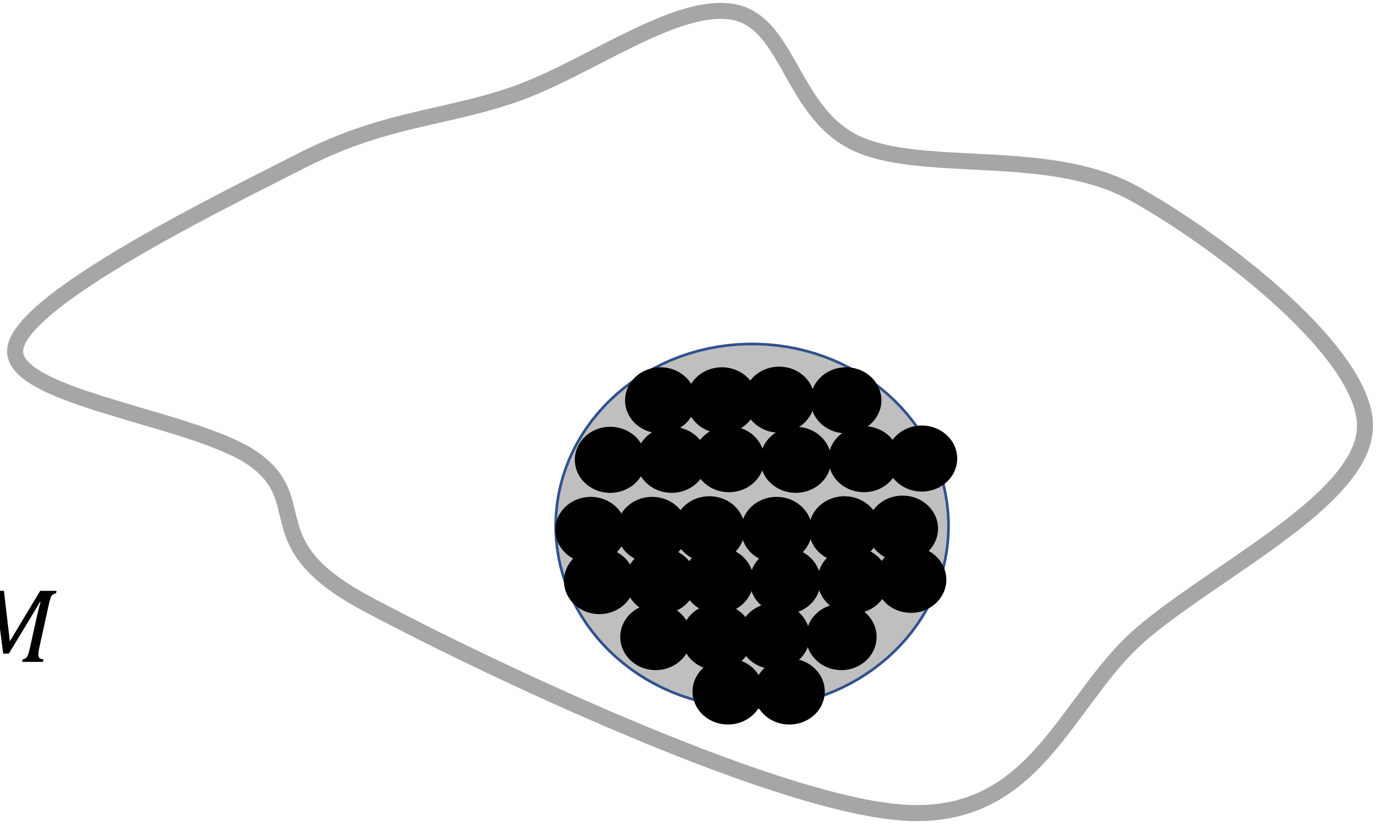
q $= -4\pi\gamma M$

size p’s sum to 1



Step 2:

construct sphere by assembling many point masses with total mass, M



flux is still

$$q = -4\pi\gamma M$$

since fluxes sum

and

flux from any one point mass is independent of its position

Step 3:

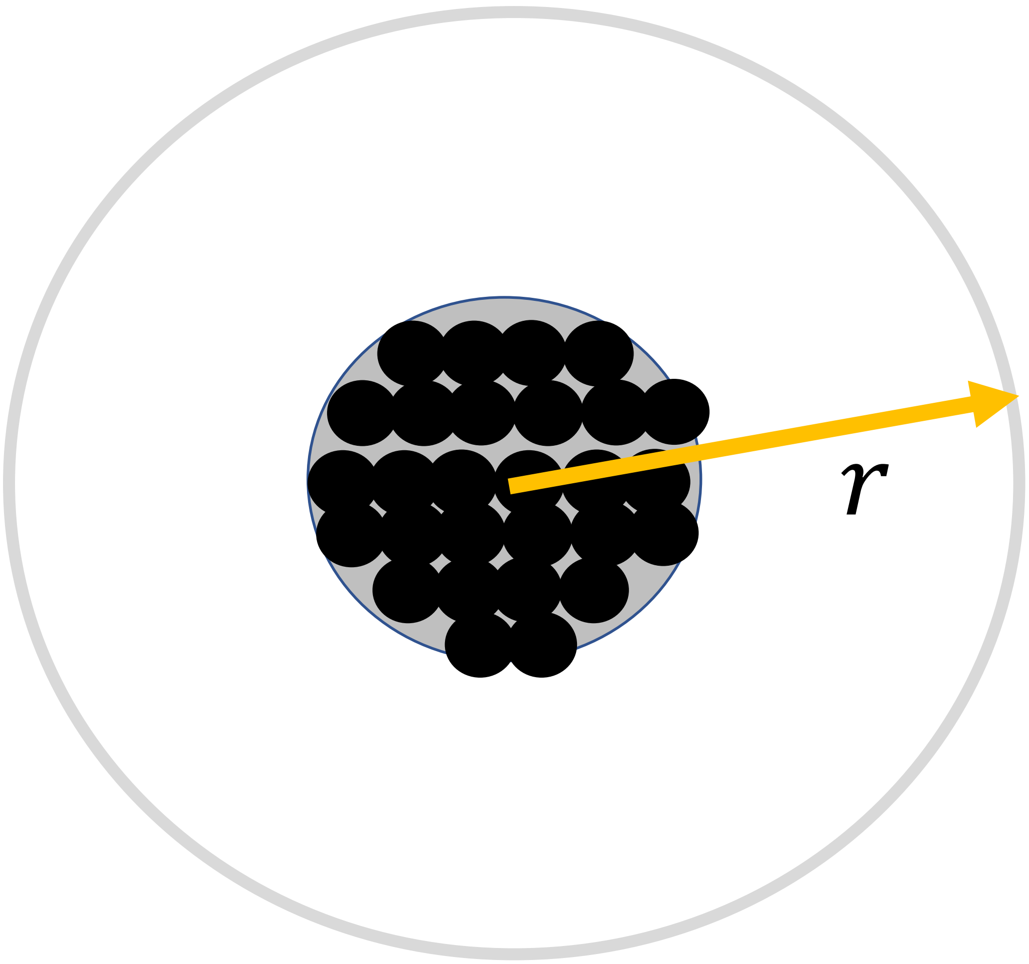
now make the surface a sphere of radius r centered at the center of the mass , so the problem is spherically symmetric

flux is still

$$q = -4\pi\gamma M$$

since flux is independent of shape of surface

furthermore, this shape is all “cap”, so $\cos\theta = 1$



Step 4:

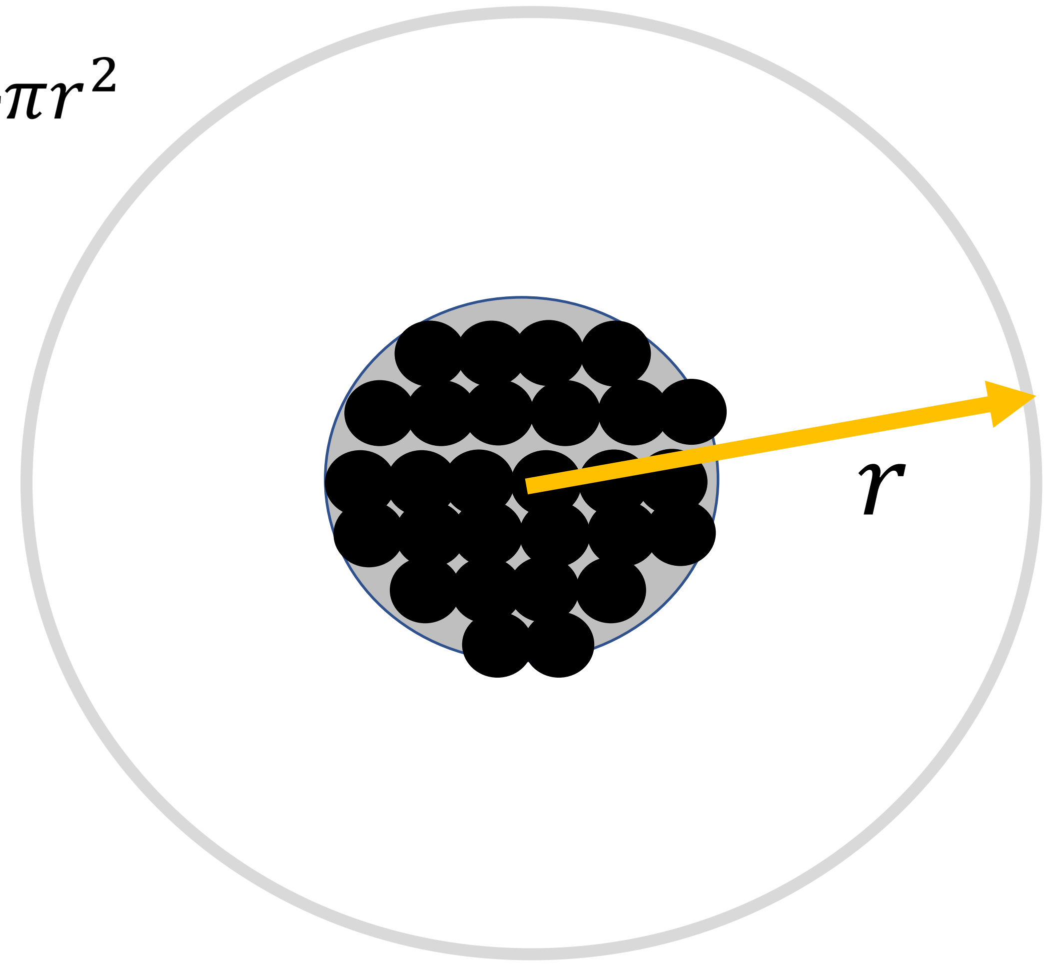
As the problem is spherically symmetric, the flux must be constant over the surface of the sphere

The sphere has area, $\mathrm{A} = 4\pi r^2$

so the flux per area is

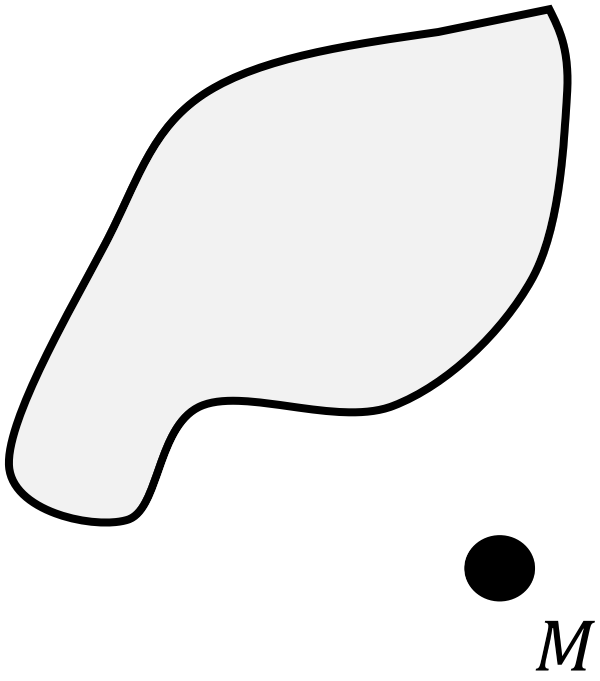
$$q/A = -\frac{\gamma M}{r^2}$$

which is equal to the force and has the form of the force from a point mass



An interesting tidbit

The flux through a surface excluding a point mass, M, is always $q = 0$:



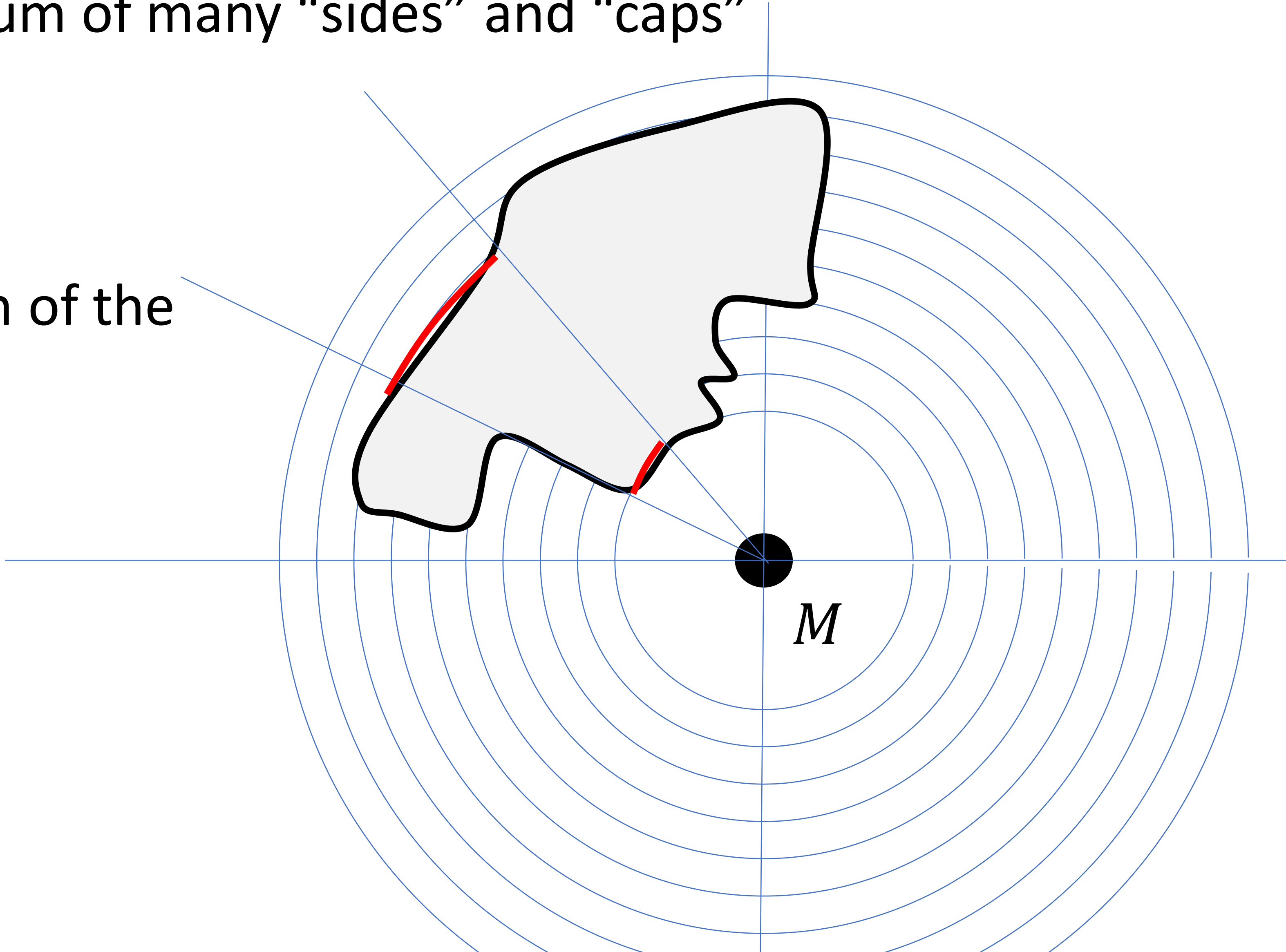
irrespective of the shape
of the surface
and
irrespective of where outside
the excluding volume the point mass is located

handled the same way

approximate the surface as sum of many “sides” and “caps”

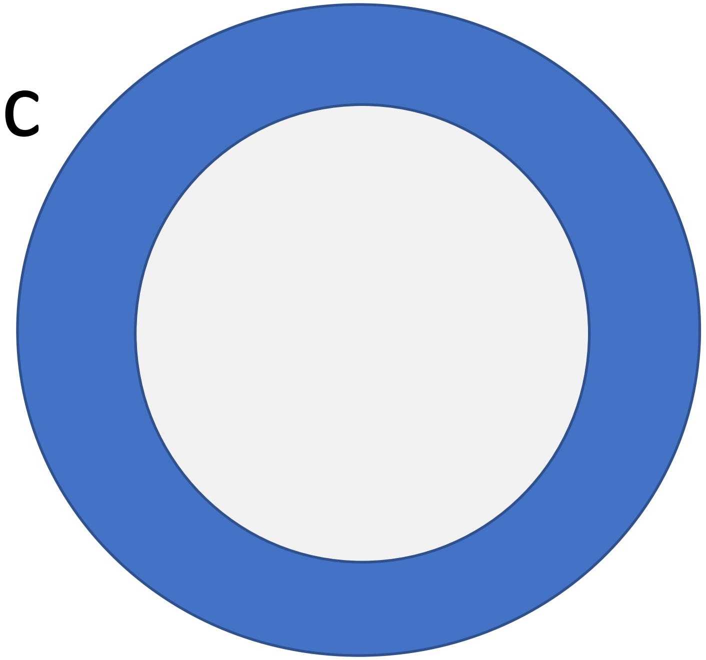
but now that caps are paired

you find that the contribution of the two caps cancel



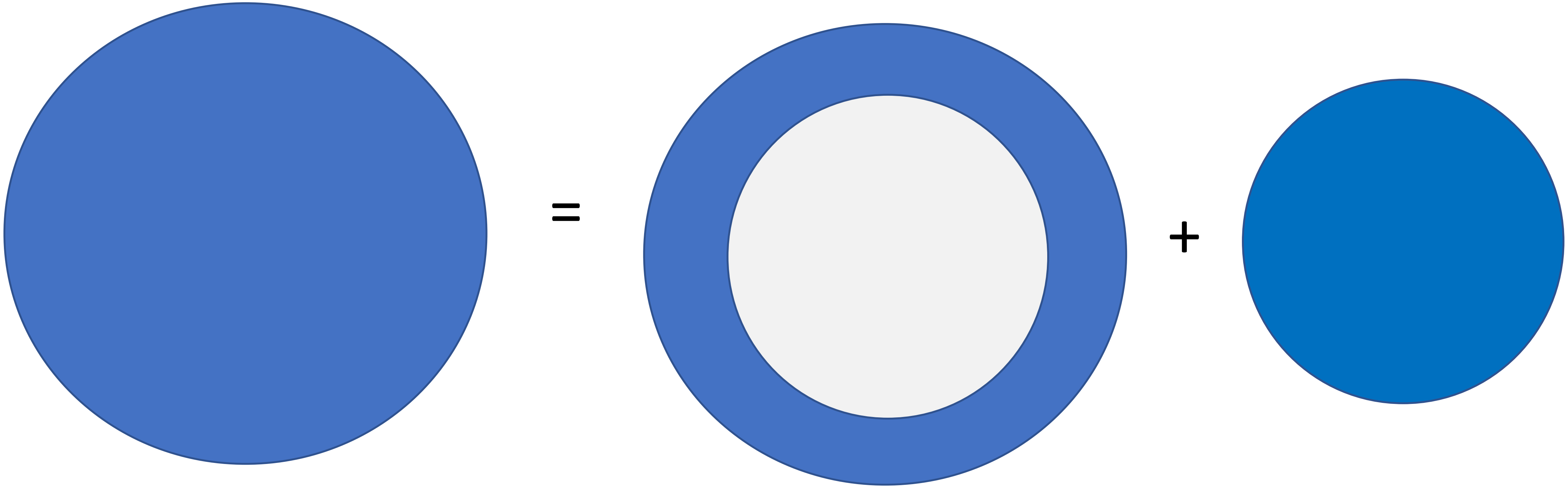
You can use this to show

force inside a cavity
inside a spherically symmetric
object is zero

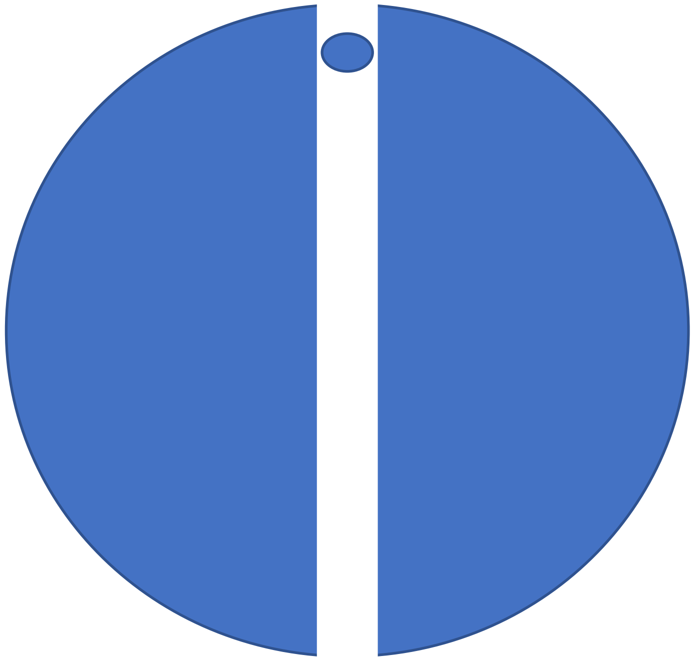


And

force at depth in a sphere depends only on mass below you



throw rock of mass m into small hole drilled thru earth

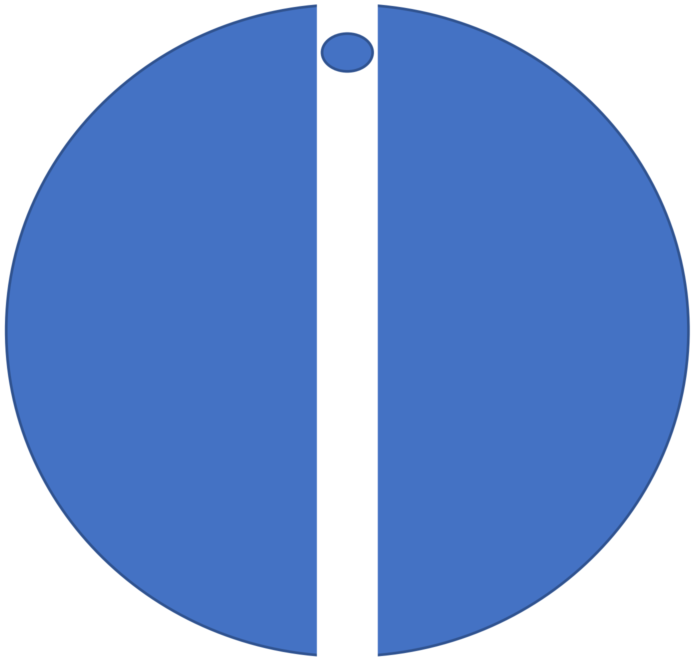


what do you think will happen?

throw rock of mass m into small hole drilled thru earth

force law

$$f_r = m\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$$

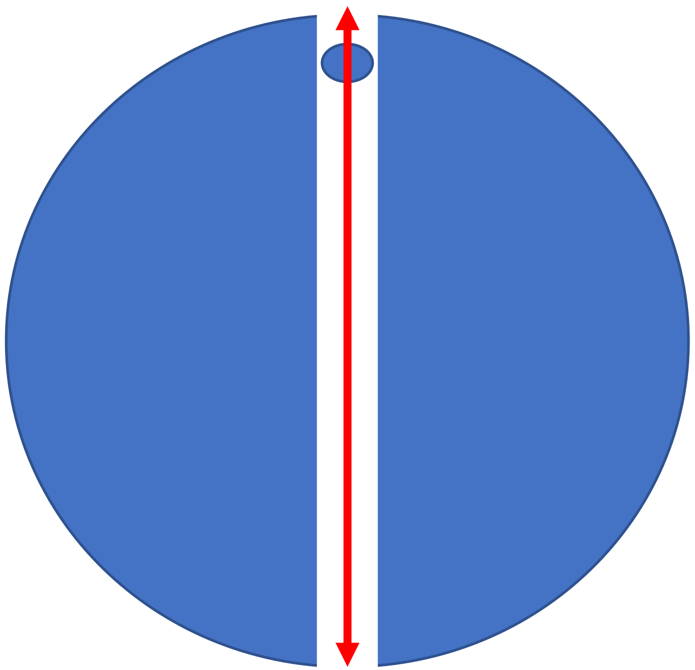


Newton's Law

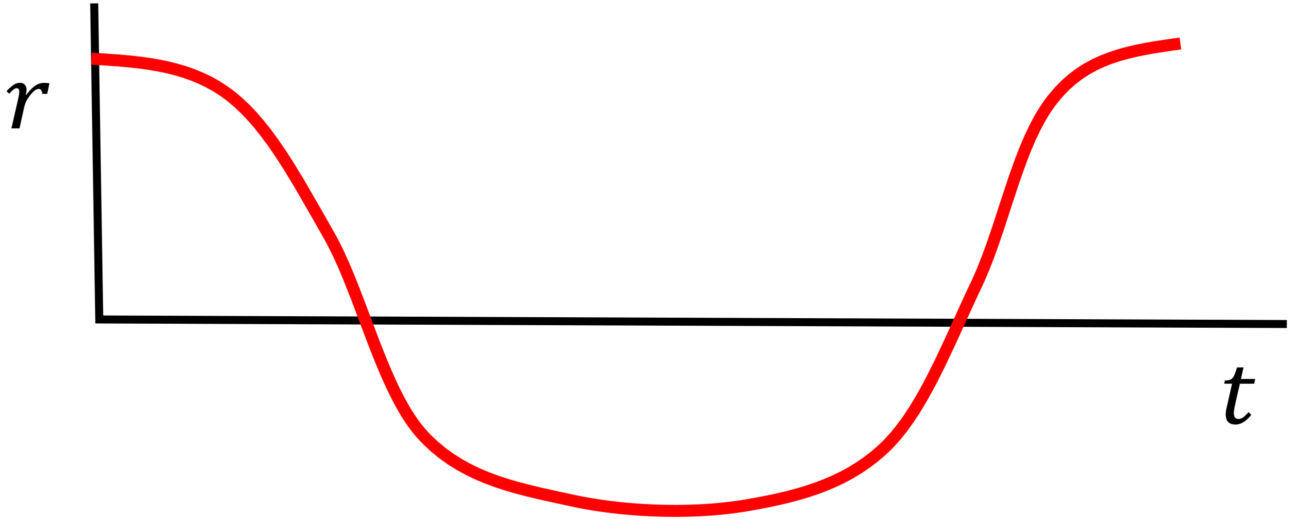
$$f_r = -\frac{\gamma m M(r)}{r^2} \quad \text{with } M(r) \approx \frac{4}{3}\pi r^3 \rho$$

$$f_r = -\frac{4}{3}\gamma m \pi r \rho = -Cmr$$

$$\frac{d^2 r}{dt^2} = -Cr$$



$$r = R_0 \cos(\sqrt{C}t)$$



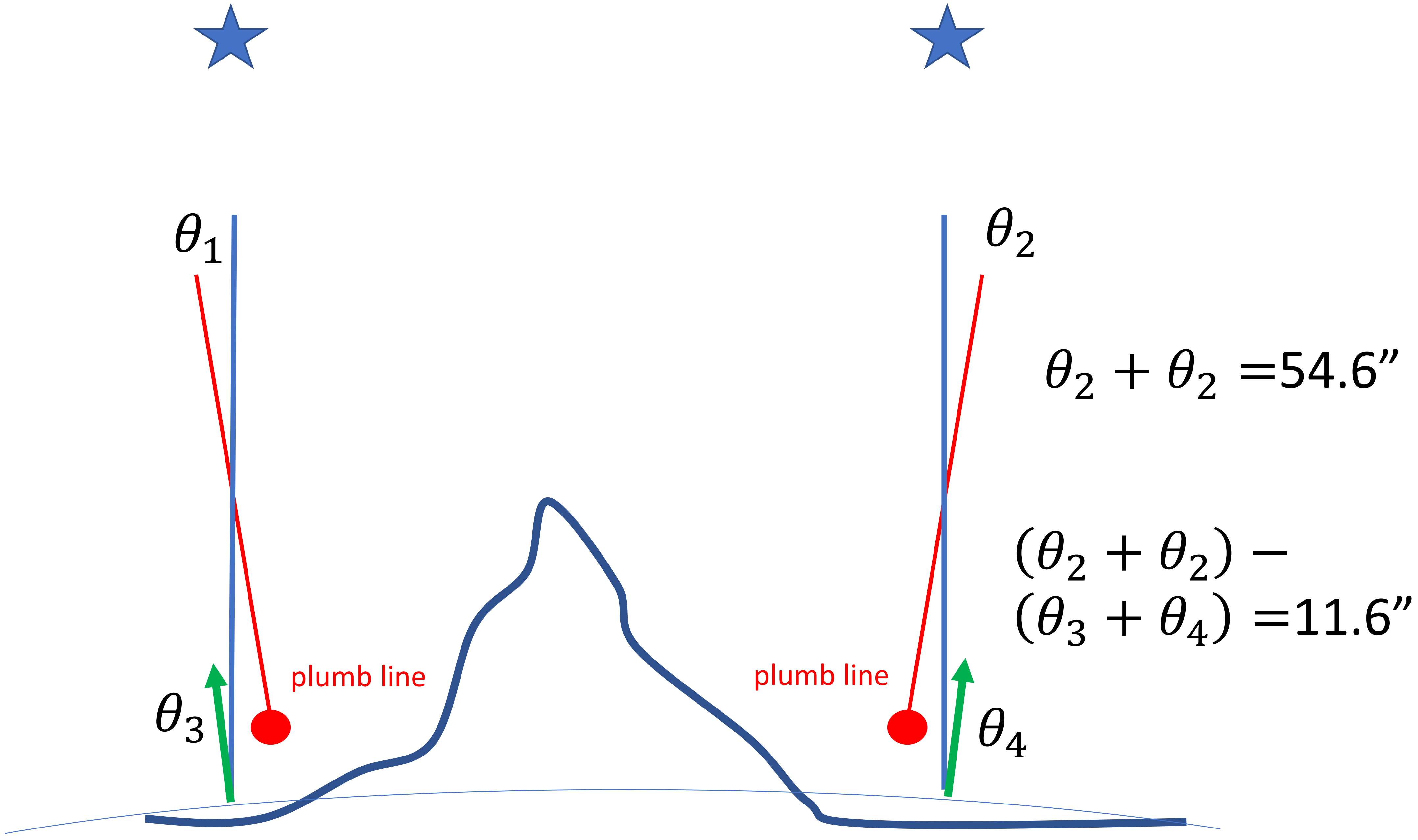
measuring gravity

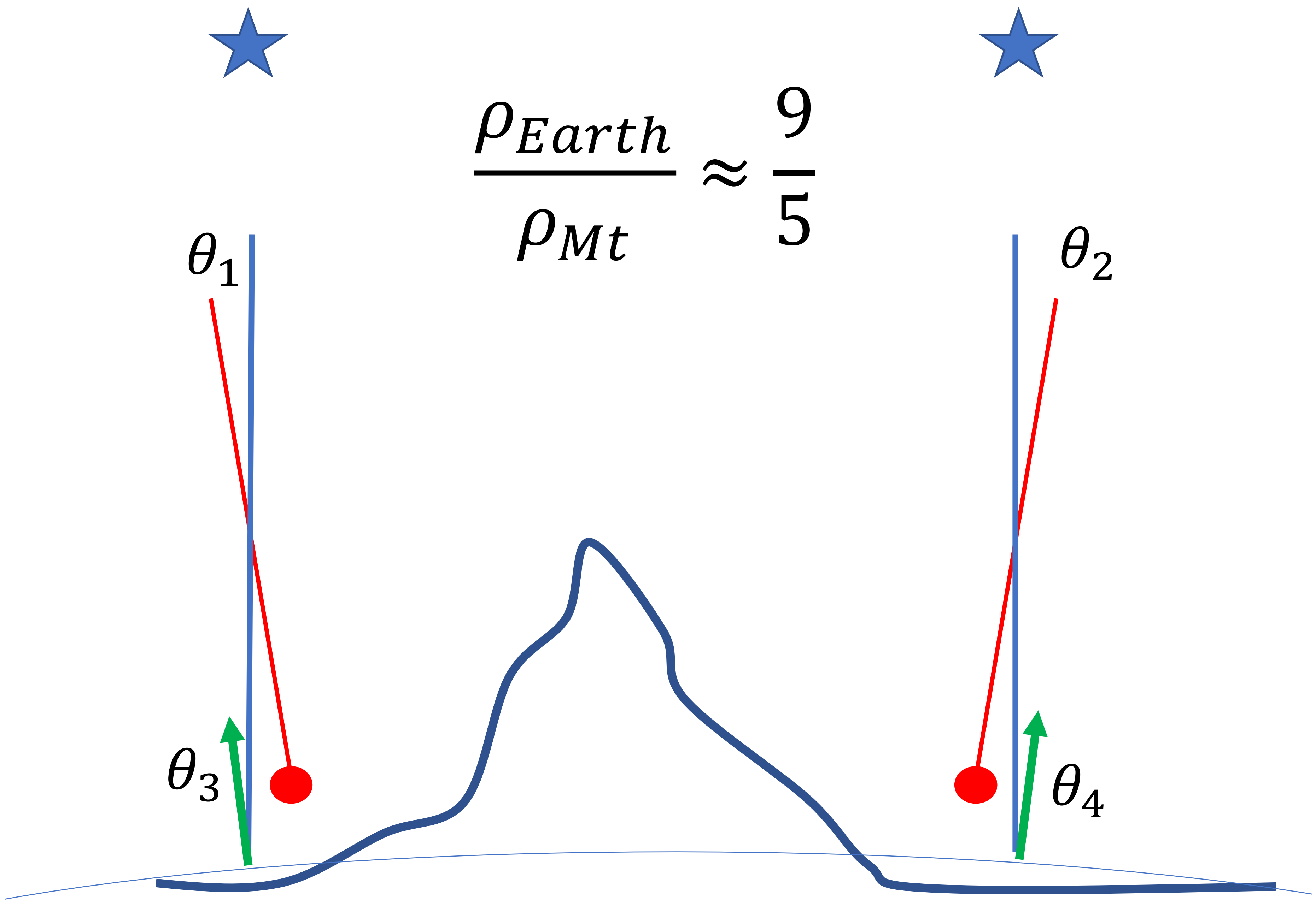
does gravity point straight down?

and if it doesn’t

how would you tell?

Mt Schiehallion Experiment, 1774


θ_1
θ_2
$\theta_2 + \theta_2$ =54.6"
$(\theta_2 + \theta_2) -$
$(\theta_3 + \theta_4)$ =11.6"
plumb line
plumb line
θ_3
θ_4

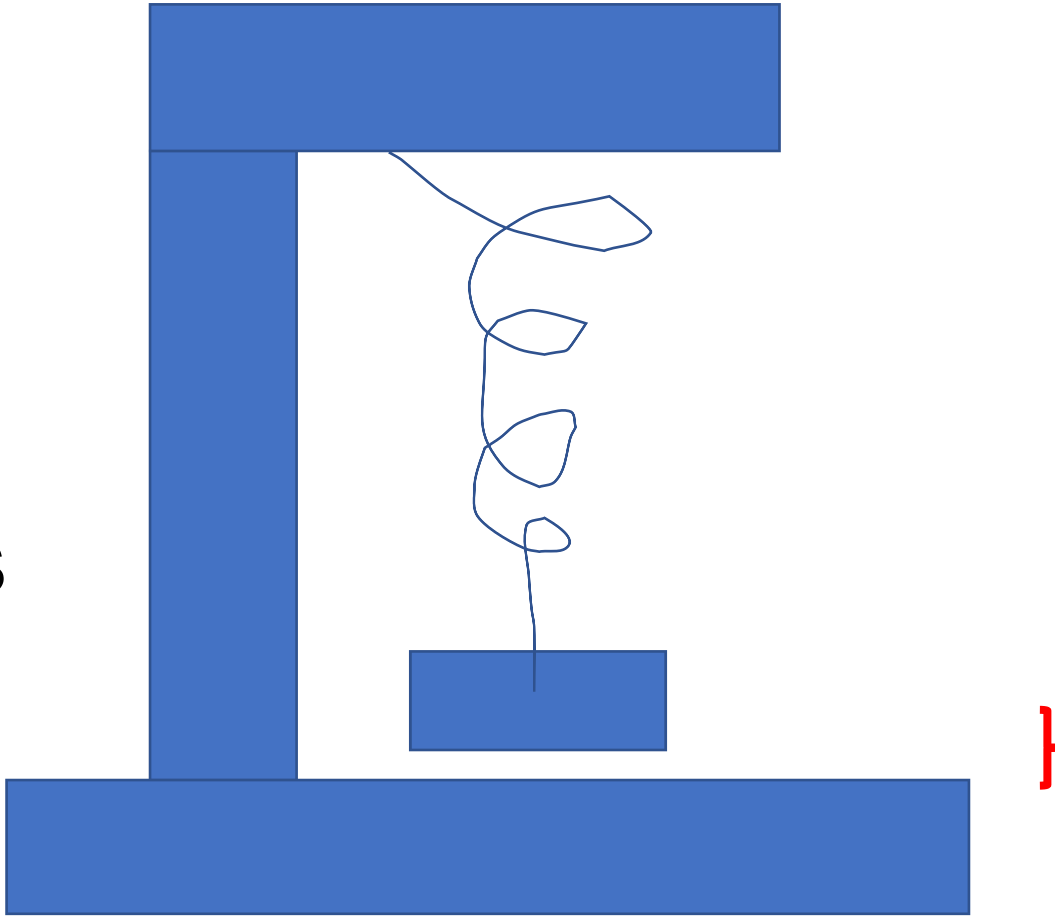


$\frac{\rho_{Earth}}{\rho_{Mt}} \approx \frac{9}{5}$
θ_1
θ_2
θ_3
θ_4

How do you measure the strength of gravity?

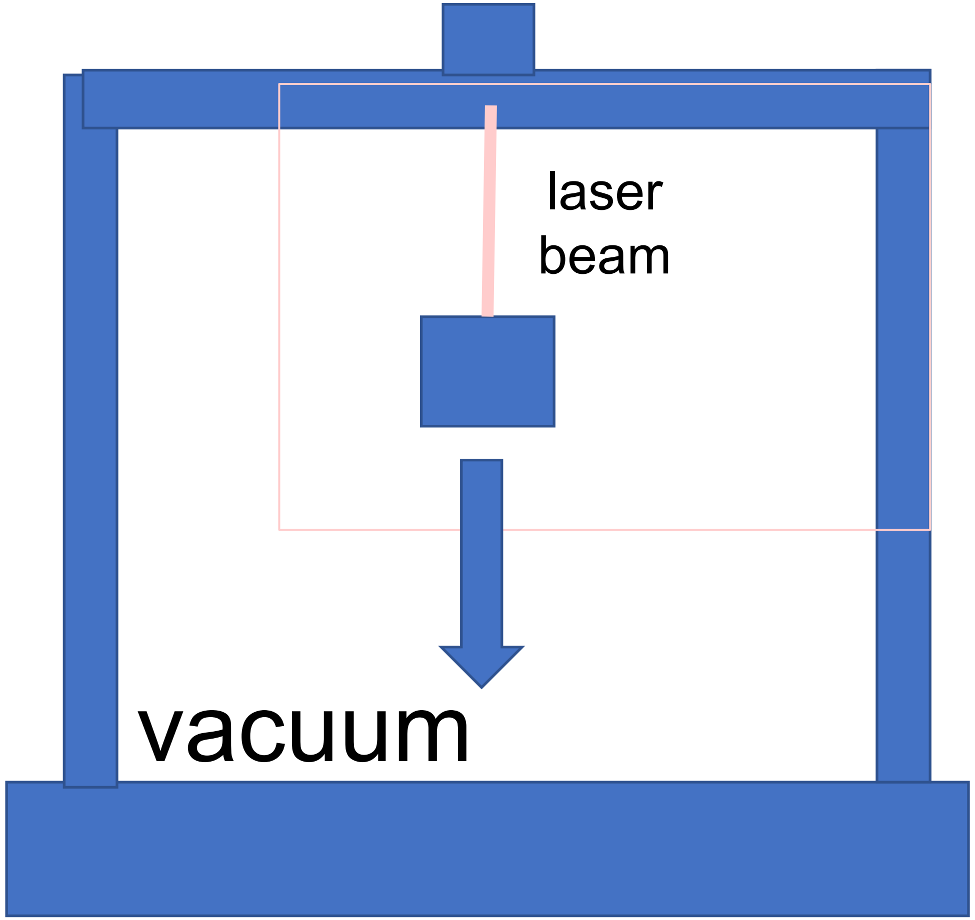
deflection of a mass weighing down a spring

best at relative measurements



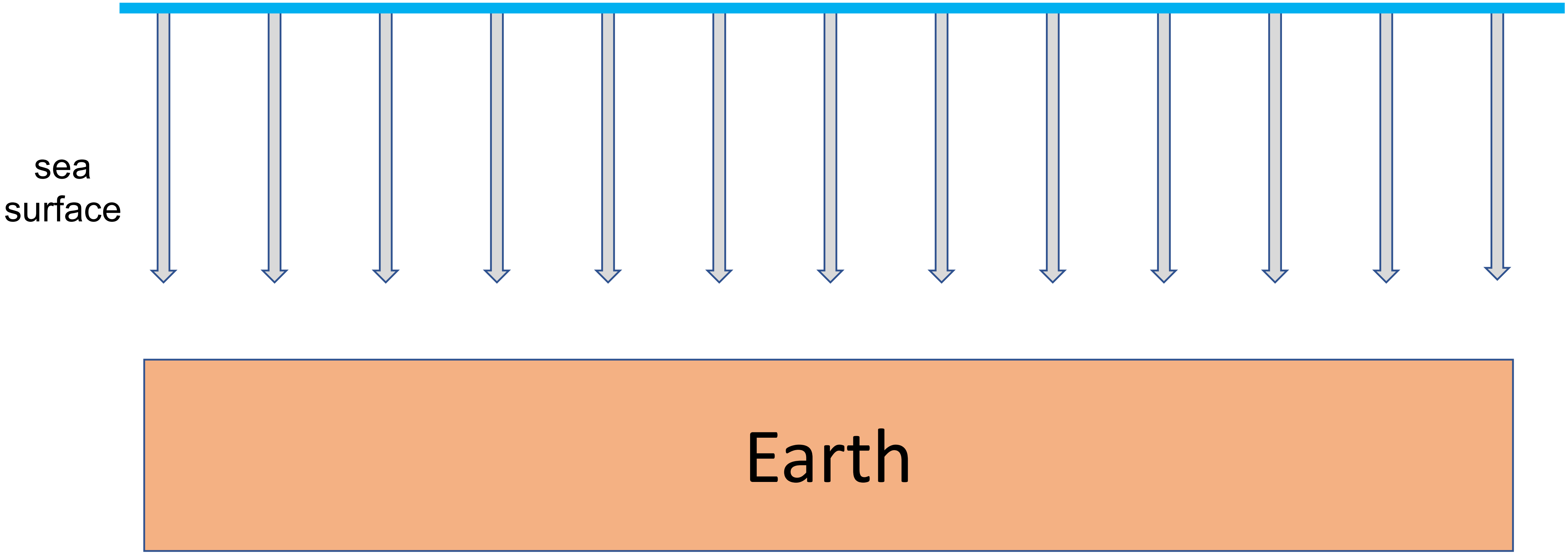
measure small changes in height

drop small mass in vacuum and measure its acceleration

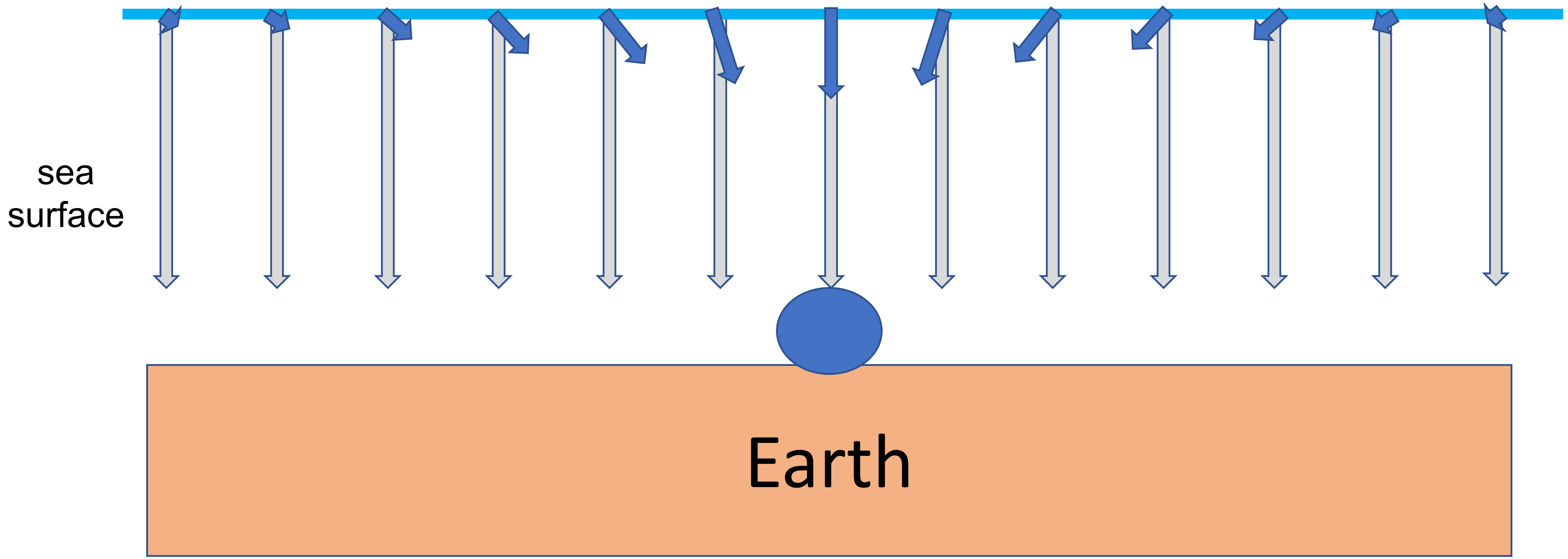


measure acceleration optically with laser interferometry

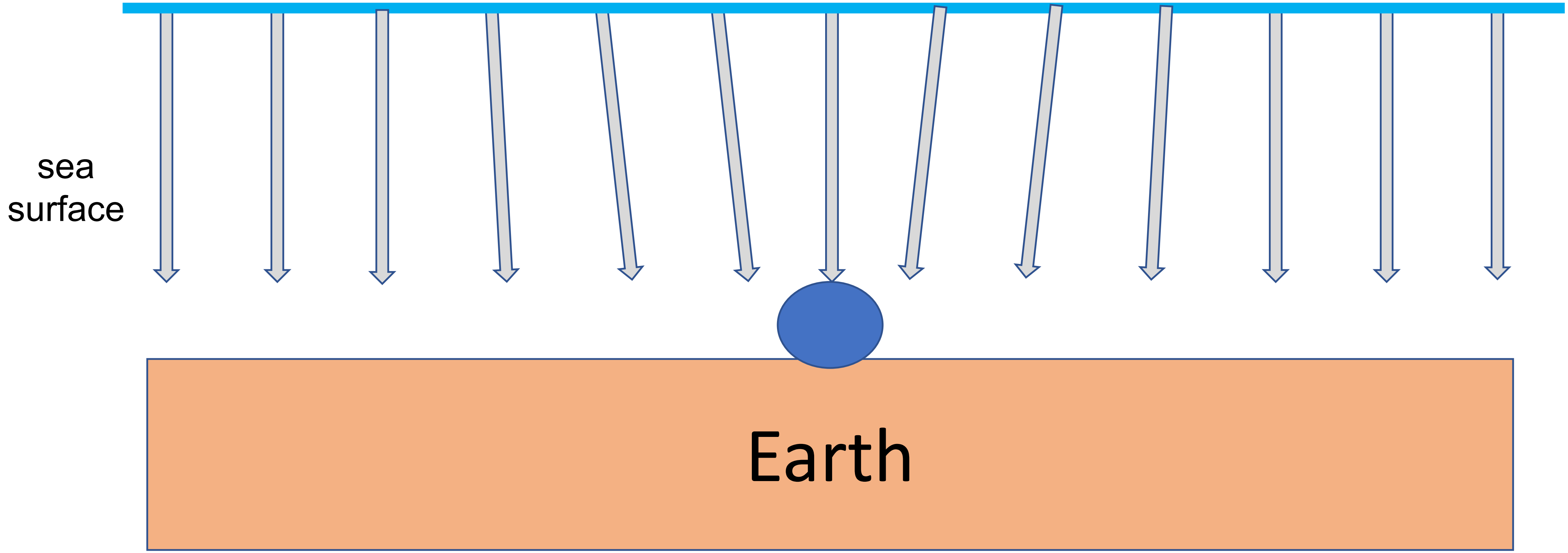
A fluid adjusts so it surface is everywhere parallel to the local gravitational field

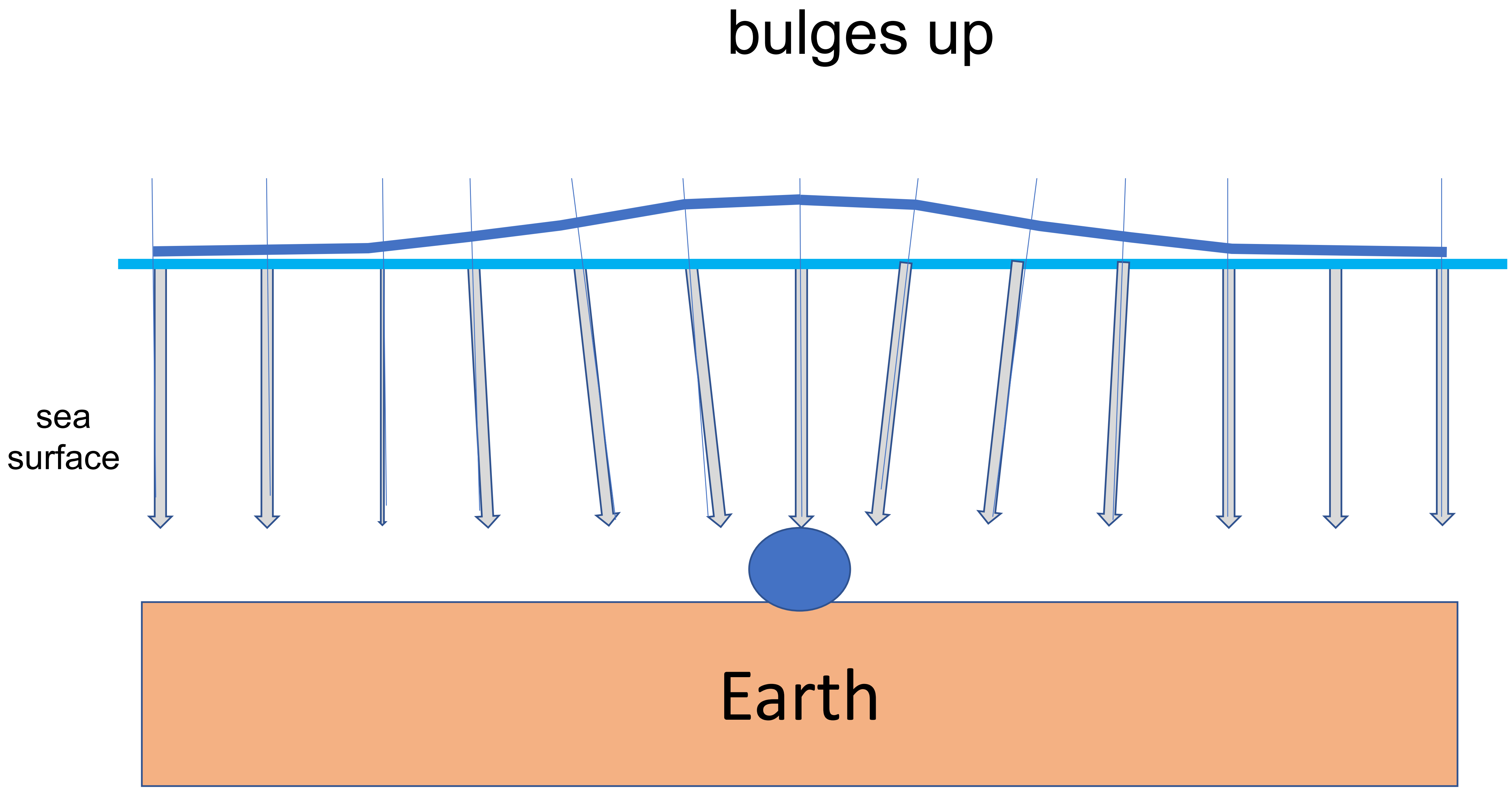


A fluid adjusts so it surface is everywhere parallel to the local gravitational field

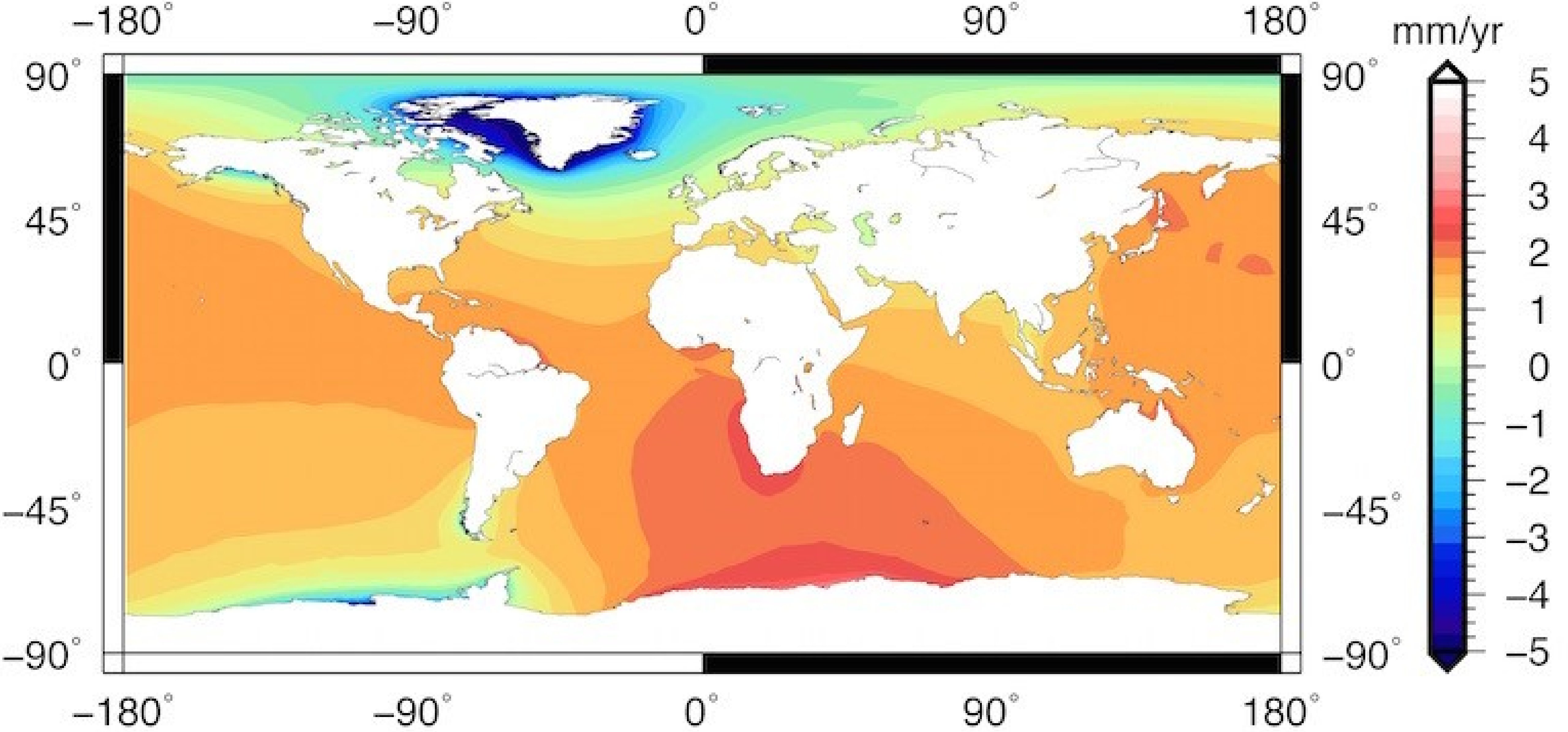


Which way does the water go?




bulges up
sea
surface
Earth

What will happen to sea level near Greenland when the ice melts?



−180°
−90°
0°
90°
180°
90°
45°
0°
−45°
−90°
mm/yr
5
4
3
2
1
0
−1
−2
−3
−4
−5